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# On social rankings and their properties

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**Abstract**

Several real-life complex systems, like human societies or economic networks, are formed by interacting units characterized by patterns of relationships that may generate a group-based social hierarchy. In this paper, we address the problem of how to rank the individuals with respect to their ability to "influence" the relative strength of groups in a society. We also analyse the effect of basic properties in the computation of a social ranking within specific classes of (ordinal) coalitional situations.

**Keywords**: social ranking, coalitional power, ordinal power, axiomatic approach.

## 1 Introduction

Ranking is a fundamental ingredient of many real-life situations, like the ranking of candidates applying to a job, the rating of universities around the world, the distribution of power in political institutions, the centrality of different actors in social networks, the accessibility of information on the web, etc. Often, the criterion used to rank the items (e.g., agents, institutions, products, services, etc.) of a set $N$ also depends on the interaction among the items within the subsets of $N$ (for instance, with respect to the users' preferences over bundles of products or services). In this paper we address the following question: given a finite set $N$ of items and a ranking over its subsets, can we derive a "social" ranking over $N$ according to the "overall importance" of its single elements?

For instance, consider a company with three employees $1, 2$ and $3$ working in the same department. According to the opinion of the manager of the company, the

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job performance of the different teams $S \subseteq N = \{1,2,3\}$ is as follows: $\{1,2,3\} \succcurlyeq \{3\} \succcurlyeq \{1,3\} \succcurlyeq \{2,3\} \succcurlyeq \{2\} \succcurlyeq \{1,2\} \succcurlyeq \{1\} \succcurlyeq \emptyset$ ($S \succcurlyeq T$, for each $S, T \subseteq N$, means that the performance of $S$ is at least as good as the performance of $T$). Based on this information, the manager asks us to make a ranking over his three employees showing their attitude to work with others as a team or autonomously. Intuitively, 3 seems to be more influential than 1 and 2, as employee 3 belongs to the most successful teams in the above ranking. Can we state more precisely the reasons driving us to this conclusion? And what can we say if we have to decide who between 1 and 2 is more productive and deserves a promotion? In this paper we analyse different properties of ordinal social rankings in order to get some answers to such questions.

In this paper, a *social ranking* is defined as a map associating to each *power relation* (i.e., a total preorder over the set of all subsets of $N$) a total preorder over the elements of $N$. The properties for social rankings that we analyse in this paper have classical interpretations, such as *anonymity* and *symmetry*, saying that the ranking should not depend on the identity of the elements of $N$, or the *dominance*, saying that an element $i \in N$ should be ranked higher than an element $j \in N$ whenever $i$ dominates $j$, i.e. a coalition $S \cup \{i\}$ is stronger than $S \cup \{j\}$ for each $S \subset N$ not containing neither $i$ nor $j$. Another property we study in this paper is the *independence of irrelevant coalitions*, saying that the social ranking between two elements $i$ and $j$ should only depend on their respective contributions when added to coalitions not containing neither $i$ nor $j$ (in other words, the information needed to rank $i$ and $j$ is provided by the relative comparison of coalitions $U, W \subset N$ such that $U \setminus \{i\} = W \setminus \{j\}$). Finally, we introduce the notion of *separability*, which specifies how to combine social rankings associated to "compatible" power relations, i.e. power relations whose intersection is still a power relation. We use these properties to axiomatically characterize social rankings on particular classes of power relations.

The structure of the paper is the following. In the next section, we present some related approaches from the literature and our main contributions. Basic notions and definitions are presented in Section 3. In Section 4 we introduce and discuss some properties for social rankings. In Section 5 we study the compatibility of certain axioms and their effect on some elementary notions of social ranking. Section 6 is devoted to the presentation of the axiomatic analysis of social rankings on a special class of compatible power relations. In Section 7 we focus on the analysis of social rankings that satisfy both the dominance property and the property of independence of irrelevant coalitions, and that, on particular power relations, are specified by the ordering of coalitions of the same size. Section 8 concludes.

# 2 Related Work and Contribution

The central question of this paper seems closely related to the well known problem of measuring the power of players in a cooperative game (see, for instance, [14]). In this context, given the information about which coalitions of players are winning or not, one can use several *power indices* from the literature on coalitional games to assess the power of the single players [6, 1, 18, 9]. However, our framework is different for at least two reasons: first, we face coalitional situations where only a qualitative (ordinal) comparison of the strength of coalitions is given; second, we look for a ranking over the single elements of $N$, and we do not require a quantitative assessment of their "power". Differently stated, we characterize social rankings starting from the very basic properties of a power relation over coalitions, and without the use of any particular coalitional game, that would necessarily require the conversion of the (purely ordinal) information about the relative strength of coalitions into a quantitative assessment of their power.

As far as we know, the only attempt in the literature to generalize the notions of coalitional game and power index within an ordinal framework has been provided in [12], where, given a total preorder representing the relative strength of coalitions, a social ranking over the player set is provided according to a notion of *ordinal influence* and using the Banzhaf index [1] of a "canonical" coalitional game. In a still different context, a model of coalition formation has been introduced in [15], where the relative strength of disjoint coalitions is represented by an exogenous binary relation and the players try to maximize their position in a social ranking.

We also noticed a connection with some kind of "inverse problems", precisely, how to derive a ranking over the set of all subsets of $N$ in a way that is "compatible" with a primitive ranking over the single elements of $N$. This question has been carried out in the tradition of the literature on extending an order on a set $N$ to its *power set* (the set of all possible subsets of $N$) with the objective to axiomatically characterize families of ordinal preferences over subsets (see, for instance, [2, 3, 4, 5, 8, 7, 10, 11]). In this context, an order $\succcurlyeq$ on the power set of $N$ is required to be an *extension* of a primitive order $P$ on $N$. This means that the relative ranking of any two singleton sets according to $\succcurlyeq$ must be the same as the relative ranking of the corresponding alternatives according to $P$. In this framework, most of the axiomatic approaches from literature focused on properties suggesting that the *interaction* among single objects should not play a relevant role in establishing the ranking among subsets [4, 17, 3]. An example is the property of *responsiveness* introduced by [17], which requires that a set $S \subseteq N$ is preferred to a set $T \subseteq N$ whenever $S$ is obtained from $T$ by replacing some object $t \in T$ with another $i \in N$ not in $T$ which is preferred to $t$ (according to the primitive ranking on the universal set $N$). In other words, the responsiveness property prevents complementarity or incompatibility effects among objects within sets of the same cardinality. The responsiveness property, together with another property called *fixed-Cardinality neutrality* and saying that the labelling of the alternatives is irrelevant in establishing the ranking among sets of a fixed cardinality, was used by [4] to characterize a particular class of extension based on a lexicographic comparison of sets of the same size.

In real life situations it seems not realistic to exclude *a priori* the possibility that items within coalitions may interact. The ranking with employees is one example, another one (and a classical one) is the problem of deciding whether to combine two distinct therapies for a disease: the combination of two treatments does not always improve the chances of success, and may provoke more serious side effects, with respect to each single treatment. More in general, according to our definition, we do not require a power relation to be responsive: even if a singleton coalition $\{x\}$ is strictly stronger than a singleton coalition $\{y\}$, we can have that for another coalition $S$ (not containing neither $x$ nor $y$) $S \cup \{y\}$ is stronger than a coalition $S \cup \{x\}$ (e.g., because of an incompatibility among $x$ and some objects in $S$).

# 3 Preliminaries and notations

A *binary relation* $R$ on a finite set $N = \{1, \dots, n\}$ is a collection of ordered pairs of elements of $N$, i.e. $R \subseteq N \times N$. $\forall x, y \in N$, the more familiar notation $xRy$ will be often used instead of the more formal one $(x, y) \in R$. We provide some standard properties for $R$. *Reflexivity*: for each $x \in N$, $xRx$; *transitivity*: for each $x, y, z \in N$, $xRy$ and $yRz \Rightarrow xRz$; *totality*: for each $x, y \in N$, $x \neq y \Rightarrow xRy$ or $yRx$; *antisymmetry*: for each $x, y \in N$, $xRy$ and $yRx \Rightarrow x = y$. A reflexive and transitive binary relation is called *preorder*. A preorder that is also total is called *total preorder*. A total preorder that also satisfies antisymmetry is called *linear order*. The notation $\neg(xRy)$ means that $xRy$ is not true. We denote by $2^N$ the power set of $N$ and we use the notations $\mathcal{T}^N$ and $\mathcal{T}^{2^N}$ to denote the set of all total preorders on $N$ and on $2^N$, respectively. Moreover, the cardinality of a set $S \in 2^N$ is denoted by $|S|$. In the remaining of the paper, we will also refer to an element $S \in 2^N$ as a *coalition* $S$.

Consider a total preorder $\succcurlyeq \subseteq 2^N \times 2^N$ over the subsets of $N$. Often we will use the notation $S \succ T$ to denote the fact that $S \succcurlyeq T$ and $\neg(T \succcurlyeq S)$ (in this case, we also say that the relation between $S$ and $T$ is 'strict'), and the notation $S \sim T$ to denote the fact that $S \succcurlyeq T$ and $T \succcurlyeq S$. For each $i, j \in N$, $i \neq j$, and all $k = 1, \dots, n-2$, we denote by $\Sigma_{ij}^k = \{S \in 2^{N \setminus \{ij\}} : |S| = k\}$ the set of all subsets of $N$ not containing neither $i$ nor $j$ with $k$ elements. Moreover, for each $i, j \in N$, we define the set $D_{ij}^k(\succcurlyeq) = \{S \in \Sigma_{ij}^k : S \cup \{i\} \succcurlyeq S \cup \{j\}\}$ as the set of coalitions $S \in 2^{N \setminus \{ij\}}$ of cardinality $k$ such that $S \cup \{i\}$ is in relation with $S \cup \{j\}$ (and, changing the ordering of $i$ and $j$, the set $D_{ji}^k(\succcurlyeq) = \{S \in \Sigma_{ij}^k : S \cup \{j\} \succcurlyeq S \cup \{i\}\}$).

# 4 Axioms for social rankings

In the following of these notes, we interpret a total preorder $\succcurlyeq$ on $2^N$ as a *power relation*, that is, for each $S, T \in 2^N$, $S \succcurlyeq T$ stands for '$S$ is considered at least as strong as $T$ according to the power relation $\succcurlyeq$'.

Given a class $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ of power relations, we call a map $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$, assigning to each power relation in $\mathcal{C}^{2^N}$ a total preorder on $N$, a *social ranking solution* or, simply, a *social ranking*. Then, given a power relation $\succcurlyeq$, we will interpret the total binary relation $\rho(\succcurlyeq)$ associated to $\succcurlyeq$ by the social ranking $\rho$, as the relative power of players in a society under relation $\succcurlyeq$. Precisely, for each $i, j \in N$, $i\rho(\succcurlyeq)j$ stands for '$i$ is considered at least as influential as $j$ according to the social ranking $\rho(\succcurlyeq)$', where the influence of an agent is intended as her/his ability to join coalitions in the strongest positions of a power relation. Note that we require that $\rho(\succcurlyeq)$ is a total preorder over the elements of $N$, that is we always want to express the relative comparison of two agents, and such a relation must be transitive.

A social ranking $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ such that $i\rho(\succcurlyeq)j \Leftrightarrow \{i\} \succcurlyeq \{j\}$ for each $\succcurlyeq \in \mathcal{C}^{2^N}$ and each $i, j \in N$ is said to be *primitive* (i.e., it neglects any information contained in $\succcurlyeq$ about the comparison of coalitions of cardinality different from 1). A social ranking $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ such that $i\rho(\succcurlyeq)j$ and $j\rho(\succcurlyeq)i$ for all $i, j \in N$ is said to be *unanimous* ($N$ is an indifference class with respect to $\rho(\succcurlyeq)$).

Now we introduce some properties for social rankings. The first axiom is the *dominance* one: if each coalition $S$ containing agent $i$ but not $j$ is stronger than coalition $S$ with $j$ in the place of $i$, then agent $i$ should be ranked higher than agent $j$ in the society, for any $i, j \in N$. Precisely, given a power relation $\succcurlyeq \in \mathcal{T}^{2^N}$ and $i, j \in N$ we say that $i$ *dominates* $j$ in $\succcurlyeq$ if $S \cup \{i\} \succcurlyeq S \cup \{j\}$ for each $S \in 2^{N\setminus\{i,j\}}$ (we also say that $i$ *strictly dominates* $j$ in $\succcurlyeq$ if $i$ dominates $j$ and in addition there exists $S \in 2^{N\setminus\{i,j\}}$ such that $S \cup \{i\} \succ S \cup \{j\}$).

**Definition 1** (DOM). *A social ranking $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ satisfies the dominance (DOM) property on $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ if and only if for all $\succcurlyeq \in \mathcal{C}^{2^N}$ and $i, j \in N$, if $i$ dominates $j$ in $\succcurlyeq$ then $i\rho(\succcurlyeq)j$ [and $\neg(j\rho(\succcurlyeq)i)$ if $i$ strictly dominates $j$ in $\succcurlyeq$].*

The following axiom states that the relative strength of two agents $i, j \in N$ in the social ranking should only depend on their effect when they are added to each possible coalition $S$ not containing neither $i$ nor $j$, and the relative ranking of the other coalitions is irrelevant. Formally:

**Definition 2** (IIC)**.** *A social ranking $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ satisfies the Independence of Irrelevant Coalitions (IIC) property on $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ iff*

$$i\rho(\succcurlyeq)j \Leftrightarrow i\rho(\sqsupseteq)j$$

*for all $i, j \in N$ and all power relations $\succcurlyeq, \sqsupseteq \in \mathcal{C}^{2^N}$ such that for each $S \in 2^{N\setminus\{i,j\}}$*

$$S \cup \{i\} \succcurlyeq S \cup \{j\} \Leftrightarrow S \cup \{i\} \sqsupseteq S \cup \{j\}.$$

For the following property we need some further notations. Let $\Pi$ be the set of all bijections $\pi : N \to N$. With a slightly abuse of notations, we also denote by $\pi(S)$ the image under $\pi$ of a coalition $S$, i.e. $\pi(S) = \{\pi(i) : i \in S\}$.

The anonymity property says that a social ranking of two agents $i$ and $j$ should not depend on the labels. We can formulate this principle as follows.

**Definition 3** (ANON)**.** *A social ranking $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ satisfies the anonymity (ANON) property on $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ iff*

$$i\rho(\succcurlyeq)j \Leftrightarrow \pi(i)\rho(\succcurlyeq)\pi(j)$$

*for all $i, j \in N$, $\pi \in \Pi$ and $\succcurlyeq \in \mathcal{C}^{2^N}$ such that for each $S \in 2^{N\setminus\{i,j\}}$*

$$S \cup \{i\} \succcurlyeq S \cup \{j\} \Leftrightarrow \pi(S) \cup \{\pi(i)\} \succcurlyeq \pi(S) \cup \{\pi(j)\}.$$

We notice that in the framework of the extension problems discussed in Sections 2, this property is very close to the neutrality property introduced in [4]. One can define a similar but more restrictive property by paying more attention to the comparisons of subsets having the same cardinality.

**Definition 4** (SYM)**.** *A social ranking $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ satisfies the symmetry (SYM) property on $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ iff*

$$i\rho(\succcurlyeq)j \Leftrightarrow p\rho(\succcurlyeq)q$$

*for all $i, j, p, q \in N$ and $\succcurlyeq \in \mathcal{C}^{2^N}$ such that $|D^k_{ij}| = |D^k_{pq}|$ and $|D^k_{ji}| = |D^k_{qp}|$ for each $k = 0, \ldots, n-2$.*

**Remark 1.** *Note that if a social ranking $\rho$ satisfies the SYM axiom on $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$, then for every $\succcurlyeq \in \mathcal{C}^{2^N}$ and $i, j \in N$, if $|D_{ij}^k| = |D_{ji}^k|$ for each $k = 0, \ldots, n-2$, then $i\rho(\succcurlyeq)j$ and $j\rho(\succcurlyeq)i$, that is $i$ and $j$ are indifferent in $\rho(\succcurlyeq)$ (to see this, simply take $p = i$ and $q = j$ in Definition 4).*

**Remark 2.** *If we want to check if a given social ranking rule satisfies DOM, IIC, ANON or SYM only partial information on $\succcurlyeq$ is needed. In fact, conditions on the ranking $\rho(\succcurlyeq)$ between two elements $\{i, j\}$ only depend on the comparisons of subsets having the same cardinality and sharing the same subset $S \in 2^{N\setminus\{i,j\}}$ not containing neither $i$ nor $j$.*

*Table 1 presents the partial information that we need in order to analyse the social ranking on three elements $1, 2,$ and $3$ for $N = \{1, 2, 3, 4\}$*[1]*. For instance, concerning the comparison $\rho(\succcurlyeq)$ between 1 and 2, only the first column of Table 1 is needed.*

Table 1: Partial information needed for the analysis of $\rho(\succcurlyeq)$ for $N = \{1, 2, 3, 4\}$

| 1 vs 2 | 2 vs. 3 | 1 vs. 3 |
|---|---|---|
| $\{1\} \succcurlyeq$ or $\preccurlyeq \{2\}$ | $\{2\} \succcurlyeq$ or $\preccurlyeq \{3\}$ | $\{1\} \succcurlyeq$ or $\preccurlyeq \{3\}$ |
| $\{1,3\} \succcurlyeq$ or $\preccurlyeq \{2,3\}$ | $\{1,2\} \succcurlyeq$ or $\preccurlyeq \{1,3\}$ | $\{1,2\} \succcurlyeq$ or $\preccurlyeq \{2,3\}$ |
| $\{1,4\} \succcurlyeq$ or $\preccurlyeq \{2,4\}$ | $\{2,4\} \succcurlyeq$ or $\preccurlyeq \{3,4\}$ | $\{1,4\} \succcurlyeq$ or $\preccurlyeq \{3,4\}$ |
| $\{1,3,4\} \succcurlyeq$ or $\preccurlyeq \{2,3,4\}$ | $\{1,2,4\} \succcurlyeq$ or $\preccurlyeq \{1,3,4\}$ | $\{1,2,4\} \succcurlyeq$ or $\preccurlyeq \{2,3,4\}$ |

From now, we will sometimes omit braces and commas to separate elements, for instance, $ij$ denotes the set $\{i, j\}$.

Table 2 is the *comparison table* of the power relation "$\{1, 2, 3\} \succcurlyeq \{3\} \succcurlyeq \{1, 3\} \succcurlyeq \{2, 3\} \succcurlyeq \{2\} \succcurlyeq \{1, 2\} \succcurlyeq \{1\} \succcurlyeq \emptyset$".

**Remark 3.** *We present in the following some properties of comparisons tables.*

[1] The complete table for the social ranking relation on $N = \{1, 2, 3, 4\}$ has three more columns : (1 vs 4), (2 vs 4) and (3 vs 4).

Table 2: Comparison table of $\{1,2,3\} \succcurlyeq \{3\} \succcurlyeq \{1,3\} \succcurlyeq \{2,3\} \succcurlyeq \{2\} \succcurlyeq \{1,2\} \succcurlyeq \{1\} \succcurlyeq \emptyset$.

| 1 vs 2 | 2 vs 3 | 1 vs 3 |
|---|---|---|
| $1 \preccurlyeq 2$ | $2 \preccurlyeq 3$ | $1 \preccurlyeq 3$ |
| $13 \succcurlyeq 23$ | $12 \preccurlyeq 13$ | $12 \preccurlyeq 23$ |

- *Let* $|N| = n$*, then the corresponding comparison table has* $2^{(n-2)} + 1$ *lines (+1 corresponds to the title line).*

- *The number of comparisons to be considered with* $k$ *elements is* $\binom{n-2}{k-1}$.

- *Only comparisons between subsets having the same cardinality being relevant, different power relations can give place to the same comparison table. For instance* $123 \succcurlyeq 32 \succcurlyeq 21 \succcurlyeq 13 \succcurlyeq 1 \succcurlyeq 2 \succcurlyeq 3 \succcurlyeq \emptyset$ *and* $32 \sqsupseteq 123 \sqsupseteq 21 \sqsupseteq 1 \sqsupseteq 2 \sqsupseteq 13 \sqsupseteq 3 \sqsupseteq \emptyset$ *will have the same comparison table.*

We start the axiomatic analysis of social rankings showing that the anonymity property and the symmetry one are not equivalent.

**Proposition 1.** *If a social ranking* $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ *satisfies SYM then it also satisfies ANON, but the converse is not true.*

*Proof.* Consider a social ranking $\rho$ that satisfies SYM. Let $\succcurlyeq \in \mathcal{T}^{2^N}$ and $i, j \in N$ be such that there exists $\pi \in \Pi$ such that

$$S \cup \{i\} \succcurlyeq S \cup \{j\} \Leftrightarrow \pi(S) \cup \{\pi(i)\} \succcurlyeq \pi(S) \cup \{\pi(j)\}$$

for all $S \in 2^{N \setminus \{i,j\}}$. It immediately follows that $|D^k_{ij}| = |D^k_{\pi(i)\pi(j)}|$ and $|D^k_{ji}| = |D^k_{\pi(j)\pi(i)}|$, for each $k = 1, \ldots, n-2$, and by SYM

$$i\rho(\succcurlyeq)j \Leftrightarrow \pi(i)\rho(\succcurlyeq)\pi(j)$$

which proves that $\rho$ satisfies also ANON.

Now, consider a social ranking $\rho$ that satisfies the ANON property and let $\succcurlyeq \in \mathcal{T}^{2^N}$, with $N = \{i, j, p, q, 1, 2, 3, 4, 5, 6\}$, be as in Table 3 and $S \sim T$ for every $S \in 2^{N \setminus \{i,j\}}$ and every $T \in 2^{N \setminus \{p,q\}}$ different from the subsets explicitly reported in Table 3. Clearly, $|D^k_{ij}| = |D^k_{pq}|$ and $|D^k_{ji}| = |D^k_{qp}|$ for each $k = 1, \ldots, n-2$.

Table 3: The ANON property does not imply the SYM property.

| $i$ vs $j$ | $p$ vs. $q$ |
|---|---|
| $\{i,1,2\} \succ \{j,1,2\}$ | $\{p,4,5\} \succ \{q,4,5\}$ |
| $\{i,1,3\} \succ \{j,1,3\}$ | $\{p,4,6\} \succ \{q,4,6\}$ |
| $\{i,1,6\} \prec \{j,1,6\}$ | $\{p,3,5\} \prec \{q,3,5\}$ |
| $S \cup \{i\} \sim S \cup \{j\}$ | $T \cup \{p\} \sim T \cup \{q\}$ |
| $\vdots$ | $\vdots$ |

On the other hand there is no permutation $\pi \in \Pi$ such that $\pi(i) = p$ and $\pi(j) = q$ and such that

$$S \cup \{i\} \succcurlyeq S \cup \{j\} \Leftrightarrow \pi(S) \cup \{\pi(i)\} \succcurlyeq \pi(S) \cup \{\pi(j)\}$$

for all $S \in 2^{N\setminus\{i,j\}}$. In fact, from the first two lines of Table 3 we necessarily have that $\pi(1) = 4$ and then $\pi(\{i,1,6\}) \neq \{p,3,5\}$. As a consequence, the ANON property cannot impose any relation between the social ranking of $i$ over $j$ and the social ranking of $p$ over $q$, and the SYM is not necessarily satisfied. $\square$

We conclude this section with an example showing that an apparently natural procedure (namely, the majority rule) to rank the agents of $N$ may fail to provide a transitive social ranking. We first formally introduce such a procedure.

**Definition 5** (Majority rule)**.** *A majority rule (denoted by $M$) is a map assigning to each power relation $\succcurlyeq \in \mathcal{T}^{2^N}$ a total binary relation $M(\succcurlyeq)$ on $N$ such that*

$$iM(\succcurlyeq)j \Leftrightarrow d_{ij}(\succcurlyeq) \geq d_{ji}(\succcurlyeq).$$

*where $d_{ij}(\succcurlyeq) = |\{S \in 2^{N\setminus\{i,j\}} : S \cup \{i\} \succcurlyeq S \cup \{j\}\}|$ for each $i, j \in N$.*

**Example 1.** *One can easily check that the majority rule $M$ satisfies the property of DOM, IIC and SYM on the class $\mathcal{T}^{2^N}$. On the other hand, it is also easy to find an example of power relation $\succcurlyeq$ such that $M(\succcurlyeq)$ is not transitive. Consider for instance the power relation $\succcurlyeq \in \mathcal{T}^{2^N}$ with $N = \{1,2,3,4\}$ such that*

$$2 \succ 1 \succ 3$$
$$23 \succ 13 \succ 12 \succ 14 \succ 34 \succ 24$$
$$134 \sim 124 \sim 234$$

*We rewrite the relevant information about $\succcurlyeq$ by means of Table 4. Note that*

Table 4: The relevant information about $\succcurlyeq$ of Example 1.

| 1 vs. 2 | 2 vs. 3 | 1 vs. 3 |
|---|---|---|
| $1 \prec 2$ | $2 \succ 3$ | $1 \succ 3$ |
| $13 \prec 23$ | $12 \prec 13$ | $12 \prec 23$ |
| $14 \succ 24$ | $24 \prec 34$ | $14 \succ 34$ |
| $134 \sim 234$ | $124 \sim 134$ | $124 \sim 234$ |

$d_{12}(\succcurlyeq) = 2$, $d_{21}(\succcurlyeq) = 3$, $d_{23}(\succcurlyeq) = 2$, $d_{32}(\succcurlyeq) = 3$, $d_{13}(\succcurlyeq) = 3$ *and* $d_{31}(\succcurlyeq) = 2$. *So, we have that* $2M(\succcurlyeq)1$, $3M(\succcurlyeq)2$ *and* $1M(\succcurlyeq)3$, *but* $\neg(3M(\succcurlyeq)1)$*):* $M(\succcurlyeq)$ *is not a transitive relation.*

# 5 Primitive and unanimous social rankings

In this section we study the relations between the axioms introduced in the previous section and the social ranking solutions. In the following, we show that DOM and SYM are not compatible in a general case, for $N > 3$ (see Theorem 1), whereas SYM and IIC determine a unanimous social ranking on particular power relations.

We start with showing some consequences of using the axioms introduced in the previous section when the cardinality of the set $N$ is 3 or 4. The analysis for cardinality $|N| = 3$ is easy since we can enumerate all the cases. As we will present in the following, the notion of complementarity plays an important role in this case. We denote by $S^*$ the complement of the subset $S$ ($S^* = N \setminus S$), and we say that a social ranking $\rho$ such that $i\rho(\succcurlyeq)j \Leftrightarrow \{i\}^* \succcurlyeq \{j\}^*$ for each $\succcurlyeq \in \mathcal{T}^{2^N}$ and each $i, j \in N$ is *complement primitive* (i.e., it neglects any information contained in $\succcurlyeq$ about the comparison of coalitions of cardinality different from $n-1$).

**Proposition 2.** *If* $|N| = 3$*, then there are only two social ranking solutions satisfying the DOM and SYM conditions: the primitive solution and the complement primitive one.*

*Proof.* Let $N = \{1, 2, 3\}$ with $1 \succcurlyeq 2 \succcurlyeq 3$. Then six cases may occur in $\succcurlyeq$: case 1)

$13 \succcurlyeq 23 \succcurlyeq 12$, case 2) $13 \succcurlyeq 12 \succcurlyeq 23$, case 3) $23 \succcurlyeq 13 \succcurlyeq 12$, case 4) $12 \succcurlyeq 13 \succcurlyeq 23$, case 5) $23 \succcurlyeq 12 \succcurlyeq 13$ and case 6) $12 \succcurlyeq 23 \succcurlyeq 13$.

DOM and SYM impose that:

case 1) by DOM :$1\rho(\succcurlyeq)2$, by SYM ($1\rho(\succcurlyeq)3$ and $2\rho(\succcurlyeq)3$) or ($3\rho(\succcurlyeq)1$ and $3\rho(\succcurlyeq)2$). Hence we have $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3$ (primitive) or $3\rho(\succcurlyeq)1\rho(\succcurlyeq)2$ (complement primitive)

case 2) by DOM :$1\rho(\succcurlyeq)2$ and $1\rho(\succcurlyeq)3$. We can have $2\rho(\succcurlyeq)3$ or $3\rho(\succcurlyeq)2$. Hence we have $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3$ (primitive) or $1\rho(\succcurlyeq)3\rho(\succcurlyeq)2$ (complement primitive)

case 3) by SYM : ($1\rho(\succcurlyeq)2$, $1\rho(\succcurlyeq)3$ and $2\rho(\succcurlyeq)3$) or ($2\rho(\succcurlyeq)1$ , $3\rho(\succcurlyeq)1$ and $3\rho(\succcurlyeq)2$).

case 4) by DOM $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3$

case 5) by DOM :$2\rho(\succcurlyeq)3$, by SYM ($1\rho(\succcurlyeq)2$ and $1\rho(\succcurlyeq)3$) or ($2\rho(\succcurlyeq)1$ and $3\rho(\succcurlyeq)1$). Hence we have $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3$ (primitive) or $2\rho(\succcurlyeq)3\rho(\succcurlyeq)1$ (complement primitive)

case 6) by DOM :$1\rho(\succcurlyeq)3$ and $2\rho(\succcurlyeq)3$. We can have $1\rho(\succcurlyeq)2$ or $2\rho(\succcurlyeq)1$. Hence we have $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3$ (primitive) or $2\rho(\succcurlyeq)1\rho(\succcurlyeq)3$ (complement primitive)

□

A relation which provides coherent comparisons with respect to the complement of objects is said "self-reflecting" . The notion of "self-reflecting" is introduced by Fishburn [19]. More formally, if we denote by $S^*$ the complement of the subset $S$ ($S^* = N \setminus S$), we say that the power relation $\succcurlyeq$ is self-reflecting if and only if for all $S, Q \in N$, $S \succcurlyeq Q$ implies $Q^* \succcurlyeq S^*$.

**Corollary 1.** *If $|N| = 3$ and the power relation is self-reflecting, then the DOM condition is sufficient in order to determine the social ranking and it corresponds to a primitive social rule.*

*Proof.* Let $N = \{i, j, k\}$. Self-reflecting implies that $\forall i, j \in N$ $i \succcurlyeq j \Leftrightarrow j^* \succcurlyeq i^* \Leftrightarrow ik \succcurlyeq jk$. By DOM we get $\forall i, j, k \in N$ $i\rho(\succcurlyeq)j \Leftrightarrow i \succcurlyeq j \Leftrightarrow j^* \succcurlyeq i^* \Leftrightarrow ik \succcurlyeq jk$. □

Next proposition presents an impossibility for cardinality $|N| = 4$ and shows that on the class $\mathcal{T}^{2^N}$ (all possible total preorders) the properties of DOM and SYM are not compatible.

**Proposition 3.** *Let $N = \{1, 2, 3, 4\}$. There is no social ranking rule $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ which satisfies DOM and SYM on $\mathcal{T}^{2^N}$.*

*Proof.* We show a particular situation where DOM and SYM are not compatible. Consider a power relation $\succcurlyeq \in \mathcal{T}^{2^N}$ with $N = \{1, 2, 3, 4\}$ and such that

$$1 \sim 2 \sim 3$$
$$13 \succ 23 \succ 12 \succ 24 \sim 14 \succ 34$$
$$1234 \sim 123 \sim 124 \sim 134 \sim 234$$

We rewrite the relevant informations about $\succcurlyeq$ and the elements $1, 2$ and $3$ by means of the following Table 5. By Remark 1, a social ranking rule $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ which

Table 5: The relevant informations about $\succcurlyeq$ and the elements $1, 2$ and $3$.

| 1 vs. 2 | 2 vs. 3 | 1 vs. 3 |
|---|---|---|
| $1 \sim 2$ | $2 \sim 3$ | $1 \sim 3$ |
| $13 \succ 23$ | $12 \prec 13$ | $12 \prec 23$ |
| $14 \sim 24$ | $24 \succ 34$ | $14 \succ 34$ |
| $134 \sim 234$ | $124 \sim 134$ | $124 \sim 234$ |

satisfies SYM should be such that $2\rho(\succcurlyeq)3$, $3\rho(\succcurlyeq)2$, $1\rho(\succcurlyeq)3$, $3\rho(\succcurlyeq)1$.
By the DOM property, we should have $1\rho(\succcurlyeq)2$, and $\neg(2\rho(\succcurlyeq)1)$, which yields a contradiction with the transitivity of the ranking $\rho(\succcurlyeq)$. $\square$

Proposition 3 can be easily generalized to the case $|N| > 4$

**Theorem 1.** *Let $|N| > 3$. There is no social ranking rule $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ which satisfies DOM and SYM on $\mathcal{T}^{2^N}$.*

*Proof.* Simply consider power relations in $\mathcal{T}^{2^N}$, $N \supseteq \{1, 2, 3, 4\}$, that are obtained from the power relation $\succcurlyeq$ defined in the proof of Proposition 3 and assigning all the additional subsets of $N$ not contained in $\{1, 2, 3, 4\}$ in the same indifference class.

More precisely, let $N \supseteq \{1,2,3,4\}$, and take $\succcurlyeq' \in \mathcal{T}^{2^N}$ such that $U \succcurlyeq' W :\Leftrightarrow U \succcurlyeq W$ (where $\succcurlyeq$ is the power relation considered in the proof of Proposition 3) for all the subsets $U, W \subseteq \{1,2,3,4\}$, and $U \succcurlyeq' W$, $W \succcurlyeq' U$ for all the other subsets of $N$ not included in $\{1,2,3,4\}$. $\square$

One could argue that the incompatibility between the properties of DOM and SYM follows from the particular instance of power relation $\succcurlyeq$ used in the proof of Proposition 3, where the fact that $|D^1_{23}(\succcurlyeq)| = |D^1_{32}(\succcurlyeq)|$ and $|D^1_{13}(\succcurlyeq)| = |D^1_{31}(\succcurlyeq)|$ (and $D^t_{ji}(\succcurlyeq) = D^t_{ij}(\succcurlyeq)$ for $t = 0, 2$ and $i, j \in \{1,2,3\}$) implies, by the SYM axiom, that 1, 2 and 3 must be indifferent in $\rho(\succcurlyeq)$. On the other hand, the following proposition shows that the adoption of properties IIC and SYM yields a unanimous social ranking over all those power relations $\succcurlyeq \in \mathcal{T}^N$ such that, for some $k \in \{0, \ldots, |N|-2\}$, $D^t_{ji}(\succcurlyeq) = D^t_{ij}(\succcurlyeq)$ for all cardinalities $t \neq k$ and all $i, j \in N$, and $|D^k_{ji}(\succcurlyeq)|$ is not necessarily equal to $|D^k_{ij}(\succcurlyeq)|$ (provided that $D^k_{ij}(\succcurlyeq) \setminus D^k_{ji}(\succcurlyeq) \neq \emptyset$ and $D^k_{ji}(\succcurlyeq) \setminus D^k_{ij}(\succcurlyeq) \neq \emptyset$).

**Proposition 4.** *Let $\rho : \mathcal{T}^{2^N} \longrightarrow \mathcal{T}^N$ be a social ranking satisfying IIC and SYM. Let $\succcurlyeq \in \mathcal{T}^{2^N}$ and $k \in \{0, \ldots, |N|-2\}$ be such that $S \cup \{i\} \succcurlyeq S \cup \{j\}$ and $S \cup \{j\} \succcurlyeq S \cup \{i\}$, for all $S \in 2^{N \setminus \{i,j\}}$ with $|S| \neq k$, $D^k_{ij}(\succcurlyeq) \setminus D^k_{ji}(\succcurlyeq) \neq \emptyset$ and $D^k_{ji}(\succcurlyeq) \setminus D^k_{ij}(\succcurlyeq) \neq \emptyset$ for all $i, j \in N$. Then $i\rho(\succcurlyeq)j$ and $j\rho(\succcurlyeq)i$ for each $i, j \in N$.*

*Proof.* Take $i, j \in N$ such that $|D^k_{ij}(\succcurlyeq)| \geq |D^k_{ji}(\succcurlyeq)|$. Define another power relation $\sqsupseteq \in \mathcal{T}^{2^N}$ such that

$$S \cup \{i\} \succcurlyeq S \cup \{j\} \Leftrightarrow S \cup \{i\} \sqsupseteq S \cup \{j\}$$

for each $S \in 2^{N \setminus \{i,j\}}$ with $|S| = k$, and $S \sqsupseteq T$ and $T \sqsupseteq S$ for all the other coalitions $S, T \in 2^N$ with $|S| = |T| \neq k+1$. We still need to define relation $\sqsupseteq$ on the remaining coalitions of size $k$.

Take $l \in N \setminus \{i, j\}$. Let $\mathcal{D} \subseteq D^k_{ij}(\succcurlyeq)$ be such that $|\mathcal{D}| = |D^k_{ji}(\succcurlyeq)|$. By Remark 4 (see Section 10 Appendix), define the remaining comparisons in $\sqsupseteq$ as follows (an illustrative example of these cases are given in Table 6):

**case 1)** for each $S \in D^k_{ji}(\succcurlyeq)$ with $l \in S$, let

$$S \cup \{i,j\} \setminus \{l\} \sqsubseteq S \cup \{i\} \text{ and } S \cup \{i,j\} \setminus \{l\} \sqsubseteq S \cup \{j\};$$

**case 2)** for each $S \in D^k_{ji}(\succcurlyeq)$ with $l \notin S$, let

$$S \cup \{i\} \sqsubseteq S \cup \{l\} \text{ and } S \cup \{j\} \sqsubseteq S \cup \{l\};$$

**case 3)** For each $S \in \mathcal{D}$ with $l \in S$, let

$$S \cup \{i,j\} \setminus \{l\} \sqsupseteq S \cup \{i\} \text{ and } S \cup \{i,j\} \setminus \{l\} \sqsubseteq S \cup \{j\};$$

**case 4)** for each $S \in \mathcal{D}$ with $l \notin S$, let

$$S \cup \{i\} \sqsubseteq S \cup \{l\} \text{ and } S \cup \{j\} \sqsupseteq S \cup \{l\};$$

**case 5)** for each $S \in D^k_{ij} \setminus \mathcal{D}$ with $l \in S$, let

$$S \cup \{i,j\} \setminus \{l\} \sqsupseteq S \cup \{i\} \text{ and } S \cup \{i,j\} \setminus \{l\} \sqsupseteq S \cup \{j\};$$

**case 6)** for each $S \in D^k_{ij} \setminus \mathcal{D}$ with $l \notin S$, let

$$S \cup \{i\} \sqsupseteq S \cup \{l\} \text{ and } S \cup \{j\} \sqsupseteq S \cup \{l\}.$$

Table 6: An illustrative example of the six possible cases for a power relation $\sqsupseteq$ as the one considered in Proposition 4 with $N = \{1,2,3,i,j,l\}$, $k = 2$ and $\mathcal{D} = \{\{1,2\},\{2,l\}\}$.

| | $i$ vs $j$ | $i$ vs. $l$ | $j$ vs. $l$ |
|---|---|---|---|
| case 4): $S = \{1,2\}$ | $\{1,2,i\} \sqsupseteq \{1,2,j\}$ | $\{1,2,i\} \sqsubseteq \{1,2,l\}$ | $\{1,2,j\} \sqsupseteq \{1,2,l\}$ |
| case 6): $S = \{1,3\}$ | $\{1,3,i\} \sqsupseteq \{1,3,j\}$ | $\{1,3,i\} \sqsupseteq \{1,3,l\}$ | $\{1,3,j\} \sqsupseteq \{1,3,l\}$ |
| case 2): $S = \{2,3\}$ | $\{2,3,i\} \sqsubseteq \{2,3,j\}$ | $\{2,3,i\} \sqsubseteq \{2,3,l\}$ | $\{2,3,j\} \sqsubseteq \{2,3,l\}$ |
| case 5): $S = \{1,l\}$ | $\{1,l,i\} \sqsupseteq \{1,i,j\}$ | $\{1,i,j\} \sqsupseteq \{1,j,l\}$ | $\{1,i,j\} \sqsupseteq \{1,i,l\}$ |
| case 3): $S = \{2,l\}$ | $\{2,l,i\} \sqsupseteq \{2,i,j\}$ | $\{2,i,j\} \sqsubseteq \{2,j,l\}$ | $\{2,i,j\} \sqsupseteq \{2,i,l\}$ |
| case 1): $S = \{3,l\}$ | $\{3,l,i\} \sqsubseteq \{3,i,j\}$ | $\{3,i,j\} \sqsubseteq \{3,j,l\}$ | $\{3,i,j\} \sqsubseteq \{3,i,l\}$ |
| | $\lvert D_{ij}(\sqsupseteq)\rvert = 4$ | $\lvert D_{il}(\sqsupseteq)\rvert = 2$ | $\lvert D_{jl}(\sqsupseteq)\rvert = 4$ |
| | $\lvert D_{ji}(\sqsupseteq)\rvert = 2$ | $D_{li}(\sqsupseteq)\rvert = 4$ | $\lvert D_{lj}(\sqsupseteq)\rvert = 2$ |

Note that $|D^k_{ji}(\succcurlyeq)| = |D^k_{li}(\sqsupseteq)| = |D^k_{jl}(\sqsupseteq)|$ and $|D^k_{ij}(\succcurlyeq)| = |D^k_{il}(\sqsupseteq)| = |D^k_{lj}(\sqsupseteq)|$. Suppose now that $i\rho(\succcurlyeq)j$. By IIC, we have $i\rho(\sqsupseteq)j$. By SYM, $j\rho(\sqsupseteq)l$ and $l\rho(\sqsupseteq)i$.

By transitivity of $\rho(\sqsupseteq)$, $j\rho(\sqsupseteq)i$. By IIC we conclude that $j\rho(\succcurlyeq)i$ too. In a similar way, if we suppose $j\rho(\succcurlyeq)i$, then we end up with the conclusion that $i\rho(\succcurlyeq)j$ too, and the proof follows. $\square$

An interesting consequence of Proposition 4 is that if the only information making a difference between two objects is given by comparisons of a fixed cardinality, then it is sufficient to have one discordance in order to declare an indifference (with IIC and SYM). Proposition 4 suggests how to deal with situations where coalitions are of a fixed size (such situations are not so eccentric in real life). For instance, let us imagine that we have committees with a given number ($k$) of persons and that we have a ranking on them (for instance $N = \{1, 2, 3, 4\}$ and $k = 2$, with $12 \succcurlyeq 13 \succcurlyeq 14 \succcurlyeq 34 \succcurlyeq 24 \succcurlyeq 23$). Since committees are always formed by two persons, no information is available on subsets of $N$ with $l \neq k$ elements (or such information is irrelevant). How to define a social ranking in this case? One solution could be to consider all the other comparisons indifferent. Then, by Proposition 4, we know that SYM and IIC properties can be used in order to support a unanimous social ranking.

**Example 2.** *Consider a power relation* $\succcurlyeq \in \mathcal{T}^{2^N}$ *with* $N = \{1, 2, 3, 4, 5\}$ *and*

$$13 \succ 23 \succ 12 \succ 24 \succ 14 \succ 34 \succ 15 \sim 25 \succ 35 \succ 45,$$

*all the other coalitions of the same size being indifferent (i.e.,* $S \cup \{i\} \succcurlyeq S \cup \{j\}$ *and* $S \cup \{j\} \succcurlyeq S \cup \{i\}$*, for all* $S \in 2^{N \setminus \{i,j\}}$ *with* $|S| \neq 1$ *and* $i, j \in \{1, 2, 3\}$*). We rewrite the relevant informations about* $\succcurlyeq$ *and elements* $1, 2$ *and* $3$ *by means of Table 7.*

*If a social ranking* $\rho$ *satisfies both SYM and DOM, then by Proposition 4, all the elements in* $\{1, 2, 3\}$ *are in relation with each other in* $\rho(\succcurlyeq)$ *(i.e. they are all indifferent).*

## 6 A property driven approach

In the previous section, we have shown that a social ranking cannot satisfy both DOM and SYM axioms on $\mathcal{T}^{2^N}$. Therefore, it seems natural to look at a restricted

Table 7: The relevant informations about $\succcurlyeq$ of Example 2 and the elements $1, 2$ and 3.

| 1 vs. 2 | 2 vs. 3 | 1 vs. 3 |
|---|---|---|
| $1 \sim 2$ | $2 \sim 3$ | $1 \sim 3$ |
| $13 \succ 23$ | $12 \prec 13$ | $12 \prec 23$ |
| $14 \prec 24$ | $24 \succ 34$ | $14 \succ 34$ |
| $15 \sim 25$ | $25 \succ 35$ | $15 \succ 35$ |
| $134 \sim 234$ | $124 \sim 134$ | $124 \sim 234$ |
| $135 \sim 235$ | $125 \sim 135$ | $125 \sim 235$ |
| $1345 \sim 2345$ | $1245 \sim 1345$ | $1245 \sim 2345$ |

domain of power relations where the two properties are compatible (for instance, avoiding power relations like $\succcurlyeq$ in the proof of Proposition 3). To this end, in the remaining of this section we consider a particular class of power relations $\mathcal{R}^{2^N} \subseteq \mathcal{T}^{2^N}$ as provided in the following definitions. For such a restriction, our intuition is the following (and inspired by Proposition 4): it can be interesting to analyse in a separate way the comparisons with different size of coalitions and the "local" dominance for a given size can play a role. We first need some further notations.

**Definition 6.** *Let $\succcurlyeq \in \mathcal{T}^{2^N}$, $i, j \in N$ and $s \in \{0, \dots, n-2\}$. We say that $i$ $s$-dominates $j$ in $\succcurlyeq$, iff*

$$S \cup \{i\} \succcurlyeq S \cup \{j\} \text{ for each } S \in 2^{N \setminus \{i,j\}} \text{ with } |S| = s. \tag{1}$$

Given a total preorder $\succcurlyeq \in \mathcal{T}^{2^N}$, let $\mathbf{P}(\succcurlyeq) \subseteq \{0, \dots, n-2\}$ be such that for each $i, j \in N$ and $s \in \mathbf{P}(\succcurlyeq)$ either $i$ $s$-dominates $j$ or $j$ $s$-dominates $i$. In other words, $\mathbf{P}(\succcurlyeq)$ represents those coalitions' sizes such that a *per size* dominance relation exists. Note that 0 and $n-2$ are in $\mathbf{P}(\succcurlyeq)$ for every $\succcurlyeq \in \mathcal{T}^{2^N}$.

**Definition 7.** *The set of compatible power relations is defined as the set $\mathcal{R}^{2^N} \subseteq \mathcal{T}^{2^N}$ such that for each $\succcurlyeq \in \mathcal{R}^{2^N}$ the following two conditions hold:*

*i) for each $s, t \in \mathbf{P}(\succcurlyeq)$ and $i, j \in N$, if $i$ $s$-dominates $j$ then $i$ $t$-dominates $j$;*

*ii) for each $s \in \{0, \dots, n-2\} \setminus \mathbf{P}(\succcurlyeq)$ and all $i, j \in N$, neither $i$ $s$-dominates $j$ nor $j$ $s$-dominates $i$ (i.e., $D^k_{ij}(\succcurlyeq) \setminus D^k_{ji}(\succcurlyeq) \neq \emptyset$ and $D^k_{ji}(\succcurlyeq) \setminus D^k_{ij}(\succcurlyeq) \neq \emptyset$).*

Roughly speaking, compatible power relations are such that no opposite dominance is allowed for coalitions of different size[2]. Of course a dominance relation between coalitions of the same size does not necessarily occur: it is still possible that for many coalitions $S$ of cardinality $k$ (not containing neither $x$ nor $y$) $S \cup \{y\}$ is stronger than $S \cup \{x\}$ and for others of the same cardinality $S \cup \{x\}$ is stronger than $S \cup \{y\}$. For instance, the power relation considered in Example 2 is a compatible one. Another example of compatible power relation is provided next.

**Example 3.** *Consider a power relation $\succcurlyeq \in \mathcal{R}^{2^N}$ with $N = \{1, 2, 3, 4\}$ such that*

$$\begin{aligned}&1 \succ 2 \succ 3 \sim 4\\&14 \succ 23 \succ 24 \succ 13 \succ 34 \succ 12\\&123 \sim 124 \succ 234 \sim 134.\end{aligned}$$

*Note that $\mathbf{P}(\succcurlyeq) = \{0, 2\}$. We rewrite the relevant informations about $\succcurlyeq$ by means of the Table 8.*

Table 8: The relevant informations about $\succcurlyeq \in \mathcal{R}^{2^N}$ of Example 3.

| 1 vs. 2 | 2 vs. 3 | 1 vs. 3 | 1 vs. 4 | 2 vs. 4 | 3 vs. 4 |
|---|---|---|---|---|---|
| $1 \succ 2$ | $2 \succ 3$ | $1 \succ 3$ | $1 \succ 4$ | $2 \succ 4$ | $3 \sim 4$ |
| $13 \prec 23$ | $12 \prec 13$ | $12 \prec 23$ | $12 \prec 24$ | $12 \prec 14$ | $13 \prec 14$ |
| $14 \succ 24$ | $24 \succ 34$ | $14 \succ 34$ | $13 \succ 34$ | $23 \succ 34$ | $23 \succ 24$ |
| $134 \sim 234$ | $124 \succ 134$ | $124 \succ 234$ | $123 \succ 234$ | $123 \succ 134$ | $123 \sim 124$ |

In order to characterize a social ranking that satisfies DOM, IIC and SYM on $\mathcal{R}^{2^N}$, we also need to introduce a new and last axiom. The next axiom says how to combine social rankings of "compatible" power relations. To be more specific, we say that if the intersection of two total preorders $\succcurlyeq, \sqsupseteq$ in $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ is still a total preorder in $\mathcal{C}^{2^N}$, then the social ranking corresponding to their intersection $\succcurlyeq \cap \sqsupseteq$ must be the intersection of the individual social rankings $\rho(\succcurlyeq) \cap \rho(\sqsupseteq)$.

[2]However one can still have situations where '$i$ dominates (not strictly) $j$' on coalitions of size $t$ and '$j$ strictly dominates $i$' on coalitions of size $l$.

**Definition 8** (SEP)**.** *A social ranking* $\rho : \mathcal{C}^{2^N} \longrightarrow \mathcal{T}^N$ *satisfies the separability (SEP) property on* $\mathcal{C}^{2^N} \subseteq \mathcal{T}^{2^N}$ *iff*

$$\rho(\succcurlyeq \cap \sqsupseteq) \equiv \rho(\succcurlyeq) \cap \rho(\sqsupseteq)$$

*for all power relations* $\succcurlyeq, \sqsupseteq \in \mathcal{C}^{2^N}$ *such that* $\succcurlyeq \cap \sqsupseteq \in \mathcal{C}^{2^N}$ *and* $\rho(\succcurlyeq) \cap \rho(\sqsupseteq) \in \mathcal{T}^N$*.*

We can now state the main theorem of this section.

**Theorem 2.** *A social ranking* $\rho : \mathcal{R}^{2^N} \longrightarrow \mathcal{T}^N$ *that satisfies the properties of DOM, IIC, SYM and SEP on the class of compatible relations* $\mathcal{R}^{2^N}$ *is such that* $i\rho(\succcurlyeq)j$ *and* $\neg(j\rho(\succcurlyeq)i)$*, if there exist* $k \in \mathbf{P}(\succcurlyeq)$ *and* $S \in 2^N$ *with* $|S| = k$ *such that* $S \cup \{i\} \succ S \cup \{j\}$*;* $i\rho(\succcurlyeq)j$ *and* $j\rho(\succcurlyeq)i$*, otherwise.*

*Proof.* Let $\succcurlyeq \in \mathcal{R}^{2^N}$. For all $k \in \{0, \ldots, n-2\}$ and $i, j \in N$ define a power relation $\succcurlyeq^k \in \mathcal{R}^{2^N}$ such that

$$S \cup \{i\} \succcurlyeq^k S \cup \{j\} \Leftrightarrow S \cup \{i\} \succcurlyeq S \cup \{j\}$$

for each $S \in 2^{N \setminus \{i,j\}}$ with $|S| = k$, and

$$S \cup \{i\} \succcurlyeq^k S \cup \{j\} \text{ and } S \cup \{j\} \succcurlyeq^k S \cup \{i\}$$

for each $S \in 2^{N \setminus \{i,j\}}$ with $|S| \neq k$.

Note that, for each $k, t \in \{0, \ldots, n-2\}$, the intersection $\succcurlyeq^k \cap \succcurlyeq^t$ is also a power relation in $\mathcal{R}^{2^N}$, and that $\succcurlyeq^k$ is a power relation of the type considered in Proposition 4.

By Proposition 4 and the fact that $\rho$ satisfies both SYM and IIC, we have that $i\rho(\succcurlyeq^k)j$ and $j\rho(\succcurlyeq^k)i$ for each $k \in \{0, \ldots, n-2\} \setminus \mathbf{P}(\succcurlyeq)$ and all $i, j \in N$.

Moreover, by the fact that $\rho$ also satisfies DOM, for each $k \in \mathbf{P}(\succcurlyeq)$ and all $i, j \in N$, we have that $i\rho(\succcurlyeq^k)j$ and $\neg(j\rho(\succcurlyeq^k)i)$, if there exists $S \in 2^N$ with $|S| = k$ such that $S \cup \{i\} \succ^k S \cup \{j\}$; $i\rho(\succcurlyeq^k)j$ and $j\rho(\succcurlyeq^k)i$, otherwise.

By the multiple application of the SEP property, we have that

$$\rho(\succcurlyeq) \equiv \rho(\succcurlyeq^1) \cap \ldots \cap \rho(\succcurlyeq^n),$$

which concludes the proof. $\square$

Surely Theorem 2 is based on a restriction which can be considered "strong". However it shows that "local" dominances can be important and when they are coherent between them they cancel all the other informations.

**Example 4.** *Consider the compatible power relation* $\succcurlyeq \in \mathcal{T}^{2^N}$ *with* $N = \{1, 2, 3, 4\}$ *of Example 3. By Theorem 2, we have that* $1\rho(\succcurlyeq)2$ *and* $\neg(2\rho(\succcurlyeq)1)$*;* $2\rho(\succcurlyeq)3$ *and* $\neg(3\rho(\succcurlyeq)2)$*;* $1\rho(\succcurlyeq)3$ *and* $\neg(3\rho(\succcurlyeq)1)$*;* $1\rho(\succcurlyeq)4$ *and* $\neg(4\rho(\succcurlyeq)1)$*;* $2\rho(\succcurlyeq)4$ *and* $\neg(4\rho(\succcurlyeq)2)$*. Finally,* $3\rho(\succcurlyeq)4$ *and* $4\rho(\succcurlyeq)3$*.*

# 7 Dictatorship of the coalition size

In Section 5, we have shown that, over a restricted domain of power relations that satisfy a given notion of compatibility between the rankings of coalitions of the same size (see Definition 7), SYM and DOM properties, together with SEP and IIC, determine a well defined social ranking, as shown by Theorem 2. In this section, we focus on power relations that do not necessarily satisfy the notion of compatibility introduced in Definition 7, but still present some regularity when coalitions of the same size are considered.

More precisely, we define a special class of power relations (namely, the *per size-strong dominant* relations) characterized by the fact that a relation of dominance always exists with respect to coalitions of the same size, but the dominance may change with the cardinality (for instance, an element $i$ could dominate another element $j$ when coalitions of size $s$ are considered, but $j$ could dominate $i$ over coalitions of size $t \neq s$). We first need to introduce the notion of $s$-strong dominance.

**Definition 9.** *Let* $\succcurlyeq \in \mathcal{T}^{2^N}$*,* $i, j \in N$ *and* $s \in \{0, \ldots, n-2\}$*. We say that* $i$ *s-strong dominates* $j$ *in* $\succcurlyeq$*, iff*

$$S \cup \{i\} \succ S \cup \{j\} \text{ for each } S \in 2^{N \setminus \{i,j\}} \text{ with } |S| = s. \tag{2}$$

**Definition 10.** *We say that* $\succcurlyeq \in \mathcal{T}^{2^N}$ *is per size-strong dominant (shortly, ps-sdom) iff for each* $s \in \{0, \ldots, n-2\}$ *and all* $i, j \in N$*, we have either*

$$[i \text{ } s\text{-strong dominates } j \text{ in } \succcurlyeq] \text{ or } [j \text{ } s\text{-strong dominates } i \text{ in } \succcurlyeq].$$

*The set of all ps-sdom power relations is denoted by* $\mathcal{S}^{2^N} \subseteq \mathcal{T}^{2^N}$.

We first study the effect of the combination of the properties of DOM and IIC on a specific instance of ps-sdom power relations where there exist elements that are always placed at the top or at the bottom in the rankings of coalitions of equal cardinality.

**Example 5.** *Consider a power relation* $\succcurlyeq \in \mathcal{S}^{2^N}$ *with* $N = \{1, 2, 3, 4\}$ *and such that*

$$\begin{aligned}
&1 \succ 2 \succ 3 \succ 4 \\
&34 \succ 24 \succ 14 \succ 23 \succ 13 \succ 12 \\
&123 \succ 134 \succ 124 \succ 234.
\end{aligned}$$

*We rewrite the relevant informations about* $\succcurlyeq$ *by means of Table 9.*

Table 9: The relevant informations about $\succcurlyeq$ of Example 5.

| 1 vs. 2 | 2 vs. 3 | 1 vs. 3 | 1 vs. 4 | 2 vs. 4 | 3 vs. 4 |
|---|---|---|---|---|---|
| $1 \succ 2$ | $2 \succ 3$ | $1 \succ 3$ | $1 \succ 4$ | $2 \succ 4$ | $3 \succ 4$ |
| $13 \prec 23$ | $12 \prec 13$ | $12 \prec 23$ | $12 \prec 24$ | $12 \prec 14$ | $13 \prec 14$ |
| $14 \prec 24$ | $24 \prec 34$ | $14 \prec 34$ | $13 \prec 34$ | $23 \prec 34$ | $23 \prec 24$ |
| $134 \succ 234$ | $124 \prec 134$ | $124 \succ 234$ | $123 \succ 234$ | $123 \succ 134$ | $123 \succ 124$ |

*Note that for each* $s \in \{0, 2\}$, *it holds that either* $S \cup \{1\} \succcurlyeq S \cup \{l\}$ *for each* $S \subseteq N \setminus \{1\}$ *with* $|S| = s$ *and all* $l \in N \setminus S$ *(i.e., coalitions* $S \cup \{1\}$ *are ranked above all coalitions* $S \cup \{l\}$*, with* $l \neq i$ *and* $S$ *containing* $0$ *or* $2$ *elements), and* $S \cup \{1\} \preccurlyeq S \cup \{l\}$ *for each* $S \subseteq N \setminus \{1\}$ *with* $|S| = 1$ *and all* $l \in N \setminus S$ *(i.e., coalitions* $S \cup \{1\}$ *are ranked below all coalitions* $S \cup \{l\}$*, with* $l \neq i$ *and* $S$ *containing precisely* $1$ *element). Similar considerations can be done for element* $4$*. So, elements* $1$ *and* $4$ *are two "extreme" ones. Let us remark that there can be at most two "extreme" elements of a power relation in* $\mathcal{S}^{2^N}$*. In Proposition 5 we argue that on this kind of power relations, a social ranking satisfying both DOM and IIC cannot rank "extreme" elements (in this case* $1$ *and* $4$*) in between two others.*

The following proposition shows the effect of DOM and IIC on the social position of the "extreme" elements.

**Proposition 5.** *Let $\rho : \mathcal{S}^{2^N} \longrightarrow \mathcal{T}^N$ be a social ranking satisfying IIC and DOM on $\mathcal{S}^{2^N}$. Let $\succcurlyeq \in \mathcal{S}^{2^N}$ and $i \in N$ be such that for each $s \in \{0, \ldots, n-2\}$ either*

$$[S \cup \{i\} \succ S \cup \{j\} \textit{ for all } j \in N \setminus \{i\} \textit{ and } S \in 2^{N \setminus \{i,j\}} \textit{ with } |S| = s] \tag{3}$$

*or*

$$[S \cup \{j\} \succ S \cup \{i\} \textit{ for all } j \in N \setminus \{i\} \textit{ and } S \in 2^{N \setminus \{i,j\}} \textit{ with } |S| = s]. \tag{4}$$

*Then, $[i\rho(\succcurlyeq)j$ for all $j \in N]$ OR $[j\rho(\succcurlyeq)i$ for all $j \in N]$.*

*Proof.* Suppose on the contrary that there exist $j, k \in N \setminus \{i\}$, such that

$$j\rho(\succcurlyeq)i \text{ and } i\rho(\succcurlyeq)k. \tag{5}$$

Define $\sqsupseteq \in \mathcal{T}^{2^N}$ such that

$$S \cup \{i\} \sqsupset S \cup \{j\} \Leftrightarrow S \cup \{i\} \succ S \cup \{j\} \text{ for all } S \subseteq N \setminus \{i, j\}, \tag{6}$$

$$S \cup \{i\} \sqsupset S \cup \{k\} \Leftrightarrow S \cup \{i\} \succ S \cup \{k\} \text{ for all } S \subseteq N \setminus \{i, k\}, \tag{7}$$

and

$$S \cup \{k\} \sqsupset S \cup \{j\} \text{ for all } S \subseteq N \setminus \{j, k\}. \tag{8}$$

[note that each coalition $S \cup \{i\}$, with $S \subseteq N \setminus \{i\}$, by condition (3) and (4), is ranked strictly higher or lower than each other coalition $S \cup \{j\}$, $j \neq i$, so the rearrangement of coalitions in $\succcurlyeq$ to obtain $\sqsupseteq$ is feasible.]

By IIC, we have that

$$i\rho(\succcurlyeq)j \Leftrightarrow i\rho(\sqsupseteq)j \text{ and } i\rho(\succcurlyeq)k \Leftrightarrow i\rho(\sqsupseteq)k.$$

So, by relation (5), $j\rho(\sqsupseteq)i$ and $i\rho(\sqsupseteq)k$. On the other hand, by DOM we have $k\rho(\sqsupseteq)j$ and $\neg(j\rho(\sqsupseteq)k)$, which yields a contradiction with the transitivity of $\rho(\sqsupseteq)$. $\square$

Proposition 5 shows that if there is an element $i \in N$ having "contradictory" and "radical" behavior depending on the size of coalitions (very well for size $k$ and very bad for size $l$), then the social ranking satisfying IIC and DOM can not give him an intermediate position: the element $i$ will be the "best" one or the "worst" one in the social ranking.

In the following we argue that if a power relation is in $\mathcal{S}^{2^N}$ and a social ranking satisfies both DOM and IIC on the set of ps-sdom power realtions $\mathcal{S}^{2^N}$, then it must exist a cardinality $t^* \in \{0, \dots, n-2\}$ whose relation of $t^*$-strong dominance (dictatorially) determines the social ranking. We first need to introduce the next lemma, where a given element $i$ plays an important role.

**Lemma 1.** *Let $i \in N$ and $\rho : \mathcal{S}^{2^N} \longrightarrow \mathcal{T}^N$ be a social ranking satisfying IIC and DOM on $\mathcal{S}^{2^N}$. There exists $t^* \in \{0, \dots, n-2\}$ such that*

$$j \rho(\succcurlyeq) k \Leftrightarrow j \text{ } t^*\text{-strong dominates } k \text{ in } \succcurlyeq,$$

*for all $j, k \in N \setminus \{i\}$ and $\succcurlyeq \in \mathcal{S}^{2^N}$.*

*Proof.* Given a power relation $\succcurlyeq \in \mathcal{S}^{2^N}$, define another power relation $\succcurlyeq_0 \in \mathcal{S}^{2^N}$ such that for each $S \subseteq N \setminus \{i\}$ we have

$$S \cup \{l\} \succ_0 S \cup \{i\} \text{ for all } l \in N \setminus (S \cup \{i\}), \tag{9}$$

and

$$U \succcurlyeq_0 W :\Leftrightarrow U \succcurlyeq W$$

for all the other possible pairs of coalitions $U, W$ whose comparison is not already considered in (9). Roughly speaking, the only difference between $\succcurlyeq_0$ and $\succcurlyeq$ is that coalitions of size $s$ containing $i$ are placed at the bottom of the ranking induced by $\succcurlyeq$ over the coalitions of the same size. By DOM, it follows that $l \rho(\succcurlyeq_0) i$ for every $l \in N$.

Now, for each $t \in \{0, \dots, n-2\}$, define a power relation $\succcurlyeq_t \in \mathcal{T}^{2^N}$ such that

$$S \cup \{i\} \succ_t S \cup \{l\} \text{ for each } l \in N \text{ and } S \in 2^{N \setminus \{i,l\}} \text{ with } |S| = s, \tag{10}$$

where $s \in \{0, \dots, t\}$, and

$$U \succcurlyeq_t W :\Leftrightarrow U \succcurlyeq_{t-1} W$$

for all the other possible pairs of coalitions $U, W$ whose comparison is not already considered in (10). So, the only difference between $\succcurlyeq_t$ and $\succcurlyeq_{t-1}$, for each $t \in \{1, \dots, n-2\}$, is that in $\succcurlyeq_t$ coalitions of size $t$ containing $i$ are placed at the top of the ranking induced by $\succcurlyeq_{t-1}$ over coalitions of the same size $t$, and all the remaining comparisons remain the same as in $\succcurlyeq_{t-1}$.

Note that by Proposition 5, we have that either $l\rho(\succcurlyeq_t)i$ for every $l \in N$, or $i\rho(\succcurlyeq_t)l$ for every $l \in N$. Moreover, By DOM, it follows that $i\rho(\succcurlyeq_{n-2})l$ for every $j \in N$.

Let $t^*$ be the smallest number in $\{0, \ldots, n-2\}$ such that $l\rho(\succcurlyeq_{t^*-1})i$ for every $l \in N$ and $i\rho(\succcurlyeq_{t^*})l$ for every $l \in N$ (for the considerations above such a $t^*$ must exist, being, at most, $t^* = n-2$).

Next, we argue that for every $j, k \in N \setminus \{i\}$, the social ranking between $j$ and $k$ in $\succcurlyeq$ is imposed by the relation of $t^*$-strong dominance in $\succcurlyeq$.

W.l.o.g., suppose that $S \cup \{j\} \succcurlyeq S \cup \{k\}$ (and, as a consequence, $S \cup \{j\} \succcurlyeq_{t^*} S \cup \{k\}$) for each $S \in 2^{N\setminus\{j,k\}}$, and $|S| = t^*$. Consider another power relation $\sqsupseteq \in \mathcal{T}^{2^N}$ obtained by $\succcurlyeq_{t^*}$ and such that:

$$S \cup \{j\} \sqsupset S \cup \{i\} \text{ for each } S \in 2^{N\setminus\{i,j\}} \text{ with } |S| = t^*, \tag{11}$$

$$S \cup \{i\} \sqsupset S \cup \{k\} \text{ for each } S \in 2^{N\setminus\{i,k\}} \text{ with } |S| = t^*, \tag{12}$$

$$S \cup \{j\} \sqsupset S \cup \{k\} \text{ for each } S \in 2^{N\setminus\{j,k\}} \setminus \left(2^{N\setminus\{i,j\}} \cup 2^{N\setminus\{i,k\}}\right), \text{ and } |S| = t^*, \tag{13}$$

and, finally,

$$U \sqsupseteq V :\Leftrightarrow U \succcurlyeq_{t^*} V \tag{14}$$

for all the other relevant pairs of coalitions $U, W$ of size $s \neq t^* + 1$. By IIC $j\rho(\sqsupseteq)i$ (since in $\sqsupseteq$ the comparisons between coalitions containing $i$ and $j$ are precisely as in $\succcurlyeq_{t^*-1}$ and, as previously stated, $j\rho(\succcurlyeq_{t^*-1})i$) and $i\rho(\sqsupseteq)k$ (since in $\sqsupseteq$ the comparisons between coalitions containing $i$ and $k$ are precisely as in $\succcurlyeq_{t^*}$ and, as previously stated, $i\rho(\succcurlyeq_{t^*})k$). Then, by transitivity of $\rho(\sqsupseteq)$ we have $j\rho(\sqsupseteq)k$. Note that by IIC, $j\rho(\sqsupseteq)k \Leftrightarrow j\rho(\succcurlyeq_{t^*})k \Leftrightarrow j\rho(\succcurlyeq)k$. We have then proved that whenever $j$ $t^*$-dominates $k$, then $j\rho(\succcurlyeq)k$.

□

We can now formulate the following theorem stating the “dictatorship of the coalition’s size”.

**Theorem 3.** *Let $\rho : \mathcal{S}^{2^N} \longrightarrow \mathcal{T}^N$ be a social ranking satisfying IIC and DOM on $\mathcal{S}^{2^N}$. There exists $t^* \in \{0, \ldots, n-2\}$ such that*

$$i\rho(\succcurlyeq)j \Leftrightarrow i \ t^*\text{-strong dominates } j \text{ in } \succcurlyeq,$$

*for all $i, j \in N$ and $\succcurlyeq \in \mathcal{S}^{2^N}$.*

*Proof.* Given a power relation $\succcurlyeq \in \mathcal{S}^{2^N}$, let $i \in N$ and define $\succcurlyeq_{t^*}$ starting from $\succcurlyeq$ and $i$ precisely as in the proof of Lemma 1.

Now take $k \in N \setminus \{i\}$ and apply Lemma 1 with $k$ in the role of $i$. Consequently, we have that there exists $\hat{t} \in \{0, \ldots, n-2\}$ such that

$$h\rho(\succcurlyeq)l \Leftrightarrow h \ \hat{t}\text{-strong dominates } l \text{ in } \succcurlyeq,$$

for each $h, l \in N \setminus \{k\}$, and in particular

$$i\rho(\succcurlyeq)l \Leftrightarrow i \ \hat{t}\text{-strong dominates } l \text{ in } \succcurlyeq,$$

for whatever complete power relation $\succcurlyeq \in \mathcal{S}^{2^N}$.

But in the proof of Lemma 1 we have shown that

$$i\rho(\succcurlyeq)l \Leftrightarrow i \ t^*\text{-strong dominates } l \text{ in } \succcurlyeq_{t^*}$$

(remember that $t^*$ in the proof of Lemma 1 is the smallest number in $\{0, \ldots, n-2\}$ such that $l\rho(\succcurlyeq_{t^*-1})i$ for every $l \in N$ and $i\rho(\succcurlyeq_{t^*})l$ for every $l \in N$). Then it must be $\hat{t} = t^*$, and the proof follows. □

**Example 6.** *Take again the power relation $\succcurlyeq \in \mathcal{S}^{2^N}$ with $N = \{1, 2, 3, 4\}$ of Example 5. Theorem 3 says that if a social ranking satisfies DOM and IIC on $\mathcal{S}^{2^N}$, then it must yield on $\succcurlyeq$ one of the following three possible linear orders: $1\rho(\succcurlyeq)2\rho(\succcurlyeq)3\rho(\succcurlyeq)4$ (corresponding to the relation of 0-strong dominance); $4\rho(\succcurlyeq)3\rho(\succcurlyeq)2\rho(\succcurlyeq)1$ (corresponding to the relation of 1-strong dominance); $4\rho(\succcurlyeq)1\rho(\succcurlyeq)3\rho(\succcurlyeq)2$ (corresponding to the relation of 2-strong dominance).*

*For instance, suppose that the social ranking is $4\rho(\succcurlyeq)2\rho(\succcurlyeq)3\rho(\succcurlyeq)1$. Define a new power relation $\sqsupseteq \in \mathcal{S}^{2^N}$ such that (again, the main changes with respect to $\succcurlyeq$ are shown in bold):*

$1 \sqsupset 2 \sqsupset 3 \sqsupset 4$

$34 \sqsupset \mathbf{23} \sqsupset \mathbf{24} \sqsupset \mathbf{13} \sqsupset \mathbf{14} \sqsupset 12$

$\mathbf{134} \sqsupset \mathbf{123} \sqsupset \mathbf{234} \sqsupset \mathbf{124}$

*We rewrite the relevant informations about $\sqsupseteq$ by means of Table 10. By DOM we*

Table 10: The relevant informations about $\sqsupseteq$ of Example 6.

| 1 vs. 2 | 2 vs. 3 | 1 vs. 3 | 1 vs. 4 | 2 vs. 4 | 3 vs. 4 |
|---|---|---|---|---|---|
| $1 \sqsupset 2$ | $2 \sqsupset 3$ | $1 \sqsupset 3$ | $1 \sqsupset 4$ | $2 \sqsupset 4$ | $3 \sqsupset 4$ |
| $13 \sqsubset 23$ | $12 \sqsubset 13$ | $12 \sqsubset 23$ | $12 \sqsubset 24$ | $12 \sqsubset 14$ | $\mathbf{13 \sqsupset 14}$ |
| $14 \sqsubset 24$ | $24 \sqsubset 34$ | $14 \sqsubset 34$ | $13 \sqsubset 34$ | $23 \sqsubset 34$ | $\mathbf{23 \sqsupset 24}$ |
| $134 \sqsupset 234$ | $124 \sqsubset 134$ | $\mathbf{124 \sqsubset 234}$ | $123 \sqsupset 234$ | $\mathbf{123 \sqsubset 134}$ | $123 \sqsupset 124$ |

*have that $3\rho(\sqsupseteq)4$ and $\neg(4\rho(\sqsupseteq)3)$. By IIC we have $4\rho(\sqsupseteq)2$ and $2\rho(\sqsupseteq)3$ (the columns '2 vs. 4' and '2 vs. 3' are the same in the Tables 9 and 10, respectively), which yields a contradiction with the transitivity of $\rho(\sqsupseteq)$.*

# 8 Conclusions

In this paper we introduced and studied the problem of how to rank the objects of a set $N$ according to their ability to influence the ranking over the subsets of $N$. As we discussed in Section 2, such a problem can be seen as an ordinal counterpart of the one about how to measure the power of players in a coalitional game [1, 6, 9, 18], or as the inverse problem of extending preferences to subsets of objects [2, 3, 4, 5, 8, 7, 10, 11]. As far as we know, this is the first time that a solution is proposed using an axiomatic approach (and without the quantitative notion of power index from cooperative game theory).

The aim of this paper was to analyse some "intuitive" properties for social rankings. We first notice that two natural properties, precisely, dominance and symmetry, are not compatible over the class of all power relations, despite the fact that, in some related axiomatic frameworks (see, for instance, [3]), similar axioms have been successfully used in combination. Then, we provide an axiomatic characterization of social rankings satisfying symmetry, dominance, iic and separability on a specific domain of compatible power relations (those whose intersection is still a power relation). Finally, we proved that the property of independence of irrelevant coalitions and dominance property determine a kind of 'dictatorship of the cardinality' when a relation of strong dominance among coalitions of the same size holds: in this case, the only social ranking satisfying those two properties is the one imposed by the relation of dominance of a given cardinality $s \in \{1, \dots, |N|\}$.

A possible direction for future research is the open question about which axioms could be used to characterize a social ranking over the domain of all possible power relations. In view of our results, some of the axioms we propose in this paper should be abandoned. In this respect, it is worth noting that all the properties that we analysed are based on the comparison of subsets having the same number of elements. Therefore, it would be interesting to study properties based on the comparison among subsets with different cardinalities. For instance, if $N = \{1, 2, 3\}$, the information of the type $\{1\} \succ \{2, 3\} \succ \{1, 3\} \succ \{2\}$ could be used to establish that 1 is socially stronger than 2 (note that 1 strictly dominates 2 on coalitions of size 1, and 2 strictly dominates 1 of coalitions of cardinality 2, but the "interval" between $\{2, 3\}$ and $\{1, 3\}$ is smaller than the one between $\{1\}$ and $\{2\}$).

A related question is the evaluation of the interaction among the elements of $N$. As we already noticed, we deal with power relations that do not necessarily satisfy the responsiveness property [4] or the monotonicity one [3], so objects may strongly interact (e.g., with respect to monotonicity, two objects $x$ and $y$ together could be less strong than $x$ and $y$ alone). Consequently, an interesting question to address is how to compare the interaction among pairs of objects taking into account their effects over all possible subsets (for instance, to establish whether the level of interaction between two objects $x$ and $y$ is stronger than the one between two other objects $w$ and $z$).

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# 10 Appendix

**Remark 4.** *Note that by transitivity of power relations $\succcurlyeq \in \mathcal{T}^{2^N}$, the relations between the elements of the columns of a comparison table must satisfy some constraints, as listed below.*

- *Let $i, j, k \in N$ and $S \in 2^{N \setminus \{i,j,k\}}$ with $S \cup \{i\} \succcurlyeq S \cup \{j\}$. Then, one of the following possibilities may occur:*
  - $S \cup \{i\} \succcurlyeq S \cup \{k\}$ *and* $S \cup \{j\} \succcurlyeq S \cup \{k\}$;
  - $S \cup \{i\} \preccurlyeq S \cup \{k\}$ *and* $S \cup \{j\} \preccurlyeq S \cup \{k\}$;
  - $S \cup \{i\} \succcurlyeq S \cup \{k\}$ *and* $S \cup \{j\} \preccurlyeq S \cup \{k\}$.
- *Let $i, j, k \in N$ and $S \in 2^{N \setminus \{i,j,k\}}$ with $S \cup \{i,k\} \succcurlyeq S \cup \{j,k\}$.*
  - $S \cup \{i,j\} \succcurlyeq S \cup \{i,k\}$ *and* $S \cup \{i,j\} \succcurlyeq S \cup \{j,k\}$;
  - $S \cup \{i,j\} \preccurlyeq S \cup \{i,k\}$ *and* $S \cup \{i,j\} \preccurlyeq S \cup \{j,k\}$;
  - $S \cup \{i,j\} \succcurlyeq S \cup \{i,k\}$ *and* $S \cup \{i,j\} \preccurlyeq S \cup \{j,k\}$.

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