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Abstract A Transferable Utility (TU) game with n players specifies a vector of $2^n - 1$ real numbers, i.e. a number for each non-empty coalition, and this can be difficult to handle for large n. Therefore, several models from the literature focus on interaction situations which are characterized by a compact representation of a TU-game, and such that the worth of each coalition can be easily computed. Sometimes, the worth of each coalition is computed from the values of single players by means of a mechanism describing how the individual abilities interact within groups of players. In this paper we introduce the class of *Generalized Additive Games* (*GAG*s), where the worth of a coalition $S \subseteq N$ is evaluated by means of an interaction filter, that is a map $\mathcal{M}$ which returns the valuable players involved in the cooperation among players in S. Moreover, by making further hypothesis on $\mathcal{M}$, we investigate the subclass of *basic GAGs*, where the filter $\mathcal{M}$ selects, for each coalition S, those players that have *friends* but not *enemies* in S. We also show that well-known classes of TU-games can be represented in terms of such basic GAGs.

Keywords TU-games · compact representation · airport games · peer games · maintenance problems.

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1 Introduction

Since the number of coalitions grows exponentially with respect to the number of players, it is computationally very interesting to single out classes of games that can be described in a concise way.
In the literature on coalitional games there exist several approaches for defining classes of games whose concise representation is derived by an additive pattern among coalitions. In some contexts, due to an underlying structure among the players, such as a network, an order, or a permission structure, the value of a coalition $S \subseteq N$ can be derived additively from a collection of subcoalitions $\{T_1, \cdots T_k\}$, $T_i \subseteq S \; \forall i \in \{1, \cdots, k\}$. Such situations are modeled, for example, by the graph-restricted games, introduced by Myerson (1977) and further studied by Owen (1986), the component additive games (Curiel et al. 1993) and the restricted component additive games (Curiel et al. 1997).

Sometimes, the worth of each coalition is computed from the values that single players can guarantee themselves by means of a mechanism describing the interactions of individuals within groups of players. In the simplest case we can consider that, when a coalition of players forms, each player brings his own value and the worth of the coalition is computed as the sum of the single contributions of players that form it. As an example, consider a cost game where n players want to buy online n different objects and the value of a single player in the game is defined as the price of the object he buys. Then, if the members of a group S of buyers agree to make the purchase together, the cost of the operation will simply be the sum of the $s = |S|$ prices of the objects bought by players in S, i.e. the sum of the costs that the single players in S should bear if they bought the objects separately.
This situation can be described by means of an *additive game*, where the value of a coalition is computed as the sum of the disjoint coalitions that form it. An additive game is indeed determined by the vector of the n values of the single players.
However, such a model may fail to reflect the importance of a subset of players in contributing to the value of the coalition they belong to. In the previous example, it might be the case that, by making a collective purchase, the costs of shipping will decrease, or when a certain threshold price is reached, some of the objects will be sold for free and therefore the price that a coalition S should pay will depend only on the price of a subset of purchased objects.

In fact, in some cases the procedure used to assess the worth of a coalition $S \subseteq N$ is strongly related to the sum of the individual values over another subset $S \subseteq N$, not necessarily included in S.
Several examples from the literature fall into this category. As a simple example, consider the well-known *glove game*: the set of players N is divided into two categories, the players in L that own a left-hand glove, and those in R with a right-hand glove. The worth of a coalition of players $S \subseteq N$ is defined as the number of pairs of gloves owned by the coalition S. In this context,

the valuable players in a coalition are those whose class is represented by a minority of the players, since the value of S is given by the minimum between the number of players in $S \cap L$ and in $S \cap R$. Therefore, we can represent this game by assigning value 1 to each player and by describing the worth of each coalition S as the sum of single players' values over the smaller subset among $S \cap L$ and $S \cap R$. A similar approach can be used to describe several other classes of games from the literature, among them the airport games (Littlechild and Owen 1973; Littlechild and Thompson 1977), the argumentation games (Bonzon et al. 2014) and some classes of operation research games, such as peer games (Branzei et al. 2002) and mountain situations (Moretti et al. 2002), that will be described in Section 4.

In all the aformentioned models, the value of a coalition S of players is calculated as the sum of the single values of players in a subset of S. On the other hand, in some cases the worth of a coalition might be affected by external influences and players outside the coalition might contribute, either in a positive or negative way, to the worth of the coalition itself. This is the case, for example, of the bankruptcy games (Aumann and Maschler 1985) and the maintenance problems (Koster 1999; Borm et al. 2001), that will be described, respectively, in Section 3 and 4.

In this paper we introduce a general class of additive TU-games where the worth of a coalition $S \subseteq N$ is evaluated by means of an interaction filter, that is a map $\mathcal{M}$ which returns the valuable players involved in the cooperation among players in S.
Our objective is to provide a general framework for describing several classes of games studied in the literature on coalitional games and to give a kind of taxonomy of coalitional games that are ascribable to this notion of additivity over individual values.
The general definition of the map $\mathcal{M}$ allows various and wide classes of games to be embraced, as for example the simple games and the aformentioned classes of games. Moreover, by making further hypothesis on $\mathcal{M}$, our approach enables to classify existing games based on the properties of $\mathcal{M}$. In particular, we introduce the class of *basic GAGs*, which is characterized by the fact that the valuable players in a coalition S are selected on the basis of the presence, among the players in S, of their *friends* and *enemies*, that is, a player contributes to the value of S if and only if S contains at least one of his friends and none of his enemies is present.

The paper is structured as follows. Section 2 provides the basic definitions and notations regarding coalitional games. In Section 3 we introduce the class of Generalized Additive Games (GAGs) and provide examples of games falling into this category. In Section 4 we introduce some hypothesis on the map $\mathcal{M}$ and describe the resulting subclass of basic GAGs, providing further examples from the literature. In Section 5 we describe some possible extensions and in Section 6 we draw some conclusions and present possible directions for future research.

2 Preliminaries

In this section we introduce some preliminary notation and definitions on coalitional games.

A TU-game (cooperative game with transferable utility), also referred to as *coalitional game*, is a pair (N, v), where N denotes the set of players and $v : 2^N \to \mathbb{R}$ is the *characteristic function*, with $v(\emptyset) = 0$. A group of players $S \subseteq N$ is called *coalition* and $v(S)$ is called the *value* or *worth* of the coalition S. If the set N of players is fixed, we identify a coalitional game (N, v) with its characteristic function v.
From now on, we shall assume w.l.o.g. that $N = \{1, \cdots, n\}$. Moreover, for a coalition S, we shall denote by s its cardinality $|S|$.

A particular class of games is that of *simple games*, where the characteristic function v can only assume values in $\{0, 1\}$.
A game (N, v) is said to be *monotonic* if it holds that $v(S) \leq v(T)$ for all $S, T \subseteq N$ such that $S \subseteq T$ and it is said to be *superadditive* if it holds that

$$v(S \cup T) \geq v(S) + v(T)$$

for all $S, T \subseteq N$ such that $S \cap T = \emptyset$.
Moreover, a game (N, v) is said to be *convex* if it holds that

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T)$$

for all $S, T \subseteq N$.
For a general introduction on cooperative games, see Maschler et al. (2013).

3 Generalized Additive Games (GAGs)

Definition 1 We shall call *Generalized Additive Situation* (GAS) any triple $\langle N, v, \mathcal{M} \rangle$, where N is the set of the players, $v : N \to \mathbb{R}$ is a map that assigns to each player a real value and $\mathcal{M} : 2^N \to 2^N$ is a *coalitional map*, which assigns a (possibly empty) coalition $\mathcal{M}(S)$ to each coalition $S \subseteq N$ of players and such that $\mathcal{M}(\emptyset) = \emptyset$.

Definition 2 Given the GAS $\langle N, v, \mathcal{M} \rangle$, the associated *Generalized Additive Game* (GAG) is defined as the TU-game $(N, v^{\mathcal{M}})$ assigning to each coalition the value

$$v^{\mathcal{M}}(S) = \begin{cases} \sum_{i \in \mathcal{M}(S)} v(i) & \text{if } \mathcal{M}(S) \neq \emptyset \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Example 1

1. Let w be a simple game. Then w can be described by the GAG associated to $\langle N, v, \mathcal{M}\rangle$ with $v(i) = 1$ for all i and

$$\mathcal{M}(S) = \begin{cases} \{i\} \subseteq S \text{ if } & S \in \mathcal{W} \\ \emptyset & \text{otherwise} \end{cases}$$

   where $\mathcal{W}$ is the set of the winning coalitions in w.
   In case there is a *veto player*, i.e. a player i such that $S \in \mathcal{W}$ only if $i \in S$, then the game can also be described by $v(i) = 1$, $v(j) = 0\ \forall j \neq i$ and

$$\mathcal{M}(S) = \begin{cases} T \text{ if } & S \in \mathcal{W} \\ R & \text{otherwise} \end{cases}$$

   with $T, R \subseteq N$ such that $i \in T$ and $i \notin R$.
   This in particular shows that the description of a game as GAG need not be unique.

2. Let w be the *glove game* defined in the following way. A partition $\{L, R\}$ of N is assigned. Define $w(S) = \min\{|S \cap L|, |S \cap R|\}$. Then w can be described as the GAG associated to $\langle N, v, \mathcal{M}\rangle$ with $v(i) = 1$ for all i and

$$\mathcal{M}(S) = \begin{cases} S \cap L \text{ if } & |S \cap L| \leq |S \cap R| \\ S \cap R & \text{otherwise.} \end{cases}$$

3. Consider the bankruptcy game (N, w) introduced by Aumann (1985), where the value of a coalition $S \subseteq N$ is given by

$$w(S) = \max\{E - \sum_{i \in N \setminus S} d_i, 0\}.$$

   Here $E \geq 0$ represents the estate to be divided and $d \in \mathbb{R}^N_+$ is a vector of claims satisfying the condition $\sum_{i \in N} d_i > E$. It is easy to show that a bankruptcy game is the difference $w = v_1^{\mathcal{M}} - v_2^{\mathcal{M}}$ of two GAGs $v_1^{\mathcal{M}}, v_2^{\mathcal{M}}$ arising, respectively, from $\langle N, v^1, \mathcal{M}^1\rangle$ and $\langle N, v^2, \mathcal{M}^2\rangle$ with $v^1(i) = E$ and $v^2(i) = d_i$ for all i and

$$\mathcal{M}^1(S) = \begin{cases} \{i\} \subseteq S \text{ if } & S \in B \\ \emptyset & \text{otherwise} \end{cases}$$

   and

$$\mathcal{M}^2(S) = \begin{cases} N \setminus S \text{ if } & S \in B \\ \emptyset & \text{otherwise} \end{cases}$$

   for each $S \in 2^N \setminus \{\emptyset\}$, where $B = \{S \subseteq N : \sum_{i \in N \setminus S} d_i \leq E\}$.

4. An extension of the bankruptcy game has been introduced by Pulido et al. (2002): an extended bankruptcy game (N, w) is defined as

$$w(S) = \max\{E - \sum_{i \in N \setminus S} d_i, \sum_{i \in S} r_i\}$$

where $E \geq 0$ represents the estate to be divided, $d \in \mathbb{R}_+^N$ is a vector of claims satisfying the condition $\sum_{i \in N} d_i \geq E$ and $r \in \mathbb{R}_+^N$ is a vector of objective entitlements satisfying the conditions $0 \leq r_i \leq d_i$ for all $i \in N$. The extended bankruptcy game can be represented as the linear combination $w = v_1^{\mathcal{M}} - v_2^{\mathcal{M}} + v_3^{\mathcal{M}}$ of three GAGs $v_1^{\mathcal{M}}, v_2^{\mathcal{M}}, v_3^{\mathcal{M}}$ arising, respectively, from $\langle N, v^1, \mathcal{M}^1 \rangle$, $\langle N, v^2, \mathcal{M}^2 \rangle$ and $\langle N, v^3, \mathcal{M}^3 \rangle$ with $v^1(i) = E$, $v^2(i) = d_i$ and $v^3(i) = r_i$ for all i and

$$\mathcal{M}^1(S) = \begin{cases} \{i\} \subseteq S \text{ if } & S \in R \\ \emptyset & \text{otherwise} \end{cases}$$

$$\mathcal{M}^2(S) = \begin{cases} N \setminus S \text{ if } & S \in R \\ \emptyset & \text{otherwise} \end{cases}$$

and

$$\mathcal{M}^3(S) = \begin{cases} \emptyset \text{ if } & S \in R \\ S & \text{otherwise} \end{cases}$$

for each $S \in 2^N \setminus \{\emptyset\}$, where $R = \{S \subseteq N : \sum_{i \in N \setminus S} d_i + \sum_{i \in S} r_i \leq E\}$.

Observe that, clearly, not every game can be described as a GAG. Obvious examples can be provided for all n, in particular for $n = 2$.

Some natural properties of the map $\mathcal{M}$ can be translated into classical properties for the associated GAG. Here is some example.

Definition 3 The map $\mathcal{M}$ is said to be *proper* if $\mathcal{M}(S) \subseteq S$ for each $S \subseteq N$; it is said to be *monotonic* if $\mathcal{M}(S) \subseteq \mathcal{M}(T)$ for each S, T such that $S \subseteq T \subseteq N$.

Note that a map $\mathcal{M}$ can be monotonic but not proper, or proper but not monotonic. An example of map $\mathcal{M}$ which is not monotonic is that one relative to the glove game. Maps that are not proper will be seen later.

The following results are straightforward.

Proposition 1 *Let $\langle N, v, \mathcal{M} \rangle$ be a GAS with $v \in \mathbb{R}_+^N$ and $\mathcal{M}$ monotonic. Then the associated GAG (N, v^M) is monotonic.*

Proposition 2 *Let $\langle N, v, \mathcal{M} \rangle$ be a GAS with $v \in \mathbb{R}_+^N$ and $\mathcal{M}$ proper and monotonic. Then the associated GAG $(N, v^{\mathcal{M}})$ is superadditive.*

Proof Let S and T be two coalitions such that $S \cap T = \emptyset$. By properness it is $\mathcal{M}(S) \cap \mathcal{M}(T) = \emptyset$. By monotonicity it is

$$\mathcal{M}(S) \cup \mathcal{M}(T) \subseteq \mathcal{M}(S \cup T).$$

Thus, since $v \in \mathbb{R}^N_+$,

$$v^{\mathcal{M}}(S \cup T) = \sum_{i \in \mathcal{M}(S \cup T)} v(i) \geq \sum_{i \in \mathcal{M}(S) \cup \mathcal{M}(T)} v(i) = v^{\mathcal{M}}(S) + v^{\mathcal{M}}(T).$$

Observe that Propositions 1 and 2 provide only sufficient conditions, for instance the glove game is monotonic and superadditive but the associated map $\mathcal{M}$ is not monotonic.

The following example shows that, if the map $\mathcal{M}$ is proper and monotonic, the corresponding GAG needs not be convex.

Example 2 Let $N = \{1, 2, 3, 4\}$, $v(i) \geq 0$ $\forall i \in N$, and $\mathcal{M}$ be such that $\mathcal{M}(\{2, 3\}) = \{3\}$ and $\mathcal{M}(S) = S$ for all $S \neq \{2, 3\}$. Then $\mathcal{M}$ is proper and monotonic but the corresponding GAG is not convex, since it holds that $v^{\mathcal{M}}(S \cup T) + v^{\mathcal{M}}(S \cap T) < v^{\mathcal{M}}(S) + v^{\mathcal{M}}(T)$ for $S = \{1, 2, 3\}$, $T = \{2, 3, 4\}$.

On the other hand, the next proposition shows that it is possible to provide sufficient conditions for a monotonic map $\mathcal{M}$ to generate a convex GAG.

Proposition 3 *Let $\langle N, v, \mathcal{M} \rangle$ be a GAS with $v \in \mathbb{R}^N_+$ and $\mathcal{M}$ monotonic and such that*

$$\mathcal{M}(S) \cap \mathcal{M}(T) \equiv \mathcal{M}(S \cap T), \tag{2}$$

for each $S, T \in 2^N$. Then the associated GAG $(N, v^{\mathcal{M}})$ is convex.

Proof The convexity condition for the GAG $(N, v^{\mathcal{M}})$ can be written as follows:

$$\begin{aligned} &\textstyle\sum_{i \in \mathcal{M}(S \cup T)} v(i) + \sum_{i \in \mathcal{M}(S \cap T)} v(i) \geq \sum_{i \in \mathcal{M}(S)} v(i) + \sum_{i \in M(T)} v(i) \\ &= \textstyle\sum_{i \in \mathcal{M}(S) \cup M(T)} v(i) + \sum_{i \in M(S) \cap \mathcal{M}(T)} v(i), \end{aligned} \tag{3}$$

for each $S, T \in 2^N$. By monotonicity of $\mathcal{M}$, we have that $\mathcal{M}(S) \cup \mathcal{M}(T) \subseteq \mathcal{M}(S \cup T)$ and $\mathcal{M}(S \cap T) \subseteq \mathcal{M}(S) \cap \mathcal{M}(T)$ for each $S, T \in 2^N$. Clearly, if $\mathcal{M}(S \cap T) \equiv \mathcal{M}(S) \cap \mathcal{M}(T)$ for each $S, T \in 2^N$, then relation (3) holds.

The condition provided by relation (2) can be useful to construct a monotonic map $\mathcal{M}$ such that the corresponding GAG is convex when $v \in \mathbb{R}^N_+$. The most trivial example is the identity map $\mathcal{M}(S) = S$ for each $S \in 2^N$. Another example is a map $\mathcal{M}$ of a GAS $\langle N, v, \mathcal{M} \rangle$ with $N = \{1, 2, 3\}$ and $v \in \mathbb{R}^N_+$ such that $\mathcal{M}(\{1, 2, 3\}) = \{1, 2, 3\}$, $\mathcal{M}(\{1, 2\}) = \{1, 2\}$, $\mathcal{M}(\{2, 3\}) = \{2\}$, $\mathcal{M}(\{2\}) = \{2\}$ and $\mathcal{M}(\{1\}) = \mathcal{M}(\{3\}) = \mathcal{M}(\{1, 3\}) = \emptyset$.

We conclude this section with an example showing that relation (2) is not a necessary condition to have a convex GAG.

Example 3 Let $N = \{1,2,3\}$, $v = (v(1), v(2), \alpha v(2))$ with $v(1), v(2) \geq 0$ and $\mathcal{M}$ be such that $\mathcal{M}(\{1,2,3\}) = \{1,2,3\}$, $\mathcal{M}(\{1,2\}) = \{1,2\}$, $\mathcal{M}(\{2,3\}) = \{2\}$ and $\mathcal{M}(\{1\}) = \mathcal{M}(\{2\}) = \mathcal{M}(\{3\}) = \mathcal{M}(\{1,3\}) = \emptyset$. The map $\mathcal{M}$ is monotonic, but relation (2) does not hold (to see this, just take $S = \{1,2\}$ and $T = \{2,3\}$). On the other hand, one can check that relation (3) holds for each $S, T \in 2^N$ if and only id $\alpha \geq 1$ (in particular note that for $S = \{1,2\}$ and $T = \{2,3\}$ relation (3) gives $v(1) + v(2) + \alpha v(2) \geq v(1) + 2v(2)$).

4 Basic GAGs

We now define an interesting subclass of GASs. Consider a *collection* $\mathcal{C} = \{\mathcal{C}_i\}_{i \in N}$, where $\mathcal{C}_i = \{F_i^1, \ldots, F_i^{m_i}, E_i\}$ is a collection of subsets of N such that $F_i^j \cap E_i = \emptyset$ for all $i \in N$ and for all $j = 1, \cdots, m_i$.

Definition 4 We denote by $\langle N, v, \mathcal{C} \rangle$ the *basic* GAS associated with the coalitional map $\mathcal{M}$ defined as:

$$\mathcal{M}(S) = \{i \in N : S \cap F_i^1 \neq \emptyset, \ldots, S \cap F_i^{m_i} \neq \emptyset, S \cap E_i = \emptyset\} \tag{4}$$

and by $\langle N, v^{\mathcal{C}} \rangle$ the associated GAG, that we shall call *basic* GAG.

For simplicity of exposition, we assume w.l.o.g. that $m_1 = m_2 = \cdots = m_n := m$. We shall call each F_i^k, for all $i \in N$ and all $k = 1, \ldots, m$, the k-th set of *friends* of i, while E_i is the set of *enemies* of i. Note that in the definition of the coalitional map $\mathcal{M}$ the union is made over the set N, and thus $\mathcal{M}$ is not proper in general. On the other hand, anytime one is interested in imposing the properness property in relation (4), it suffices to impose $F_i^1 = \{i\}$ for each $i \in N$.

The following proposition characterizes monotonic basic GAGs.

Proposition 4 *Let $\langle N, v, \mathcal{C} \rangle$ be a basic GAS with $v \in \mathbb{R}_+^N$ and $\mathcal{C} = \{\mathcal{C}_i\}_{i \in N}$. Then the associated GAG $(N, v^{\mathcal{C}})$ is monotonic if and only if $E_i = \emptyset$ $\forall i \in N$.*

Proof The sufficient condition is obvious. Moreover, suppose $E_i \neq \emptyset$ for some i and let $j \in E_i$. Consider $S = F_i^1 \cup \cdots \cup F_i^m$. Then $i \in \mathcal{M}(S)$, while $i \notin \mathcal{M}(S \cup j)$.

Remark 1 Consider a GAS situation $(N, (F_i^1 = \{i\}, \ldots, F_i^m, E_i = \emptyset)_{i \in N}, v)$ with $v(i) \geq 0$ for each $i \in N$. It is easy to check that $\mathcal{M}$ is both monotonic and proper. Then, by Proposition 2, the associated GAG is superadditive.

The basic GAG $v^{\mathcal{C}}$ associated with a basic GAS can be decomposed in the following sense: define the collection of n games $v^{\mathcal{C}_i}$, $i = 1, \ldots, n$, as

$$v^{\mathcal{C}_i}(S) = \begin{cases} v(i) & \text{if} \quad S \cap E_i = \emptyset, S \cap F_i^k \neq \emptyset, k = 1, \ldots, m \\ 0 & \text{otherwise.} \end{cases} \tag{5}$$

Proposition 5 *The basic GAG $v^{\mathcal{C}}$ associated with the map defined in (4) verifies:*

$$v^{\mathcal{C}} = \sum_{i=1}^{n} v^{\mathcal{C}_i}. \tag{6}$$

A particularly simple case is when $m = 1$ for all i, which means that every player has a unique set of friends, that we shall denote by F_i. Also, an important case is when each player i has at most two set of friends, one of them being the singleton $\{i\}$, i.e. the coalitional map is proper. The following examples show that this case too includes interesting classes of games.

Example 4 We provide two examples of basic GAGs where each player has a unique set of friends.

1. (airport games) (Littlechild and Owen 1973; Littlechild and Thompson 1977): Let N be the set of players. We partition N into groups $N_1, N_2, \ldots, N_k$ (according to the original interpretation (Littlechild and Owen 1973; Littlechild and Thompson 1977), representing groups of players who need landing strips of the same length) such that to each N_j, $j = 1, \ldots, k$, is associated a positive real number c_j with $c_1 \leq c_2 \leq \cdots \leq c_k$ (representing the cost associated to the k different landing strips). Consider the game $w(S) = \max\{c_i : i \in S\}$. This type of game (and variants) can be described by a basic GAS $\langle N, (\mathcal{C}_i = \{F_i, E_i\})_{i \in N}, v\rangle$ by setting for each $i \in N_j$ and each $j = 1, \ldots, k-1$:
   - the value $v(i) = \frac{c_j}{|N_j|}$,
   - the set of friends $F_i = N_j$,
   - the set of enemies $E_i = N_{j+1} \cup \ldots \cup N_k$,

   and with the sets $F_l = N_k$ and $E_l = \emptyset$ for each $l \in N_k$.
2. (argumentation games) (Bonzon et al. 2014): Consider a directed graph $\langle N, \mathcal{R}\rangle$, where the set of nodes N is a finite set of *arguments* and the set of arcs $\mathcal{R} \subseteq N \times N$ is a *binary defeat* (or *attack*) *relation* (see Dung 1995). For each argument i we define the set of *attackers* of i in $\langle N, \mathcal{R}\rangle$ as the set $P(i) = \{j \in N : (j, i) \in \mathcal{R}\}$. The meaning is the following: N is a set of arguments, if $j \in P(i)$ this means that argument j attacks argument i. The value of a coalition S is the number of arguments in the opinion S which are not attacked by another argument of S. This type of game (and variants) can be described as a basic GAS $\langle N, v, \{F_i, E_i\}\rangle$ by setting $v(i) = 1$, the set of friends $F_i = \{i\}$ and the set of enemies $E_i = P(i)$.

In the setting of the argumentation, it is possible to consider different types of characteristic functions. For instance, it is interesting to consider the game $(N, v^{\mathcal{M}})$ such that for each $S \subseteq N$, $v^{\mathcal{M}}(S)$ is the sum of $v(i)$ over the elements of the set $D(S) = \{i \in N : P(i) \cap S = \emptyset$ and $\forall j \in P(i), P(j) \cap S \neq \emptyset\}$ of arguments that are not internally attacked by S and at the same time are defended by S from external attacks:

$$v^{\mathcal{M}}(S) = \sum_{i \in D(S)} v(i). \tag{7}$$

It is clear that such a situation cannot be described by a basic GAG where each player has a unique set of friends. The game in (7) can however be described as a basic GAG $\langle N, v^{\mathcal{C}}\rangle$, where, given a bijection $k : P(i) \to \{1, \cdots, |P(i)|\}$, $\mathcal{C}_i = \{F_i^1, \cdots, F_i^{|P(i)|}, E_i\}$ is such that $F_i^{k(j)} = P(j) \setminus P(i)$ for all $j \in P(i)$, and $E_i = P(i)$ for all $i \in N$.

Example 5 The following examples show that the unanimity and canonical games can be represented as basic GAGs with more than one set of friends.

1. (unanimity games) Let $S = \{s_1, \cdots, s_s\} \subseteq N$. Consider the unanimity game (N, u_S) defined as

$$u_S(T) = \begin{cases} 1 \text{ if} \quad S \subseteq T \\ \\ 0 \quad \text{otherwise.} \end{cases}$$

   This game can be described by a basic GAS $\langle N, v, \{F_i^1, \cdots, F_i^s, E_i\}\rangle$ by setting $v(i) = 1$ for some $i \in S$, $v(j) = 0$ $\forall j \neq i$, $F_i^k = \{s_k\}$ $\forall k = 1, \cdots s$ and $\forall i \in N$, and $E_i = \emptyset$ $\forall i \in N$.

2. (canonical games) Let $S \subseteq N$. Consider the canonical game (N, e_S) defined as

$$e_S(T) = \begin{cases} 1 \text{ if} \quad S = T \\ \\ 0 \quad \text{otherwise.} \end{cases}$$

   This game can be described as a basic GAS $\langle N, v, \{F_i^1, \cdots, F_i^s, E_i\}\rangle$ by setting $v(i) = 1$ for some $i \in S$, $v(j) = 0$ $\forall j \neq i$, $F_i^k = \{s_k\}$ $\forall k = 1, \cdots s$ and $\forall i \in N$, and $E_i = N \setminus S$.

We now provide further examples of classes of coalitional games from the literature which can be represented as basic GAGs, where in general each player can have several sets of friends.

4.1 Maintenance problems

A *maintenance problem* as introduced by Koster (1999) (see also Borm et al. (2001)) arises when a group of players N is connected by a tree T (e.g., a computer network) to a root 0 (e.g., a service provider) and each edge of the tree has an associated maintenance cost; the problem is how to share in a fair way the cost of the entire network T among the players in N. More formally, a couple (T, t) is given, where T=$(N \cup \{0\}, E)$ is a tree. $N \cup \{0\}$ represents the set of vertices (or nodes) and E is the set of edges, i.e. the pairs $\{i, j\}$ such that $i, j \in N$. 0 is the root of the tree having only one adjacent edge, and $t : E \to \mathbb{R}^+$ is a nonnegative cost function on the edges of the tree. Note that each vertex $i \in N$ is connected to the root 0 by a unique path P_i; we shall denote by e_i the edge in P_i that is incident to i. A *precedence relation* $\preceq$ is defined by: $j \preceq i$ if and only if j is on the path P_i. A *trunk* $R \subseteq N \cup \{0\}$ is a set of vertices which is closed under the relation $\preceq$, i.e. if $i \in R$ and $j \preceq i$, then $j \in R$. The set of *followers* of player $i \in N$ is given by $F(i) = \{j \in N | i \preceq j\}$ (note that $i \in F(i)$ for each $i \in N$). The cost of a trunk R is then defined as

$$C(R) = \sum_{i \in R \setminus \{0\}} t(e_i),$$

and the associated *maintenance cost game* (N, c) is defined by

$$c(S) = \min\{C(R) : S \subseteq R \text{ and } R \text{ is a trunk}\}.$$

Note that edge e_i is present in the cheapest trunk containing all members of S whenever a member of S is a follower of player i, i.e. $S \cap F(i) \neq \emptyset$. Therefore, we can represent the cost game (N, c) as the GAG associated to the basic GAS on N where $v(i) = t(e_i)$ and where $\mathcal{M}$ is defined by relation (4) with collections $\mathcal{C}_i = \{F_i, E_i\}$ such that $F_i = F(i)$ and $E_i = \emptyset$ for every $i \in N$.

4.2 Peer situations

In *peer games* (Branzei et al. 2002) over a player set N, the economic relationships among players are represented by a hierarchy described by a directed rooted tree T with N as the set of nodes and with 1 as the root (representing the leader of the entire group). The individual features are agents' potential economic possibilities, described by a vector $a \in \mathbb{R}^N$, where a_i is the gain that player i can generate if all players at un upper level in the hierarchy cooperate with him. In other words, player i becomes effective and may produce a gain a_i only if his superiors cooperate with him.
For every $i \in N$, we denote by $S(i)$ the set of all agents in the unique directed path connecting 1 to i, i.e. the set of superiors of i. Given a *peer group situation* (N, T, a) as described above, a *peer game* is defined as the game (N, v^P) such that for each non-empty coalition $S \subseteq N$

$$v^P(S) = \sum_{i \in N : S(i) \subseteq S} a_i.$$

A peer game (N, v^P) can be represented as the GAG associated to the basic GAS on N where $v(i) = a_i$ and where $\mathcal{M}$ is defined by relation (4) with collections $\mathcal{C}_i = \{F_i^1, \ldots, F_i^n, E_i\}$ such that:

$$F_i^j = \begin{cases} \{j\} & \text{if } j \in S(i) \\ \{i\} & \text{otherwise} \end{cases}$$

and $E_i = \emptyset$ for all $i \in N$.

An interesting example of peer games (Branzei et al. 2002) (and indeed of GAGs) are coalitional games arising from sealed bid second price auctions, where there is a seller who wish to sell an object at price not smaller than a given $r > 0$ (reservation price). Each player $i \in N$ has his own evaluation w_i of the object and can submit a bid b_i in an envelope (not necessarily equal to w_i). The mechanism of the auction is such that after the opening of the envelopes, the object is given to the player with the highest bid at the second highest price[1]. Suppose that $w_1 > w_2 > \ldots > w_n \geq r$. It is easy to check that in such a situation, a dominant strategy for each player i who acts alone (i.e., without colluding with the other players) is to bid his own value $b_i = w_i$. This leads to a situation where player 1 obtains the object at the price w_2, so the players' payoffs are $v(1) = w_1 - w_2$ and $v(i) = 0$ if $i \neq 1$.

Now, consider the possibility of collusion among the players, which means that players may form coalitions and agree on the bid each player should put in the respective envelope. For a coalition S, the dominant strategy is that the player $i(S) \in N$ with the highest evaluation in S bids $w_{i(S)}$, and the other players in S bid r, the reservation price. In this way, if all players collude, the worth of coalition N is $v(N) = w_1 - r$. In general, for every coalition $S \subseteq N$, we have that $v(S) = 0$ if $1 \notin S$ (it is still dominant for players in $N \setminus S$ to play their true evaluation, and then $1 \in N \setminus S$ gets the object), and $v(S) = w_1 - w_{i(N \setminus S)}$ if $1 \in S$, where $i(N \setminus S)$ is the player with highest evaluation in $N \setminus S$. Such a game (N, v) can be seen as the GAG associated to the basic GAS with collections $\mathcal{C}_i = (F_i^1, \ldots, F_i^i, E_i)$ where $F_i^j = \{j\}$ for every j in $\{1, \ldots, i\}$, $E_i = \emptyset$ and $v(i) = w_i - w_{i+1}$ for every $i \in N$.

4.3 Mountain situations

Consider a special version of a directed minimum cost spanning tree situation, introduced in Moretti et al. (2002), characterized by a group N of persons whose houses lie on mountains and are not yet connected to a water purifier downhill. It is possible but not necessary for every person (house) to be connected directly with the water purifier; being connected via others is sufficient. Only connections from houses to strictly lower ones are allowed. Such a situation can be represented by a rooted directed graph $< N \cup \{0\}, A >$ with $N \cup \{0\}$ as set of vertices, $A \subset N \times (N \cup \{0\})$ as set of edges and 0 as the root, and a weight function $w : A \to \mathbb{R}^+$, representing the cost associated to each edge. We assume that for each $k \in N$, $(k, 0) \in A$ (i.e., every node has the possibility to be directly connected with the source) and in order to impose the fact that only connection to lower houses are possible, no cycles are allowed.

Given such a mountain situation, the corresponding cooperative cost game (N, c) is given by $c(\emptyset) = 0$ and the cost $c(T) = \sum_{a \in \Gamma(T)} w(a)$ of a non-empty coalition T is the cost of an optimal 0-connecting tree $\Gamma(T) \subseteq A(T)$ in the mountain problem on the directed graph $< T \cup \{0\}, A(T) >$, i.e. $\Gamma(T)$ is a tree of minimum cost connecting all players in T to the source 0.

It can be checked that for each optimal 0-connecting tree $\Gamma(T) \subseteq A(T)$, each node $i \in T$ is directly connected with his best connection in $T \cup \{0\}$, that

[1] We do not consider here the case where players may submit equal bids.

is a node $b_T(i) \in \arg\min_{l \in T \cup \{0\}:(i,l) \in A} w(i,l)$ (see Moretti (2008)) for further details).

Now, assume that for each $a \in A$, $w(a)$ can assume only two values, let's say m or 0, with $m > 0$, and let $B_i = \{j \in N : w(i,j) = 0\}$ the set of best connections for $i \in N$ (actually, every mountain situation can be decomposed as a sum of simple dichotomous mountain situations like that). We can represent the cost game (N, c) as the GAG associated to the basic GAS on N where $v(i) = 0$ if $w(i,0) = 0$, and $v(i) = m$ otherwise, and where $\mathcal{M}$ is defined by relation (4) with collections $\mathcal{C}_i = \{F_i, E_i\}$ such that $F_i = \{i\}$ and $E_i = B_i$ for every $i \in N$.

4.4 A characterization of basic GASs

As it has been shown in the previous sections, a variety of classes of games that have been widely investigated in the literature can be described using the formalism provided by basic GASs. It is therefore interesting to study under which conditions a GAS can be described as a basic one. To this purpose, the following proposition provides a necessary and sufficient condition when the set of enemies of each player is empty.

Proposition 6 *Let $\langle N, v, \mathcal{M}\rangle$ be a GAS. The map $\mathcal{M}$ can be obtained by relation (4) via collections $\mathcal{C}_i = \{F_i^1, \ldots, F_i^{m_i}, E_i = \emptyset\}$, for each $i \in N$, if and only if $\mathcal{M}$ is monotonic.*

Proof It is obvious that every map $\mathcal{M}$ obtained by relation (4) over a collections $\mathcal{C}_i = \{F_i^1, \ldots, F_i^{m_i}, E_i = \emptyset\}$, for each $i \in N$, is monotonic.

Now, consider a monotonic map $\mathcal{M}$ and, for each $i \in N$, define the set $\mathcal{M}_i^{-1} = \{S \subseteq N : i \in \mathcal{M}(S)\}$ of all coalitions whose image in $\mathcal{M}$ contains i. Let $\mathcal{S}^{\mathcal{M},i}$ be the collection of minimal (with respect to set inclusion) coalitions in $\mathcal{M}_i^{-1}$, formally:

$$\mathcal{S}^{\mathcal{M},i} = \{S \in \mathcal{M}_i^{-1} : \text{ it does not exist } T \in \mathcal{M}_i^{-1} \text{ with } T \subset S\}.$$

For each $i \in N$, consider a collection $\mathcal{C}_i = \{F_i^1, \ldots, F_i^{m_i}, E_i = \emptyset\}$ such that

$$\{F_i^1, \ldots, F_i^{m_i}\} = \{T \subseteq N : |T \cap S| = 1 \text{ for each } S \in \mathcal{S}^{\mathcal{M},i} \text{ and } |T| \leq |\mathcal{S}^{\mathcal{M},i}|\}, \tag{8}$$

where each set of friends F_i^k, $k \in \{1, \ldots, m_i\}$, contains precisely one element with each coalition in $T \in \mathcal{S}^{\mathcal{M},i}$ and no more than $|\mathcal{S}^{\mathcal{M},i}|$ elements.

Denote by $\mathcal{M}^*$ the map obtained by relation (4) over such collections $\mathcal{C}_i$, $i \in N$. We need to prove that $\mathcal{M}(S) = \mathcal{M}^*(S)$ for each $S \in 2^N$, $S \neq \emptyset$.

First note that for each $i \in N$ and for every coalition $S \in \mathcal{M}_i^{-1}$, we have $i \in \mathcal{M}^*(S)$. Let us prove now that $i \notin \mathcal{M}^*(S)$ for each $S \notin \mathcal{S}^{\mathcal{M},i}$. Suppose, by contradiction, that there exists $T \subseteq N$ with $T \notin \mathcal{M}_i^{-1}$ such that $F_i^k \cap T \neq \emptyset$, for each $k \in \{1, \ldots, m_i\}$. This means that for every $S \in \mathcal{S}^{\mathcal{M},i}$, $|S \setminus T| \neq \emptyset$. Define a coalition $U \subseteq N$ such that $|U \cap (S \setminus T)| = 1$ for each $S \in \mathcal{S}^{\mathcal{M},i}$, i.e. U contains precisely one element S that is not in T, for each $S \in \mathcal{S}^{\mathcal{M},i}$

Then U must be a set of friends in the collection $\{F_i^1, \dots, F_i^{m_i}\}$, which yields a contradiction with the fact that $U \cap T = \emptyset$. It follows that for each $i \in N$, $i \in \mathcal{M}^*(S)$ if and only if $i \in \mathcal{M}(S)$ for each $S \subseteq N$, which concludes the proof.

Based on the arguments provided in the proof of Proposition 6, the following example shows a procedure to represent a GAS with a monotonic map $\mathcal{M}$ as a basic GAS.

Example 6 Consider a GAS $\langle N, v, \mathcal{M}\rangle$ with $N = \{1, 2, 3, 4\}$ and $\mathcal{M}$ monotonic such that $\mathcal{M}(\{1, 2, 3\}) = \{3\}$, $\mathcal{M}(\{3, 4\}) = \{2, 3\}$, $\mathcal{M}(\{1, 3, 4\}) = \{2, 3, 4\}$, $\mathcal{M}(\{1, 2, 3, 4) = \{2, 3, 4\}$, and $\mathcal{M}(S) = \emptyset$ or all other coalitions. The sets of minimal coalitions are as follows: $\mathcal{S}^{\mathcal{M},1} = \emptyset$, $\mathcal{S}^{\mathcal{M},2} = \{\{3, 4\}\}$, $\mathcal{S}^{\mathcal{M},3} = \{\{1, 2, 3\}, \{3, 4\}\}$, $\mathcal{S}^{\mathcal{M},4} = \{\{2, 3, 4\}\}$. Such a map can be represented via relation (4) with the collections: $F_1^1 = \emptyset$, $\{F_2^1, F_2^2\} = \{\{3\}, \{4\}\}$, $\{F_3^1, \dots, F_3^5\} = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}$, $\{F_4^1, \dots, F_4^3\} = \{\{2\}, \{3\}, \{4\}\}$, where such collections of friends are obtained via relation (8).

Moreover, by Propositions 1, 4 and 6 we have the following corollary.

Corollary 1 *Let $\langle N, v, \mathcal{M}\rangle$ be a basic GAS with $v \in \mathbb{R}_+^N$. Then the associated basic GAG (N, v^C) is monotonic if and only if $\mathcal{M}$ is monotonic.*

Proof The sufficient condition follows directly from Proposition 1. Moreover, by Proposition 4, if $v^{\mathcal{C}}$ is monotonic, then $E_i = \emptyset$ for all $i \in N$ and, by Proposition 6 it follows that $\mathcal{M}$ is monotonic, which concludes the proof.

5 Possible extensions

Further extensions can be introduced, in order to embrace a wider range of games that can be represented in a compact way into our framework.

Observe that the definition of GAG is based on the coalitional map $\mathcal{M}$, which is a multimap that assigns a coalition $\mathcal{M}(S) \subseteq N$ to each coalition $S \subseteq N$ of players. A further generalization is possible: think of the *graph-restricted game* introduced by Myerson (1977), where the worth of a coalition is evaluated on the connected components induced by an underlying graph. This class of games can be represented by considering a multimap $\mathcal{M} : 2^N \to 2^{2^N}$, which assigns to each coalition $S \subseteq N$, a subset $\mathcal{M}(S) \in 2^N$.

Moreover, every TU-game (N, v) can be described as a sum of k GAGs. For example, every game with 3 players can be represented as the sum of at most 3 GAGs. To see this, it suffices to consider 3 GAGs with maps, respectively, $\mathcal{M}^1$, $\mathcal{M}^2$ and $\mathcal{M}^3$ such that $\mathcal{M}^1(i) = \{i\}$ for all i, $M^1(\{i, j\}) = \emptyset$ for all i, j, $M^1(\{1, 2, 3\}) = \emptyset$, $\mathcal{M}^2(i) = \emptyset$ for all i, $M^2(\{1, 2\}) = \{1\}$, $M^2(\{1, 3\}) = \{3\}$, $M^2(\{2, 3\}) = \{2\}$, $M^2(\{1, 2, 3\}) = \emptyset$, $\mathcal{M}^3(i) = \emptyset$ for all i, $M^3(\{i, j\}) = \emptyset$ for all i, j, $M^3(\{1, 2, 3\}) = \{1\}$ and v^1, v^2, v^3 such that $v^1(i) = v(\{i\})$ for all i, $v^2(1) = v(\{1, 2\})$, $v^2(2) = v(\{2, 3\})$, $v^2(3) = v(\{1, 3\})$, $v^3(1) = v(\{1, 2, 3\})$ and

$v^3(2) = v^3(3) = 0$. In general at most $\frac{2^n-1}{n}$ GAGs are needed, but even less are sufficient if there are some "additive" coalitions, i.e. coalitions such that their value can be derived as the sum of values of other coalitions.

Another extension is the following. The definition of a basic GAG is based on a collection of sets $\mathcal{C} = \{\mathcal{C}_i\}_{i \in N}$, one for each player, where $\mathcal{C}_i = \{F_i^1, \ldots, F_i^m, E_i\}$ is a collection of subsets of N that satisfy some particular properties.
If we provide each player i with multiple collections $\{\mathcal{C}_i^1, \cdots, \mathcal{C}_i^k\}$, with $\mathcal{C}_i^k = \{F_i^{k1}, \ldots, F_i^{km}, E_i^k\}$, we are then able to represent those games that are associated to marginal contribution nets (MC-nets), introduced by Ieong and Shoham (2005).
The basic idea behind marginal contribution nets is to represent in a compact way the characteristic function of a game, as a set of rules of the form: *pattern* $\longrightarrow$ *value*, where a *pattern* is a Boolean formula over a set of n variables (one for each player) and a *value* is a real number. Here we restrict our attention to Boolean formulas that are conjunctions of literals, i.e. variables or their negations, and we shall call the corresponding rule a *basic* rule. A basic rule is said to *apply* to a coalition S if S contains all players whose variables appear unnegated in the pattern (represented by the literal x_i), and does not contain any of the players that appear negated (represented by the literal $\neg x_i$). For example, a rule with pattern $x_1 \wedge x_2 \wedge \neg x_3$ applies to the coalition $\{1, 2\}$ and $\{1, 2, 3, 4\}$ but not to the coalitions $\{1\}$ or $\{1, 2, 3\}$.
More formally, consider a collection of rules $R = \{r^1, \cdots, r^m\}$, where $r^k = \psi^k \longrightarrow x^k$ for each $k = 1, \cdots, m$, with ψ^i is being a basic rule over $\{x_1, \cdots, x_n\}$ and $x^k \in \mathbb{R}$ for all k. A coalition S is said to *satisfy* ψ (and we write $S \models \psi$) if and only if $r : \psi \longrightarrow x$ applies to S. The set R defines a coalitional game (N, v^R), introduced by Ieong and Shoham (2005), where $N = \{1, \cdots, n\}$ and the value of a coalition is computed by summing the values of all the rules that apply to it, i.e. v^R is given by

$$v^R(S) = \sum_{r^i \in R : S \models \psi^i} x^i.$$

As an example, consider the following MC-net:

$$r^1 : x_1 \wedge x_2 \wedge \neg x_3 \longrightarrow 3$$
$$r^2 : x_2 \longrightarrow 4$$

The corresponding game is (N, v), with $N = \{1, 2, 3\}$ and $v = 3e_{\{1,2\}} + 4u_{\{2\}}$, where e_S and u_S are defined as in Example 5.

Indeed, every coalitional game can be represented through MC-nets by defining one rule for each coalition $S \subseteq N$, where the pattern contains all the variables corresponding to the players in S and the negation of all the other variables, and the corresponding value is equal to the value of S in the game.

We show here how a game deriving from a MC-nets representation can be described as a generalization of a basic GAG, where each player has multiple collections of set of friends and enemies. For each rule $r^k = \psi^k \longrightarrow x^k$ such that the variable x_i appears unnegated in ψ^k, we provide player i with a collection of sets of friends and enemies $\mathcal{C}_i^k = \{F_i^{k1}, \dots, F_i^{km_k}, E_i^k\}$, defined as follows: $F_i^{kj} = \{j\}$ if x_j appears unnegated in ψ^k and $E_i^k = \{j \in N$ such that x_j appears negated in $\psi^k\}$, where m_k is the number of variables that appear unnegated in the pattern ψ^k. Moreover, for each player i we define a vector of values $v(i) = \{v_1(i), \cdots, v_m(i)\}$, where $v_k(i) = 0$ if x_i appears negated or does not appear in ψ^k and, for each rule r^k where x_i appears unnegated, $v_k(i)$ is given by:

$$v_k(i) = \frac{x^k}{c_k},$$

where c_k is the number of players whose corresponding variables appear unnegated in rule r^k. With the aformentioned definitions, a MC-nets game v^R can be described as

$$v^R = \sum_{i \in N} v^{C_i^R},$$

where $v^{\mathcal{C}_i^R}$ is defined as follows for every $i \in N$:

$$v^{\mathcal{C}_i^R}(S) = \begin{cases} 0 & \text{if } S \cap F_i^{kj} = \emptyset \, \forall j, k \text{ or } S \cap E_i^k \neq \emptyset \, \forall k \\ \sum\limits_{k=1}^{m} v_k(i) & \text{otherwise.} \end{cases}$$

For the previous example, the collections of friends and enemies would be defined as:

$$\mathcal{C}_1^1 = \{F_1^{11} = \{1\}, F_1^{12} = \{2\}, E_1^1 = \{3\}\}$$
$$\mathcal{C}_2^1 = \{F_1^{21} = \{1\}, F_1^{22} = \{2\}, E_1^2 = \{3\}\}$$
$$\mathcal{C}_2^2 = \{F_2^{21} = \{2\}, E_2^2 = \emptyset\}$$

Moreover, $v(1) = (\frac{3}{2}, 0)$, $v(2) = (\frac{3}{2}, 4)$ and $v(3) = (0, 0)$.

In this way, we are indeed able to describe every TU-game, since the representation of MC-nets is complete. The computational complexity of such representation is in general high. However, when a game can be described by a small collection of rules, and therefore the associated extended GAS is described in a relatively compact way, the complexity of its representation and of the computation of solutions is consequently reduced.

6 Conclusions

In the present paper, the class of generalized additive games is introduced, where the worth of a coalition of players is evaluated by means of a map $\mathcal{M}$ that selects the valuable players in the coalition.
Several examples from the literature of classical coalitional games that can be described within our approach are presented.

An interesting direction for future research is that of coalition formation, since for generic basic GAGs associated to GASs with nonnegative v, where the sets of enemies are not empty, the grand coalition is not likely to form. As an example of coalition formation problem that can be well represented in these terms, consider the following "triangle" situation on three researchers, namely, Alice, Bob and Carol. Due to their characters' affinities, Alice and Bob love to do research together, but they do not like at all to be involved in research projects with Carol. Instead, Carol loves to do research with Bob, but not with Alice. On the other hand, in order to make a successful research, they need to perform a certain number of expensive experiments. Because of the bad financial status of their respective departments, Alice and Bob's personal research funds are very limited, whereas Carol can rely on a conspicuous international funding. Such a situation can be represented by as basic GAS on the three researchers $N = \{\text{Alice}, \text{Bob}, \text{Carol}\}$ where the set of friends of Alice contains only Bob, the set of friends of Bob and Carol is the same and coincides with the singleton Alice, the set of enemies of Alice and Bob is the singleton Carol and, finally, the set of enemies of Carol is the singleton Alice. In addition, the function $v = (v(\text{Alice}), v(\text{Bob}), v(\text{Carol}))$ of their individual contribution is given by their respective research funds. The corresponding basic GAG is $v^{\mathcal{C}}(\text{Alice}, \text{Bob}) = v(\text{Alice}) + v(\text{Bob})$, $v^{\mathcal{C}}(\text{Bob}, \text{Carol}) = v(\text{Carol})$ and $v^{\mathcal{C}}(S) = 0$ for all the other coalitions $S \subseteq N$. It is quite natural to expect that if $v(\text{Carol})$ is quite larger than $v(\text{Alice}) + v(\text{Bob})$, then the coalition $\{\text{Bob}, \text{Carol}\}$ will form, despite the reciprocal friendship between Alice and Bob. In general, we believe that the issue about which coalitions are more likely to form in a basic GAG is not trivial and deserves to be further explored.

Another interesting problem is related to the analysis of classical solutions for GAGs, that is partially addressed in the forthcoming paper by Cesari et al. (2015), where the behaviour of certain solutions like semivalues and core allocations are studied in connection with the properties of the filtering map $\mathcal{M}$ introduced and discussed in this article.

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