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Coalitional Nash Equilibria

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Abstract

Let G be an N-player game in strategic form and $\mathcal{C}$ be a set of permissible coalition of players (exogenously given). A strategy profile σ is a coalitional-equilibrium if no permissible coalition in $\mathcal{C}$ has a unilateral deviation that profits to all its members. At the two extremes: when $\mathcal{C}$ contains only singleton players, σ reduces to a Nash equilibrium and when $\mathcal{C}$ consists on all coalitions of players, σ is a strong Nash equilibrium. Our paper provides conditions for existence of coalitional equilibria that combine quasi-concavity and balancedness.

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1 Introduction

Nash [18] and strong Nash [2] equilibria are among the main solution concepts in non-cooperative game theory. The first asks the stability of the strategy profile against all single player unilateral deviations while the second asks the stability against all coalitions unilateral deviations. The usual conditions that are assumed to show existence seems of different kind. For the existence of Nash equilibria, the game is supposed to be quasi-concave while for the existence of strong equilibria, it is assumed to be balanced [4] (formal definitions are given in the next sections).

Our paper proposes to unify both notions in one single model and provides a condition that is reduced to quasi-concavity for Nash equilibria and is related to balancedness for strong equilibria. More precisely, the concept of coalitional equilibrium is defined and a sufficient geometric condition is provided for its existence. To do so, some classical fixed point theorems are revisited, related to each others and sometimes extended.

The concept of coalitional equilibrium lies between Nash equilibrium and strong equilibrium. An exogenous coalitional structure defines which coalitions are permissible to

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deviate. A strategy profile is a coalitional-equilibrium if it is stable against all permissible coalitions unilateral deviations. When only single-player coalitions are permissible it is Nash-equilibrium and when all coalitions are permissible it is strong-equilibrium. The motivation is straightforward. In many applications (voting, council of Europe, or market competition) some coalitions are not natural and so cannot be expected to coordinate (extreme-leftists and -rightists, western and eastern countries, or firms of different areas).

Ray and Vohra [19] consider a solution concept that uses coalitional equilibria but only for partition structures. First, for each partition of the players, they associate the set of coalitional equilibria (as in our definition). Second, given a coalitional-equilibrium for a given partition, one can define internal stability with respect to stable coalitional equilibria of all finer partitions. The set of stable equilibria for a given partition is constructed recursively. Starting from the finest partition structure (that contains only singleton), one can construct stable coalitional equilibria for larger partition structures, and so on. Clearly, their construction could be extended naturally by relaxing the partition assumption.

In section 2.1 Sonnenschein's [24] classical fixed point theorem is established in topological vector spaces (TVS) without the Hausdorff and the local convexity usual assumptions[1]. This theorem provides concavity and continuity conditions on a non-transitive preference relation to have a maximal element. Secondly, it is shown that Gale and Mas-Collel's theorem [6] still hold under the same topological assumptions and so also Nash-Glicksberg's theorem. This conclusion has already been reached by Reny [20]. Interestingly, Nash-Glicksberg's theorem still valid even when there are infinitely many players. The proofs follow the one in Border [4]. Our unique contribution relies on the observation that (1) Fan-KKM's lemma, which is known to hold without the local convexity [1] assumption, still hold without the Hausdorff hypothesis and (2) that Fan-KKM's lemma implies all the fixed point theorems of section 2.1.

In section 2.2, a new version of Sonnenschein's theorem is established for locally convex, not necessary Hausdorff TVS. The new version reinforces the concavity assumption and relaxes the continuity hypothesis of the classical version. The new continuity requirement merely asks some correspondence to be lower-hemi-continuous, while the classical version asks the lower sections of the correspondence to be open. Sections 2.3 and 2.4 studies some related fixed point theorems (dealing with lower-hemi-continuous correspondences) in Hausdorff, metrizable, Banach and finite-dimensional spaces.

The fixed point theorems for hemi-continous preferences established in sections 2.2 and 2.4 are used in section 3 to obtain a condition for the existence of coalitional equilibria in strategic games. This new condition is shown to be equivalent to quasi-concavity for Nash equilibria and to be closely related to balancedness for strong equilibria.

[1]It was already known that local convexity is not necessary [1].

2 Some Fixed Point Theorems Revisited

2.1 General Topological Vector Spaces

S is a compact convex set of a topological vector space (TVS). The interior of $X \subset S$ relatively to S is denoted $intX$, its convex envelope coX and the closed convex hull $\overline{co}X$.

The classical results described in this section usually assumes the TVS to be Hausdorff. We will show that this assumption is not needed. Our proofs are slight adaptations of the corresponding one in [1] or [4].

Assume the following classical KKM lemma that could easily be proved using the famous Sperner's lemma (see [4]).

Lemma 1 (Knaster-Kuratowski-Mazurkiewicz [7]) *Let $\{x_1, ..., x_k\} \in \mathbf{R}^m$ and $\{F_k\}$ be a family of closed subset of the Euclidean linear vector space $\mathbf{R}^m$. If for any $K \in \{1, ..., k\}$, $co\{x_i, i \in K\} \subset \cup_{i \in K} F_i$, then $\cap_{i=1,...,k} F_i$ is non-empty and compact.*

Let us show that following Fan's lemma (see [1] theorem 17.46) does not need the Hausdorff assumption.

Lemma 2 (Fan) *If the correspondence F from S to S has closed values and if for any finite family $\{x_1, ..., x_k\}$, $co\{x_1, ..., x_k\} \subset \cup_{i=1,...,k} F(x_i)$ then $\cap_{x \in S} F(x) \neq \emptyset$.*

Proof. For each finite family $\{x_1, ..., x_k\}$, let $G(x_i) = F(x_i) \cap co\{x_1, ..., x_k\}$. Since each $G(x_i)$ is finite dimensional, and since the Euclidean topology is finer than the initial topology [21] on the vectorial space generated by $\{x_1, ..., x_k\}$, each $G(x_i)$ is also closed in the Euclidean topology. KKM lemma then implies that $\cap_{i=1,...,k} G(x_i) \neq \emptyset$ and consequently $\cap_{i=1,...,k} F(x_i) \neq \emptyset$. Since a family of compact sets with the finite intersection property has a nonempty intersection, $\cap_{x \in S} F(x) \neq \emptyset$. ■

Let A be a correspondence on S (i.e. from S to S), best viewed as a (not necessarily transitive) preference where $A(x) \subset S$ is interpreted as the set of points in S strictly better than x (that is $A(x) = \{y \in X : y \succ x\}$ where $\succ$ is a preference relation). The set of maximal elements of A is $E = \{x \in S \text{ such that } A(x) = \emptyset\}$.

A famous implication of Fan's lemma is the following important result, first established in Sonnenschein [24]. For its proof and the numerous applications, see [1] and [4]. This theorem is equivalent to other fixed point theorems of the literature [15]. In general, the theorem is stated with the Hausdorff assumption. This is not needed.

Theorem 3 (Sonnenschein [24]) *Let A be a correspondence on S. **If** (i) for all $x \in S$, $x \notin coA(x)$ and (ii) for any $y \in A^{-1}(x)$ there exists $x' \in S$ (possibly $x' = x$) such that $y \in intA^{-1}(x')$, **then** the set of maximal elements of A is compact and non-empty.*

Let us first extends the useful lemma 17.47 in [1] to non Hausdorff TVS (using exactly the same proof).

Lemma 4 (Aliprantis Border [1]) *If for each $x \in S$, $x \notin coA(x)$, then $\cap_{x \in S} F(x) \neq \emptyset$ where $F(x)$ is the closure of the complementary of $A^{-1}(x)$: $F(x) = \overline{S \backslash A^{-1}(x)}$.*

Proof. From Fan's lemma, it is sufficient to show that for any finite family $\{x_1, ..., x_k\}$ and any $y \in co\{x_1, ..., x_k\}$, $y \in \cup_{i=1,...,k} F(x_i)$. Suppose not. This implies that for any i, $y \notin F(x_i)$ implying that $x_i \in A(y)$ so that $y \in coA(y)$, a contradiction. ■

Now we can know prove Sonnenschein's theorem in without the Hausdorff assumption of the TVS.

Proof. We follow the proof in Border [4]. Note that $E = \cap_{x \in S} S/A^{-1}(x)$. By (ii), $E = \cap_{x' \in S} S \backslash intA^{-1}(x')$ and so it is compact (as the intersection of a family of compact sets). Thus $\overline{S \backslash A^{-1}(x)} \subset \overline{S \backslash intA^{-1}}(x)$. Hence, $\cap_{x \in S} \overline{S \backslash A^{-1}(x)} \subset \cap_{x \in S} S \backslash intA^{-1}(x)$. By (i) and the lemma 4, $\cap_{x \in S} \overline{S/A^{-1}(x)} \neq \emptyset$. ■

Assumption (i) of Sonnenschein asks that coA does not have a fixed point. The condition may be viewed as a quasi-concavity assumption on the preference relation. Assumption (ii) imposes a kind of continuity on the preference relation but allows the possibility of having some discontinuities. It is satisfied if for example the lower-sections $A^{-1}(x)$ are open for every x. However, (ii) is not satisfied if A is only lower-hemi-continuous (i.e. $A^{-1}(W)$ open for any W open).

Let us now extend Gale and Mas-Collel theorem to non Hausdorff TVS as well.

A correspondence A on S is called[2] FS if (i) for all $x \in S$, $x \notin coA(x)$ and (ii-o) it has an open graph. A is locally FS-majorized at x if there is a FS-correspondence B^x on S and a neighborhood V^x of x such that $A(y) \subset B^x(y)$ for all $y \in V^x$. A is globally FS-majorized if B^x is the same for all x.

Observe that for a correspondence, open graph implies open lower sections which implies lower-hemi-continuity.

Lemma 5 (Borling/Keiding [5]) *If A is everywhere locally FS-majorized, it is globally FS-majorized and thus admits a maximal element.*

Proof. Follow again the proof in Border [4]. For each x, let B^x locally FS-majorizes A on a closed neighborhood V^x of x. Let $V^{x_1}, ..., V^{x_k}$ be a finite subcover of S (by compactness of S). Define the correspondence $D^i(x) = B^{x_i}(x)$ if x is in V^{x_i} and S if not. Let $B(x) = \cap_i D^i(x)$. Then B is a FS-correspondence that globally majores A. ■

Assume now that $S = \prod_{i \in N} S_i$ where each S_i is a compact-convex subset of a TVS (typically, the strategy set of player i where N is the set of players). By Tychonoff's theorem [1], S is convex and compact for the product topology. N may be any set, not necessarily finite. As usual, let $S_{-i} = \prod_{j \neq i} S_j$ be the set of profiles of players other than i. Let $\{A_i\}_{i \in N}$ be a family of correspondences, where for each i in N, A_i is a correspondence from S to S_i. The set of maximal elements of $\{A_i\}$ is $\{x$ such that $A_i(x) = \emptyset$ for all i in $N\}$.

Theorem 6 (Gale/Mas-Colell [6] Extended) ***If** for each $i \in N$ and $x \in S$ (i) $x_i \notin coA_i(x)$ and (ii) A_i has an open graph then the set of maximal elements of $\{A\}$ is nonempty and compact.*

[2] FS is the name given in Border [4]. F stands for Fan and S for Sonnenschein.

Proof. Follow the proof of [5]. For each i, define $\tilde{A}_i(x) = S_{-i} \times A_i(x)$. Let $I(x) = \{i : \tilde{A}_i(x) \neq \emptyset\}$ et let $B(x) = \cap_{i \in I(x)} \tilde{A}_i(x)$. Since each $B(x)$ is locally majorized by some $\tilde{A}_i(x)$ for each x, it follows from the last lemma that there is an x such that $B(x) = \emptyset$. ■

Observe that Gale and Mas-Colell's proved their theorem in finite dimensions. A direct application is the following extension of the Nash-Glicksberg's [11] theorem to non Hausdorff locally convex TVS.

Let N be a finite set of players. Let $G = (N, \{S_i\}_{i \in N}, \{g_i\}_{i \in N})$ be a strategic game. Assume that for each i in N, S_i, the strategy set of player i, is a compact subset of a TVS and that the payoff function of player i, $g_i : S = \prod_{i \in N} S_i \to \mathbf{R}$ is continuous for the product topology and quasi-concave in $s_i \in S_i$. A strategy profile $s \in S$ is a Nash equilibrium if for all $t \in S$ and all $i \in N$, $g_i(s) \geq g_i(t_i, s_{-i})$. Define $A_i(s) = \{t_i \in S_i : g_i(t_i, s_{-i}) > g_i(s)\}$. Then A_i satisfies (i) and (ii) of the last theorem. Consequently, one has the following extension of Nash-Glicksberg's theorem to any TVS and infinitely many players.

Theorem 7 (Nash-Glicksberg's [11] Extended) *Under the assumptions described above, the set of Nash equilibria of the strategic game is non-empty and compact.*

Reny [20] proved the above theorem in general TVS for finitely many players without assuming continuity but a better-reply security assumption.

Existence of a Nash equilibrium for infinitely many player games could be used to show the following extension of Kreps and Wislon's result to extensive form games with infinite duration (we assume the reader to be familiar with the extensive form game literature).

Theorem 8 (Kreps and Wilson [8] Extended) *Any extensive form game with finitely many players, perfect recall, with at most M actions at each information set, infinite duration and continuous payoff functions with respect to the product topology, has a sequential equilibrium.*

Let us give the sketch of the proof, following essentially Kreps and Wislon's proof. The agent normal form game is the game with infinitely many players where each player is duplicated to as many agents as there are informations sets of that player. Each agent of a player is maximizing his payoff, conditionally to some Bayes-consistent belief that his information set is reached. Let the ϵ-perturbed agent normal form game be the game where all agents are constrained to choose their actions with a probability at least $\frac{1}{M} > \epsilon > 0$. In this game, all information sets are reached with positive probability so the belief system is well defined. Moreover, the belief system moves continuously with respect to the strategies in the ϵ-perturbed game. The last theorem can thus be applied to this perturbed game to obtain existence of Nash equilibrium. As ϵ goes to zero, one obtains, by compactness and diagonal extraction, a limiting strategy profile and an associated consistent belief system where, at each information set, the one-shot deviation principle is satisfied for the agent playing at that information set. This defines a sequential equilibrium.

2.2 Locally Convex TVS

Assume now that the Topological vector space is locally convex (LC). Under this additional assumption, the fixed point theorem that follows reinforces slightly the quasi-concavity assumption (i) and relaxes sufficiently the continuity assumption (ii). This allows the possibility of considering lower-hemi-continuous correspondences.

Theorem 9 *Let A be a correspondence on S.* ***If*** *there is a convex open neighborhood W of zero such that (i') $x \notin coA(x) - W$ for all x and (ii') for any $y \in A^{-1}(x)$, there exists $x' \in S$ (possibly $x' = x$) such that $y \in intA^{-1}(x' + W)$,* ***then*** *the set E of maximal elements of A is compact and non-empty.*

Thus, the theorem provides new conditions for a nontransitive preference to have a maximal element. We first establish a useful lemma (to be compared with lemma 4).

Lemma 10 *If there is a convex open neighborhood W of zero such that for each $x \in S$, $x \notin coA(x) - W$, then $\cap_{x \in S} F(x) \neq \emptyset$ where $F(x)$ is the closure of the complement of $A^{-1}(x + W)$: $F(x) = \overline{S \backslash A^{-1}(x + W)}$.*

Proof. From Ky Fan's lemma, it is sufficient to show that for any finite family $\{x_1, ..., x_k\}$ and any $y \in co\{x_1, ..., x_k\}$, $y \in \cup_{i=1,...,k} F(x_i)$. Suppose not. This implies that for any i, $y \notin F(x_i)$ implying that $x_i \in A(y) - W$. Since W is convex, conclude that $y \in coA(y) - W$, a contradiction. ■

We now prove the fixed point theorem (to compare with the proof of Sonnenschein's theorem).

Proof. By assumption, there is an open and convex neighborhood W of zero such that, for all $x \in S$, $x + W \cap coA(x) = \emptyset$. Note that $E = \cap_{x \in S} S \backslash A^{-1}(x + W)$. By (ii'), $E = \cap_{x' \in S} S \backslash intA^{-1}(x' + W)$, so that it is compact (as the intersection of a family of compact sets). Thus $\overline{S \backslash A^{-1}(x + W)} \subset S \backslash intA^{-1}(x + W)$. Hence, $\cap_{x \in S} \overline{S \backslash A^{-1}(x + W)} \subset \cap_{x \in S} S \backslash intA^{-1}(x + W)$. By (i') and the last lemma, $\cap_{x \in S} \overline{S \backslash A^{-1}(x + W)} \neq \emptyset$. ■

This version may be useful in other contexts such as competitive equilibria as in Sonnenschein [24] or social coalitional equilibria as in Ichiishi [12].

2.3 Locally Convex and Hausdorff TVS

The topological vector space is assumed now to be locally convex and Hausdorff (LCH). Theorem 9 allows to deduce that:

Corollary 11 (Tychonoff's theorem) ***If*** *A is continuous and has non-empty values,* ***then*** *there is $x \in S$ such that $x \in \overline{co}A(x)$.*

Proof. If there is a convex neighborhood of zero such that for all $x \in S$, $x + W \cap \{coA(x)\} = \emptyset$, since $A(x)$ is lower-hemi-continuous, theorem 9 implies that A has a maximal element: a contradiction. Thus, for each W, there exists $x^W \in S$ such that $x^W + W \cap \{coA(x^W)\} \neq \emptyset$. Since $\overline{co}A$ is upper-hemi-continuous when A is (see [1]), by compactness of S and by the Hausdorff and local convexity assumptions, tending W to zero implies that there $x \in \overline{co}A(x)$. ■

This provides a short proof of Tychonoff's theorem [1]. Actually, if f is continuous from S to S, and if $A(x) = \{f(x)\}$ then there is $x \in \overline{co}\{A(x)\} = \{f(x)\}$.

Schauder conjectured in 1935 that local convexity is not necessary in Tychonoff's theorem. The conjecture has been proved only recently by Robert Cauty [10].

In the next section we prove that some fixed point theorems closely related to theorem 9 from Tychonoff's theorem when S is metrizable, some are known.

2.4 Metric and Banach Spaces

From now on, the topological vector space is always assumed to be Hausdorff and locally convex. Some of the following theorems are known ([6], [25], [26]). They are established from two Michael's selection theorems and Tychonoff's theorem (established in the previous section).

Theorem 12 (Wu [25]) *Suppose S metrizable.* ***If*** *(a) $x \notin \overline{co}A(x)$ for all x and (b) A is lower-hemi-continuous,* ***then*** *the set E of maximal elements of A is non-empty and compact.*

Proof. Observe that $\overline{co}A$ is lower-hemi-continuous when A is (see [1]). If A has no maximal element, from Michael's [16] selection theorem (3.2"), there is a continuous selection f from S to S such that $f(s) \in \overline{co}A(s)$. By Tychonoff's theorem, f has a fixed point $s = f(s) \in \overline{co}A(s)$, contradicting (a). Compacity follows because $E = \cap_{x \in S} S \backslash A^{-1}(x+W)$. ■

Alternatively, Wu's theorem says that metrizable spaces, if A is lower-hemi-continuous with convex-closed and non-empty values, it has a fixed point.

In finite dimensions, Yannelis and Prabhakar [26] (theorem 5.2) proved a powerful result. Under only the assumptions (i) $x \notin coA(x)$ and (b) A is lower-hemi-continuous, they show that A has a maximal element. The proof is simple: if there was no maximal element, Michael's [17] theorem (3.2"') implies the existence of a continuous selection f such that for all x, $f(x) \in coA(x)$. By Brouwer's fixed point theorem, f admits a fixed point: a contradiction with (i).

Michael's [17] theorem (3.2"') is more powerful and it implies the existence of a maximal element under the same conditions (i) and (b) when S is a subset of a Banach space and for all $x \in S$, $coA(x)$ has an inside point.

Recall that if K is a convex subset of a Banach space, then a supporting set of K is a closed, convex subset L of $\overline{K}$ different from $\overline{K}$ such that if an interior point of a segment in $\overline{K}$ is in L, then the whole segment is in L. The set of all elements of K which are not in any supporting set of K will be "Inside of K". This is always non-empty if K is finite dimensional or is closed (see [17]).

Theorem 13 (Yannelis and Prabhakar [26] Extended) *Suppose S is a subset of a Banach space.* ***If*** *(i) $x \notin coA(x)$ for all x, (b) A is lower-hemi-continuous, (c) $coA(x)$ has* *an inside point for all x such that $coA(x)$ is non-empty,* ***then*** *set of maximal elements of A is non-empty and compact.*

3 Coalitional Nash Equilibria

Let N be a set of players (not necessarily finite). Let $G = (N, \{S_i\}_{i\in N}, \{g_i\}_{i\in N})$ be a strategic game. Assume that for each i in N, S_i, is a compact subset of a *Hausdorff and locally convex TVS* and that the payoff function of player i, g_i is continuous. This defines a compact-convex-continuous strategic game.

Let $\mathcal{C} \subset 2^N$ be an exogenously given set of what we will call permissible coalitions. As usual, $S = \prod_{i\in N} S_i$ is the set of strategy profiles. For a coalition of players C, let $S_C = \prod_{j\in C} S_j$ and let $N\backslash C$ denote the set of players outside C.

Definition 14 *s is a $\mathcal{C}$-coalitional-equilibrium of G if no permissible coalition in $\mathcal{C}$ has a unilateral deviation that profits to all its members; That is, there is no C in $\mathcal{C}$ and no $t_C \in S_C$ such that for any $i \in C$, $g_i(t_C, s_{N\backslash C}) > g_i(s)$.*

Definition 15 *G is $\mathcal{C}$-quasi-concave if for all $s \in S$, $\epsilon > 0$ and any family of permissible coalitions $(C_k)_{k\in K}$ with corresponding strategies $t_{C_k} \in S_{C_k}$,* ***if*** *for all k and $i \in C_k$ $g_i(t_{C_k}, s_{N\backslash C_k}) \geq g_i(s) + \varepsilon$,* ***then*** *$s \notin \overline{co}\{(t_{C_k}, s_{N\backslash C_k}), k \in K\}$.*

In finite dimensional strategy spaces, Caratheodory's theorem implies that $\overline{co}$ above could be replaced by co and that finitely many deviating coalitions are sufficient.

When only single player coalitions are permissible, the condition is reduced to the quasi-concavity assumption in Nash-Glicksberg's [11] theorem. When all coalitions are permissible, the quasi-concavity condition is related to the balanced-condition (defined just below) under which Ichiishi [14] proved the existence of a strong equilibrium[3]. Thus $\mathcal{C}$-quasi-concavity may be viewed as a mixture of quasi-concavity and the balancedness conditions.

Theorem 16 *If a compact-convex-continuous strategic game is $\mathcal{C}$-quasi-concave, the set of $\mathcal{C}$-coalitional-equilibria is compact and non-empty.*

Proof. Let $A_C(s) = \{(t_C, s_{N\backslash C})$ such that for all $i \in C$, $g_i(t_C, s_{N\backslash C}) > g_i(s) + \varepsilon\}$ and let $A = \bigcup_C A_C$. From the continuity of the game, A is lower-hemi-continuous. Suppose that for each convex and open neighborhood W of zero there is s^W such that $s^W + W \in coA(s)$. Thus there exists a finite family of permissible coalitions $(C_k^W)_{k\in K}$ and strategies $(t_{C_k}^W \in S_{C_k})_{k\in K}$ such that $s^W + W \in co\{(t_{C_k}, s_{N\backslash C_k}), k \in K\}$ and $g_i(t_{C_k}, s_{N\backslash C_k}) > g_i(s) + \varepsilon$ for all k and $i \in C_k$. Using compactness of the strategy space, Hausdorff assumption and continuity of the payoff functions, by tending W to zero one deduces the existence of s and of a family of permissible coalitions $(C_k)_{k\in K}$ and strategies $(t_{C_k} \in S_{C_k})_{k\in K}$ such that $s \in \overline{co}\{(t_{C_k}, s_{N\backslash C_k}), k \in K\}$ and $g_i(t_{C_k}, s_{N\backslash C_k}) \geq g_i(s) + \varepsilon$ for all k and $i \in C_k$: a

[3]The result may be found in definition 23.3 and theorem 23.7 in Border [4].

contradiction. Thus, (i') and (b) are satisfied. Since (b) implies (ii'), theorem 9 implies that A has a maximal element s_ε whose accumulation points are coalitional-equilibria (thanks to compactness of the strategy spaces and continuity of the payoff functions). ■

Observe that the correspondences A and A_C in the proof above do not have open lower-sections: they are only lower-hemi-continuous. Thus, Sonnenschein's [24] fixed point theorem cannot be used. This justify the need of a fixed point theorem that deals with lower-hemi-continuous correspondences. In metrizable or Banach spaces, theorems 12 and 13 may well be used to establish the result.

Interestingly, another definition of A_C in the last proof would lead to another well known equilibrium concept. For example, if one defines $A_C(s) = \{t_C \in S_C : \text{there is } i \in C$ such that $g_i(t_C, s_{-C}) > g_i(s) + \epsilon\}$, the underlying concept requires that a permissible coalition has a deviation if at least one of its members profits even if all the other players inside the coalition lose (this is of course too demanding!). Hence, at equilibrium, players outside a permissible coalition forces all the players inside the coalition to play according to the equilibrium. The concept was defined by Berge [3] but only for permissible coalitions of the form $N\backslash\{i\}$, $i \in N$. A coalitional Berge-equilibrium exists if G is compact-convex-continuous and *for all* $s \in S$, $\epsilon > 0$ *and any family of permissible coalitions* $(C_k)_{k\in K}$ *with corresponding strategies* $t_{C_k} \in S_{C_k}$, ***if*** *for all* k *<u>there is</u>* $i \in C_k$ *such that* $g_i(t_{C_k}, s_{N\backslash C_k}) > g_i(s) + \epsilon$, ***then*** $s \notin \overline{co}\{(t_{C_k}, s_{N\backslash C_k}), k \in K\}$.

Recall that a finite family of nonempty subsets $\mathcal{B}$ of N is *balanced* if for each $B \in \mathcal{B}$, there are nonnegative real numbers λ_B (balancing weights) such that for each i in N, $\sum_{B:i\in B} \lambda_B = 1$.

Definition 17 *G is balanced if for all $\alpha \in \mathbf{R}^N$ and any finite balanced family of coalitions $\{C_k\}$ with weights $\{\lambda_k\}$ and corresponding strategies $\{t^{C_k}\} \in S$,* ***if*** *$g_i(t^{C_k}) > \alpha_i$ for all k and $i \in C_k$* ***then*** $g_i(s) > \alpha_i$ for all players $i \in N$, *where* $s_i := \sum_{k:i\in C_k} \lambda_k t_i^{C_k}$.

A quite similar condition[4] is:

Definition 18 *G is quasi-balanced if for all $\alpha \in \mathbf{R}^N$ and any finite balanced family of coalitions $\{C_k\}$ with weights $\{\lambda_k\}$ and corresponding strategies $\{t^{C_k}\} \in S$,* ***if*** *$g_i(t^{C_k}) \geq \alpha_i$ for all k and $i \in C_k$ and $g_i(t^{C_k}) > \alpha_i$ for some k and $i \in C_k$* ***then*** $g_i(s) \neq \alpha_i$ for some player $i \in N$, *where* $s_i := \sum_{k:i\in C_k} \lambda_k t_i^{C_k}$.

Lemma 19 *In finite dimensional strategy spaces, G quasi-balanced implies it is $\mathcal{C}$-quasi-concave for every $\mathcal{C}$.*

Proof. Take $(C_k)_{k\in K}$ to be a finite family of permissible coalitions, let $t_{C_k} \in S_{C_k}$ and suppose $s = \sum_k \alpha_k(t_{C_k}, s_{N/C_k})$. The family $\{\{C_k\} \cup \{N/C_k\}\}_{k\in K}$ with weights $\{(\alpha_k, \alpha_k)\}_{k\in K}$ is balanced. Define $t^{C_k} = (t_{C_k}, s_{N/C_k})$ and $t^{N/C_k} = s$. Then, $s_i = \sum_{k:i\in C_k} \alpha_k t_i^{C_k} + \sum_{k:i\in N/C_k} \alpha_k t_i^{N/C_k}$. Suppose that for each k and $i \in C_k$, $g_i(t^{C_k}) > g_i(s)$. Since for all $i \in N/C_k$, $g_i(t^{N/C_k}) = g_i(s)$, taking $\alpha = g(s)$ leads to a contradiction. ■

[4] We expect that every balanced game could be approximated by a quasi-balanced game.

4 Conclusion

To establish the existence of Nash equilibria, Glicksberg [11] needs quasi-concavity of the payoff functions and uses a standard fixed point theorem (Brouwer/Kakutani). Scarf [22] establishes the existence of the core of a NTU game using the Lemke-Howson algorithm. Shapley [23] establishes the KKMS lemma (22.4 in [4]) using the Lemke-Howson algorithm and apply it to prove the non-vacuity of the core. Around that time, there was no successful work who proves the non-vacuity of the core using an abstract fixed point theorem and no sufficient conditions for the existence of strong equilibria. Ichiishi [13] proves the KKMS lemma from Fan's [9] coincidence theorem and from the same theorem he provides the existence of strong equilibria in balanced strategic games. However, the Fan coincidence theorem does not permit a deep understanding of the link between the quasi-concavity condition for Nash equilibria and the balanced condition for strong equilibria. Our paper shows that they are but two sides of the same coin.

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