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Abstract

This paper presents a new method for multiple criteria ordinal classification (sorting) problems. This type of problem requires the different classes or categories to be pre-defined and ordered, from the best to the worst or from the worst to the best. The actions (not necessarily known *a priori*) are assigned to the different ordered classes. Several ELECTRE type methods were designed to deal with this kind of problem. However, none proposes characterization of the categories through a set of limiting profiles. This is the novelty of the current method, which may be considered as an extension of ELECTRE TRI-B. It fulfills a set of structural requirements: Uniqueness of the assignments, independence, monotonicity, homogeneity, conformity, and stability with respect to merging and splitting operations. All these features will be presented in the current paper as well as three illustrative examples.

Key-words: Multiple criteria decision aiding, Decision support, Sorting, ELECTRE TRI-nB.

1. Introduction

In this paper we are interested in decision aiding contexts in which the objects of a decision (actions, alternatives, options,…) must be assigned to ordered categories, defined *a priori*. Assignment of actions to categories is based on the evaluation of each action according to several criteria. The objective of a multiple criteria sorting method is to help decision-makers to assign each action to a single category or a range of categories. Each category is defined to receive actions, which will or might be processed in the same way (at least in a first step).

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The term “decision-maker” (DM) represents those in whose name or for whom the decision aiding must be provided. In what follows, the term “analyst” denotes a facilitator of the decision aiding process, who must perform her/his role in interaction with the DM.

Many multiple criteria decision aiding methods have been developed for multiple criteria sorting problems. Most of them can be classified as follows (see Doumpos *et al*., 2009):

- Those based on the development of value functions (*e.g*., Jacquet-Lagrèze, 1995; Köksalan and Ulu, 2003; Köksalan and Özpeynirci, 2009; Greco *et al*., 2010);
- Those based on decision rules also called symbolic models (*e.g*., Greco *et al*., 2001; Błaszczyński *et al*., 2007; Fortemps *et al*., 2008; Dembczyński *et al*., 2009); and
- Those based on the construction of outranking relations (*e.g*., Massaglia and Ostanello, 1991; Yu, 1992; Perny, 1998; Belacel, 2000; Fernandez *et al*., 2008; Tervonen *et al*., 2009; Fernandez and Navarro, 2011; Bouyssou and Marchant, 2015).

In the framework of outranking approaches, the most widely used method is ELECTRE TRI, recently renamed as ELECTRE TRI-B (see Almeida-Dias *et al*., 2010). This method was introduced in the Ph.D. thesis of W. Yu (1992) (supervised by B. Roy) and was detailed in Roy and Bouyssou (1993). ELECTRE TRI-B has generated much interest. Some of its applications can be found in several papers (see, for example, Dimitras *et al*., 1995; Raju *et al*., 2000; Arondel and Girardin, 2000; Joerin *et al*., 2001; Zopounidis and Doumpos, 2002; Mavrotas *et al*., 2003; Georgopoulou *et al*., 2003; Merad *et al*., 2004; Siskos *et al*., 2007; Xidonas *et al*., 2009; Brito *et al*., 2010; Norese and Carbone, 2014; Dias de Oliveira and Gomes, 2015; Govindan and Jepsen, 2016).

In ELECTRE TRI-B, categories are defined by the introduction of a single limiting profile (reference or boundary action) between adjacent categories. Such limiting profiles can be defined in a co-construction interactive process between the analyst and the DM. ELECTRE TRI-B makes use of an outranking relation to compare the actions to limiting profiles. From this comparison the method proposes a category for each action.

Almeida-Dias *et al*. (2010) proposed the ELECTRE TRI-C method, in which each category is defined through a single characteristic (typical or representative) reference action. As in ELECTRE TRI-B, the set of characteristic actions should be co-constructed through an interactive process between the analyst and the DM. The relations between these methods have been studied by Bouyssou and Marchant (2015), who have also proposed a new variant of ELECTRE TRI-B.

As stated by Fernandez and Navarro (2011), it is questionable whether a single limiting profile or reference action is sufficient for an acceptable characterization of its related category. If the actions to be sorted were incomparable with most actions in the set of reference actions, an outranking-based sorting method would suggest ill-determined assignments. According to Figueira *et al*. (2013), if each category is defined by a set of several reference actions, it enriches the definition of categories and allows for narrower ranges of categories to which an action can be assigned. With this purpose in mind, ELECTRE TRI-nC was proposed as a generalization of ELECTRE TRI-C, using several reference actions to characterize each category (see Almeida-Dias *et al*., 2012).

The purpose of this paper is to present a generalization of ELECTRE TRI-B, called ELECTRE TRI-nB, similar to the generalization of ELECTRE TRI-C which gave rise to ELECTRE TRI-nC. In ELECTRE TRI-nB, characterization of the adjacent categories is done by a set of limiting profiles. Compared to ELECTRE TRI-B, the assignment suggestions in ELECTRE TRI-nB are made taking into account richer preference relations among actions and limiting profiles, and this enrichment can lead to more appropriate assignments. This new method is not a competitor of other methods for sorting problems. It was designed to model different situations in different contexts. The methods are more or less context-dependent. For the choice of multiple criteria methods, see, for example, the paper by Roy and Słowiński (2013).

ELECTRE TRI-nB is fundamentally different from ELECTRE TRI-nC. Their differences are easy to explain from their particular cases, *i.e.*, ELECTRE TRI-B and ELECTRE TRI-C. Almeida-Dias *et al*. (2010) show that ELECTRE TRI-C is not theoretically equivalent to ELECTRE TRI-B because the assignments obtained by these two methods are different.

It should be noted that ELECTRE TRI-B has been criticized because selection of the limiting profile placed on the exact lower boundary of each category is a hard task. Certainly, an ELECTRE TRI-B assignment is very sensitive to the choice of profiles. In real-world problems the DM may be able to identify several actions that fulfil the requirements to be limiting profiles. ELECTRE TRI-B requires the DM to select the single “best” profile, because other profiles can lead to different assignments. That choice could be a very difficult task. With ELECTRE TRI-nB this difficulty disappears. Thus, ELECTRE TRI-nB helps to “solve” the problem of the multiplicity of limiting profiles. Now, that multiplicity of possible profiles ceases to be an inconvenience and becomes a benefit. The way in which we have proposed the extension has an additional advantage: Many of the theoretical studies related to ELECTRE TRI-B can now be extended to the ELECTRE TRI-nB method. This extension is not trivial. It is necessary to define the relations among actions and boundaries, and the characteristics of the boundaries for keeping the properties of ELECTRE TRI-B at a more general level. These properties come from rational arguments and should be retained. There are alternative ways to define boundaries and relations among them and actions, but they fail to fulfil the rational requirements as an extension of ELECTRE TRI-B. In our proposal ELECTRE TRI-B can be seen from a more general perspective. The amount of information the DM has (or can obtain) about the boundaries is the decisive factor. More information should be better than less, but the method must be able to adequately organize and process the additional information. ELECTRE TRI-nB does this. ELECTRE TRI-B is its particular case when the DM has minimal information about the sets of limiting profiles.

This paper is organized as follows. Section 2 is devoted to presentation of the new method, ELECTRE TRI-nB. Section 3 presents the main theorem proving some main properties of this method. Section 4 provides three examples illustrating the new method and its capacity to identify more appropriate assignments. Finally, Section 5 is devoted to outlining the main conclusions of the paper and future lines of research.

2. The ELECTRE TRI-nB method

In order to describe ELECTRE TRI-nB four points should be addressed: 1) some notation and fundamental definitions; 2) characterization of the limiting profiles between adjacent categories; 3) some preference relations between actions and limiting boundaries; and 4) the assignment procedures (including the relationship between them).

2.1. Some notation and fundamental definitions

Let U denote the universe of actions (objects) x characterized by a coherent family of N real valued criteria, denoted $G= \{g_1,...,g_j,...,g_N\}$, with $N \geq 3$ (ELECTRE TRI-nB like many other methods does not have a real interest for $N=2$). Without loss of generality, we assume that increasing the performance on any criterion increases preference. Consider also a finite set of ordered categories $C= \{C_1,\ldots,C_k,...,C_M\}$, ($M \geq 2$); C_M contains the best (most preferred) actions. The term "preferred" is related to each particular sorting problem (for instance, "higher quality", "more consensual", "less risky", and so on). A fuzzy outranking relation $\sigma(x,y)$ is built on $U \times U$. Its value models the degree of credibility of the statement "x is at least as good as y" from the DM's perspective in each particular sorting problem (for instance, "x

is at least as consensual as y", or "x is at most as risky as y"). The formula for $\sigma(x,y)$ can be modeled as in the ELECTRE III or ELECTRE TRI method (see Roy, 1991).

In what follows, P is used to represent the preference relation in a broader sense, including weak and strict preferences as they are defined in Roy (1996).

Let us notice that in all ELECTRE methods, given the credibility index, $\sigma(x, y)$, the following crisp binary relations are defined:

i. $xS(\lambda)y$ iff $\sigma(x, y) \geq \lambda$ (λ-outranking);

ii. $xP(\lambda)y$ iff $\sigma(x, y) \geq \lambda$ and $\sigma(y, x) < \lambda$ (λ- preference);

iii. $xI(\lambda)y$ iff $\sigma(x, y) \geq \lambda$ and $\sigma(y, x) \geq \lambda$ (λ-indifference);

iv. $xR(\lambda)y$ iff $\sigma(x, y) < \lambda$ and $\sigma(y, x) < \lambda$(λ-incomparability).

(If $xS(\lambda)y$ then either $xP(\lambda)y$ or $xI(\lambda)y$.)

For all ordered pairs $(x,y) \in U\times U$, we say that x *dominates* y iff $g_i(x) \geq g_i(y)$ for i=1,…,N and $g_j(x) > g_j(y)$ for at least one $g_j \in G$. Let D denote this dominance relation.

The following properties of the above relations hold:

i. $xDy \Rightarrow xS(\lambda)y$; (2.1)

ii. $xS(\lambda)y$ and $yDz \Rightarrow xS(\lambda)z$; (2.2)

iii. $xP(\lambda)y$ and $yDz \Rightarrow xP(\lambda)z$; (2.3)

iv. xDy and $yS(\lambda)z \Rightarrow xS(\lambda)z$; (2.4)

v. xDy and $yP(\lambda)z \Rightarrow xP(\lambda)z$. (2.5)

2.2. Characterization of the limiting profiles

This subsection introduces the definition of the sets of limiting profiles as well as the assumptions on these subsets.

Definition 1 (The set of limiting profiles)

The set of limiting profiles of all the categories is $B=\{B_0, B_1,\ldots,B_{M-1}, B_M\}$, where B_0 (resp. B_M) is a single lower (resp. upper) limiting profile of C_1 (resp. C_M) chosen as in ELECTRE TRI-B.

The following assumptions generalize the ELECTRE TRI-B method.

Condition 1 (Basic assumptions on the set *B*)

The boundaries between C_k and C_{k+1} are characterized by a set of limiting profiles, $B_k = \{b_{k,j}\}$, such that, for k=1,…,M-1:

i. Category C_k is characterized by a set of upper limiting profiles, B_k, and by a set of lower limiting profiles, B_{k-1}. By hypothesis the elements $b_{k,j}$ of B_k belong to C_{k+1} (this hypothesis states that categories are closed from below);

ii. For all ordered pairs ($b_{k,j}$, $b_{k,i}$) such that $b_{k,j}$, $b_{k,i} \in B_k$ there is no λ-preference between $b_{k,j}$ and $b_{k,i}$ (this implies that we have either $b_{k,j} I(\lambda) b_{k,i}$ or $b_{k,j} R(\lambda) b_{k,i}$);

iii. For all ordered pairs ($b_{h,j}$, $b_{k,i}$) such that $b_{k,j} \in B_k$ and $b_{h,i} \in B_h$ ($k > h$) we cannot have $b_{h,i} P(\lambda) b_{k,j}$.

Condition 2 (Separability conditions on the set *B*)

i. Dominance-based separability condition:
   - Primal: For any limiting profile $z \in B_h$, with $h < k$, there is at least a limiting profile $w \in B_k$ such that wDz;
   - Dual: For any limiting profile $z \in B_h$, with $h > k$, there is at least a limiting profile $w \in B_k$ such that zDw.

ii. Preference-based separability condition:
   - Primal: For any limiting profile $z \in B_h$, with $h < k$, there is at least a limiting profile $w \in B_k$ such that $wP(\lambda)z$;
   - Dual: For any limiting profile $z \in B_h$, with $h > k$, there is at least a limiting profile $w \in B_k$ such that $zP(\lambda)w$.

iii. Hyper-separability condition:
   - The dominance-based separability condition holds;
   - The preference-based separability condition holds.

2.3. Relations between actions and limiting profiles

This subsection is mainly devoted to introducing the relations between an action and the set of limiting profiles.

Definition 2: (λ-relations between an action and a set of limiting profiles)

i. $xS(\lambda)B_k$ iff, for all $b_{k,j}\in B_k$, we have either $xR(\lambda)b_{k,j}$ or $xS(\lambda)b_{k,j}$; the latter relation being fulfilled by at least one $b_{k,j}\in B_k$ (for all $b_{k,j}\in B_k$ we cannot have $b_{k,j}P(\lambda)x$);

ii. $xP(\lambda)B_k$ iff, for all $b_{k,j}\in B_k$, we have either $xR(\lambda)b_{k,j}$ or $xI(\lambda)b_{k,j}$ or, $xP(\lambda)b_{k,j}$; the latter relation being fulfilled by at least one $b_{k,j}\in B_k$ (for all $b_{k,j}\in B_k$ we cannot have $b_{k,j}P(\lambda)x$);

iii. $B_kS(\lambda)x$ iff, for all $b_{k,j}\in B_k$, we have either $b_{k,j}R(\lambda)x$ or $b_{k,j}S(\lambda)x$; the latter relation being fulfilled by at least one $b_{k,j}\in B_k$ (for all $b_{k,j}\in B_k$ we cannot have $xP(\lambda)b_{k,j}$);

iv. $B_kP(\lambda)x$ iff, for all $b_{k,j}\in B_k$, we have either $b_{k,j}R(\lambda)x$ or $b_{k,j}I(\lambda)x$ or $b_{k,j}P(\lambda)x$; the latter relation being fulfilled by at least one $b_{k,j}\in B_k$ (for all $b_{k,j}\in B_k$ we cannot have $xP(\lambda)b_{k,j}$);

v. $xI(\lambda)B_k$ iff, for all $b_{k,j}\in B_k$, we have either $xR(\lambda)b_{k,j}$ or $xI(\lambda)b_{k,j}$; the latter relation being fulfilled by at least one $b_{k,j}\in B_k$ (since it is a symmetric relation, we cannot have $b_{k,j}P(\lambda)x$ or $xP(\lambda)b_{k,j}$);

vi. $xR(\lambda)B_k$ iff, for all $b_{k,j}\in B_k$, we have $xR(\lambda)b_{k,j}$ or when $xP(\lambda)b_{k,j}$ for some j and $b_{k,i}P(\lambda)x$ for some i different from j (some examples showed this case can exist) (since it is a symmetric relation, we cannot have $xS(\lambda)B_k$ or $B_kS(\lambda)x$).

Remark 1

When $card(B_k)$ =1 ($B_k=\{b_{k,1}\}$),$k=1,\ldots,M$-1, the above definitions are equivalent to the outranking, preference, indifference, and incomparability relations between x and $b_{k,1}$ as in ELECTRE TRI-B.

Remark 2

i. $xP(\lambda)B_k \Rightarrow xS(\lambda)B_k$;

ii. $xP(\lambda)B_k \Rightarrow not(B_kS(\lambda)x)$, and consequently $xP(\lambda)B_k \Rightarrow not(B_kP(\lambda)x)$;

iii. $B_kP(\lambda)x \Rightarrow not(xS(\lambda)B_k)$, and consequently $B_kP(\lambda)x \Rightarrow not(xP(\lambda)B_k)$;

iv. $B_kP(\lambda)x \Rightarrow B_kS(\lambda)x$;

v. $B_kP(\lambda)x$ and $xDy \Rightarrow B_kP(\lambda)y$;

vi. xDy and $yS(\lambda)B_k \Rightarrow xS(\lambda)B_k$;

vii. xDy and $yP(\lambda)B_k \Rightarrow xP(\lambda)B_k$.

Proposition 1 (Comparisons of the actions against the sets of limiting profiles)

If the set B fulfills:

i. The primal dominance-based separability condition, then for all $x \in U$, $xS(\lambda)B_k \Rightarrow not(B_hP(\lambda)x)$, for $h < k$;

ii. The primal dominance-based separability condition, then for all $x \in U$, $B_kP(\lambda)x \Rightarrow not(xS(\lambda)B_h)$, for $h > k$;

iii. The (primal and the dual) dominance-based separability condition, then for all $x \in U$, $xS(\lambda)B_k \Rightarrow xS(\lambda)B_h$, for $h < k$;

iv. The (primal and the dual) dominance-based separability condition, then for all, $x \in U$, $B_kP(\lambda)x \Rightarrow B_hP(\lambda)x$, for $h > k$;

v. The dual dominance-based separability condition, then for all, $b_{k,j} \in B_k$, $b_{k,j}S(\lambda)B_h$ for $h \leq k$;

vi. The primal preference-based separability condition, then for all, $b_{k,j} \in B_k$, $B_hP(\lambda)b_{k,j}$ for $h > k$.

Proof:

i. If with the primal dominance-based separability condition we have $xS(\lambda)B_k$, then there is no $z \in B_h$, with $h < k$, such that $zP(\lambda)x$. This can be easily proved. Suppose there exists $z \in B_h$ such that $zP(\lambda)x$. This contradicts $xS(\lambda)B_k$, since $w \in B_k$, wDz, and $zP(\lambda)x$ imply $wP(\lambda)x$. From Definition 2.iv, it follows that $not(B_hP(\lambda)x)$, for $h < k$.

ii. Suppose that $xS(\lambda)B_h$, for $h > k$; there is no $w \in B_h$ such that $wP(\lambda)x$ (from Definition 2.i). Then, there is no $z \in B_k$ such that $zP(\lambda)x$, since if $zP(\lambda)x$ and wDz (from the primal dominance-based separability condition), then $wP(\lambda)x$. Hence, $not(zP(\lambda)x)$, for all $z \in B_k$, implies $not(B_kP(\lambda)x)$ (from Definition 2.iv), which contradicts the hypothesis.

iii. Suppose we have both $xS(\lambda)B_k$ and $not(xS(\lambda)B_h)$, for $h < k$. We have two possible situations: a) there is $z \in B_h$ such that $zP(\lambda)x$; or b) $not(xS(\lambda)z)$, for all $z \in B_h$. In case a) and under the primal dominance-based separability condition, there is $w \in B_k$ such that wDz, hence $wP(\lambda)x$, which contradicts $xS(\lambda)B_k$. In case b) and under the dual dominance-based separability condition, $not(xS(\lambda)z)$, for all $z \in B_h$ implies $not(xS(\lambda)w)$, for all $w \in B_k$, since from $xS(\lambda)w$ and wDz it follows that $xS(\lambda)z$. Thus, $not(xS(\lambda)w)$ for all $w \in B_k$ is in contradiction with $xS(\lambda)B_k$ (Definition 2.i).

iv. Suppose that $not(B_hP(\lambda)x)$. We have two possible situations: a) there is $w \in B_h$ such that $xP(\lambda)w$; or b) $not(wS(\lambda)x)$, for all $w \in B_h$. In case a) and under the dual dominance-based separability condition, there is $z \in B_k$ such that wDz; this implies that $xP(\lambda)z$, which contradicts $B_kP(\lambda)x$ (Definition 2.iv). In case b) and with the primal dominance-based separability condition $not(wS(\lambda)x)$, for all $w \in B_h$, implies $not(zS(\lambda)x)$, for all $z \in B_k$ since from $zS(\lambda)x$ and wDz it follows that $wS(\lambda)x$. Finally, $not(zS(\lambda)x)$, for all $z \in B_k$, contradicts $B_kP(\lambda)x$.

v. The proof is obvious from Conditions 1.ii and 1.iii, Definition 2.i and the dual dominance-based separability condition.

vi. The proof is obvious from Condition 1.iii, Definition 2.iv and the primal preference-based separability condition.

2.4. Assignment procedures

In order to take into account multiple limiting profiles, the following two procedures were designed and constitute the ELECTRE TRI-nB method.

Definition 3 (Pseudo-conjunctive procedure)

Given the chosen $\lambda(> 0.5)$:

i. Compare x to B_k for $k=M,\ldots,1$;

ii. Let B_{k-1} be the first profile such that:

1) $xS(\lambda)B_{k-1}$;

2) There is no $h<k-1$ such that $B_hP(\lambda)x$;

iii. Assign x to category C_k.

Definition 4 (Pseudo-disjunctive procedure)

Given the chosen $\lambda(> 0.5)$:

i. Compare x to B_k for $k=1,2,\ldots,M$;

ii. Let B_k be the first profile such that:

1) $B_kP(\lambda)x$;

2) There is no $h>k$ such that $xS(\lambda)B_h$;

iii. Assign x to category C_k.

Proposition 2 (Relationship between the two assignment procedures)

Let C_{k*} and C_{k**} denote the categories $x \in U$ is assigned to, respectively, by the pseudo-conjunctive procedure and by the pseudo-disjunctive procedure. Then, we have $k^* \leq k^{**}$.

Proof:

Since x in C_{k*} represents the pseudo-conjunctive assignment, then $xS(\lambda)B_{k^*-1}$ and there is no $h < k^*-1$ such that $B_hP(\lambda)x$. Hence, in the pseudo-disjunctive procedure the first time for which $B_hP(\lambda)x$ must hold $h \geq k^*$. Thus, x is assigned to C_{k**} with $k^{**} \geq k^*$ by the pseudo-disjunctive procedure.

Proposition 3 (ELECTRE TRI-B is a particular case of ELECTRE TRI-nB)

If $card(B_k)=1$, for $k=1,\ldots,M-1$, and the (primal or dual) dominance-based separability condition (Condition 2.i-primal or 2.i-dual) holds, then ELECTRE TRI-nB corresponds to ELECTRE TRI-B.

Proof:

If $card(B_k)=1$, for $k=1,\ldots,M-1$ (in what follows consider this unique profile as b_k, for $k=1,\ldots,M-1$), and the (primal or dual) dominance-based separability condition (Condition 2.i-primal or 2.i-dual) holds, then b_kDb_{k-1}, for $k=1,\ldots,M-1$. The assignment with ELECTRE TRI-nB procedures requires the additional conditions 2) of Definitions 3.ii and 4.ii. with respect to the original ELECTRE TRI-B method. These additional conditions are automatically fulfilled:

- Condition 2) from Definitions 3.ii: There is no $h < k-1$ such that $b_hP(\lambda)x$. If $xS(\lambda)b_{k-1}$, then we cannot have both, $b_{k-1}Db_h$ and $b_hP(\lambda)x$.
- Condition 2) from Definitions 4.ii: There is no $h > k$ such that $xS(\lambda)b_h$. If $b_kP(\lambda)x$, then we cannot have both b_hDb_k and $xS(\lambda)b_h$.

Remark 3

ELECTRE TRI-nB allows, in general, reducing the number of incomparabilities by enhancing the boundaries. For an illustration of this important fact, suppose two categories and an action x whose appropriate assignment is C_2. Let us analyze when the pseudo-conjunctive procedure with a single profile fails. It happens when $not(xS(\lambda)b_{1,1})$. Suppose that $not(b_{1,1}P(\lambda)x)$; then the inappropriate assignment can be corrected. If we add $b_{1,2}$, $b_{1,3},\ldots,b_{1,j},\ldots$ it is likely that $xS(\lambda)b_{1,j}$ and $not(b_{1,i}P(\lambda)x)$ for the other limiting profiles; hence

$xS(\lambda)B_1$ and x is assigned to the appropriate category C_2. Suppose that the appropriate category is C_1. The pseudo-conjunctive procedure fails when $xS(\lambda)b_{1,1}$ and x is incorrectly assigned to C_2. If we add $b_{1,2}$, $b_{1,3},\ldots,b_{1,j},\ldots$, it is likely that $b_{1,j}P(\lambda)x$, hence $not(xS(\lambda)B_1)$; then x is assigned to the appropriate category C_1.

In the pseudo-disjunctive procedure, failure happens when a) the appropriate assignment is C_1 and $not(b_{1,1}P(\lambda)x)$, and b) the appropriate assignment is C_2 and, however, $b_{1,1}P(\lambda)x$. In case a), when $not(xP(\lambda)b_{1,1})$ the inappropriate assignment can be corrected. If we add $b_{1,2}$, $b_{1,3},\ldots,b_{1,j},\ldots$ it is likely that $b_{1,,j}P(\lambda)x$ for some $b_{1,,j}$ and $not(xP(\lambda)b_{1,i})$ for the other limiting profiles; thus, $B_1P(\lambda)x$ and x is assigned to the appropriate category C_1. In case b), if we add $b_{1,2}$, $b_{1,3},\ldots,b_{1,j},\ldots$ it is likely that $xP(\lambda)b_{1,,j}$ for some $b_{1,j}$; thus, $not(B_1P(\lambda)x)$ and x is assigned to the appropriate category.

Nevertheless, the fact of adding new limiting profiles does not necessarily improve each particular assignment. Let us analyze the effect of adding a new profile to one of the boundaries when the pseudo-conjunctive procedure is used.

Suppose that the appropriate category of x is $C_{k'}$ and, however, x is assigned to C_k ($k<k'$). It follows that $xS(\lambda)B_{k-1}$ and $not(xS(\lambda)B_k)$. The relation $not(xSB_k)$ implies: *i)* $xR(\lambda)b_{k,j}$, for all $b_{k,j}$, or *ii)* there is a $b_{k,j}$ such that $b_{k,j}P(\lambda)x$. Let $b_{k,i}$ denote a new profile added to B_k and now $B_k^* = \{B_k \cup b_{k,i}\}$. Three possible situations may occur: $xS(\lambda)b_{k,i}$, $xR(\lambda)b_{k,i}$, and $b_{k,i}P(\lambda)x$. Consider,

- Case *i)* and $xS(\lambda)b_{k,}$ imply $xS(\lambda)B_k^*$, then x is assigned to C_{k+1}. Thus, the assignment "error" $C_{k'}$ - C_{k+1} reduces or vanishes.
- Case *i)* and ($xR(\lambda)b_{k,i}$ or $b_{k,i}P(\lambda)x$) imply $not(xS(\lambda)B_k^*)$, then x keeps its assignment to C_k.
- Case *ii)* implies $not(xS(\lambda)B_k^*)$; the relation between x and $b_{k,i}$. does not matter. Thus x keeps its assignment to C_k.

(Note that either Case *ii)* or $b_{k,i}P(\lambda)x$ are unlikely situations, since the appropriate category for x is $C_{k'}$ ($k'>k$) and $b_{k,j}$ is on the lower boundary of C_{k+1}.)

Now, suppose that $k>k'$. It follows that $xS(\lambda)B_{k-1}$ and $not(B_h\, P(\lambda)x)$, for $h<k-1$; there is a $b_{k-1,j}$ such that $xS(\lambda)b_{k-1,j}$ and there is no $b_{k-1,i}$ such that $b_{k-1,i}P(\lambda)x$. Suppose that $b_{k-1,i}$ is added to B_{k-1} and now $B_{k-1}^* = \{B_{k-1} \cup b_{k-1,i}\}$. We have three possible situations: *a)* $xS(\lambda)b_{k-1,i}$; *b)* $xR(\lambda)b_{k-1,i}$; and, *c)* $b_{k-1,i}P(\lambda)x$.

In situations *a)* and *b)*, $xS(\lambda)B_{k-1}*$, then x keeps its assignment to C_k. In situation *c)* $not(xS(\lambda)B_{k-1}*)$, then x is assigned to $C_{k''}$ ($k''<k$). The assignment "error" reduces or vanishes. Note that situation *c)* is likely since x belongs to a category worse than the category of $b_{k-1,i}$. Now, suppose that x is assigned to its appropriate category $C_{k'}$. It follows that $xS(\lambda)B_{k'-1}$ and $not(xS(\lambda)B_{k'})$.

The addition to a new profile $b_{k'-1,i}$ can worsen the assignment only if $b_{k'-1,i}P(\lambda)x$ because in this case $not(xS(\lambda)B_{k'-1}*)$. However, this is unlikely since x belongs to $C_{k'}$ and $b_{k'-1,i}$ is on the lower boundary of the same category. Besides, the addition of a new profile $b_{k',i}$ can worsen the assignment only if $xS(\lambda)b_{k',i}$ and there is no $b_{k',j}$ such that $b_{k',j}P(\lambda)x$. This is also unlikely because x belongs to a category worse than the category to which those profiles belong.

Similar arguments can be given for the pseudo-disjunctive procedure.

3. Structural properties of ELECTRE TRI-nB

It is necessary to examine whether the new method, ELECTRE TRI-nB, fulfills the properties of ELECTRE TRI-B after taking into consideration the presence of sets of limiting profiles. In what follows, we introduce the required properties, adapted for the case of multiple limiting profiles, and provide the proofs of such properties in Appendix A. This needs to adapt the definition of the merging and splitting operations.

Definition 5 (**Merging and splitting operations**)

(a) Merging: Two consecutive categories, C_k and C_{k+1}, will be merged to become a new one C'_k. This is achieved by removing the limiting profile B_k. The category C'_k is bounded by the sets B_{k-1} and B_{k+1}. The new set B' of limiting profiles is defined as follows: $B'_h = B_h$ for $h = 0,\ldots,k-1$, $B'_{h-1} = B_h$, for $h= k+1,\ldots,M$.

(b) Splitting: The category C_k is split into two new consecutive categories C'_k and C'_{k+1}. This is achieved by inserting a new set of limiting profiles B'_k, such that the elements of B'_k fulfill the properties stated in Condition 1. The new set of limiting profiles B' is defined as follows: $B'_h = B_h$, for $h = 0,\ldots,k-1$, $B'_k = B'_k$, and $B'_h = B_{h-1}$, for $h = k+1,\ldots,M+1$.

Definition 6 (**Structural requirements**)

The following consistency properties are imposed *a priori* on ELECTRE TRI-nB:

i. *Uniqueness*: Each action is assigned to a unique category.

ii. *Independence*: The assignment of an action does not depend on the assignment of the other actions.

iii. *Conformity*:

   (a) If $xS(\lambda)B_k$ and $B_{k'}P(\lambda)x$ ($k'>k$), then the action x is assigned to C_f with $k+1 \leq f \leq k'$;

   (b) Each limiting profile $b_{k,j} \in B_k$ is assigned to C_{k+1}.

iv. *Monotonicity:* If an action x dominates an action y, xDy, and if y is assigned to C_k, then x is assigned to $C_{k'}$ with $k' \geq k$.

v. *Homogeneity:* If two actions compare the same way with respect to the limiting profiles, they must be assigned to the same category.

vi. *Stability:* When applying either a merging or a splitting operation, the actions previously assigned to the non-modified categories will be assigned to the same categories after modification. After merging two consecutive categories, the actions previously assigned to the merged categories are assigned to the new category. After splitting a category into two new consecutive categories, the actions previously assigned to the modified category are assigned to one of the new categories.

Theorem 1 (Structural properties of ELECTRE TRI-nB**).**

Under the basic assumptions (Condition 1) the assignment procedures fulfill the requirements of uniqueness, independence, homogeneity, and monotonicity. Adding only the primal dominance-based separability condition (Condition 2.i-primal) part a) of conformity holds. With the dominance-based separability condition (Condition 2.i) the requirement of stability is verified. Under the basic assumptions (Condition 1) and the preference-based separability condition (Condition 2.ii) part b) of conformity holds. Under the hyper-separability condition (Condition 2.iii) all the structural requirements are fulfilled.

Appendix A contains the proof of this theorem.

4. Some (illustrative) numerical examples

This section presents three illustrative examples.

4.1 Comparing ELECTRE TRI-B and ELECTRE TRI-nB

Consider a multiple criteria sorting problem in which the action x, evaluated on four criteria $g(x)$= (6,1,2,1), should be assigned to a category of the ordered set $C=\{C_1, C_2, C_3\}$ (C_3 is the most preferred).

Consider the following parameters (weights, indifference thresholds, preference thresholds, veto thresholds, and credibility level): w_i= 0.25; q_i= p_i= 0; v_i= 2.5; (i = 1,…4); and λ=0.75.

In what follows we shall show the impact of enriching the number of limiting profiles between two consecutive categories. Consider the following cases:

a) ELECTRE TRI-B (single limiting profile between two consecutive categories)

Consider the following limiting profiles.

Table 1a. Performances of limiting profiles b_k

	g_1	g_2	g_3	g_4
b_0	1	1	1	1
b_1	2	2	2	2
b_2	3	3	3	3
b_3	6	6	6	6

The λ- relations among the action and the limiting profiles are shown in Table 2a.

Table 2a. λ-preference relations

	b_0	b_1	b_2	b_3
x	P	R	R	P^{-1}

Note: $xP^{-1}b_k \equiv b_kPx$

The action x is assigned to C_1 (resp. C_3) by the pseudo-conjunctive (resp. pseudo-disjunctive) procedure.

The original set B (Table 1.a) fulfils both the dominance-based separability condition and the preference-based separability condition. Although these conditions are not necessary for ELECTRE TRI-nB, in what follows we will concentrate on adding profiles taking care of those properties.

b) Add a new profile between C_1 and C_2.

Consider $B_0 = \{b_0\}$, $B_1 = \{b_{1,1}, b_{1,2}\}$, $B_2 = \{b_2\}$, and $B_3 = \{b_3\}$. Please note that the new limiting profile $b_{1,2}$ is incomparable to $b_{1,1}$; b_2 is preferred to $b_{1,2}$, but this is not dominated by the first one. The ideal profile b_3 dominates and is preferred to any other action. The contrary is fulfilled by the anti-ideal profile.

Table 1b. Performances of limiting profiles

	g_1	g_2	g_3	g_4
b_0	1	1	1	1
$b_{1,1}=b_1$	2	2	2	2
$\boldsymbol{b_{1,2}}$	**3.2**	**2.8**	**1**	**1**
b_2	3	3	3	3
b_3	6	6	6	6

The λ-relations among the action and the single limiting profiles and the sets of limiting profiles are shown in Table 2b.

Table 2b. λ-preference relations

	$B_0=\{b_0\}$	$b_{1,1}$	$b_{1,2}$	$B_1=\{b_{1,1}, b_{1,2}\}$	$B_2=\{b_2\}$	$B_3=\{b_3\}$
x	P	R	P	P	R	P^{-1}

Note: $xP^{-1}B_k \equiv B_kPx$

Adding the new profile leads to:

- Assigning x to C_2 by the pseudo-conjunctive procedure (instead of C_1);
- Assigning x to the same category (C_3) by the pseudo-disjunctive procedure;
- Losing the primal dominance-based separability condition since $b_{1,2}$ is not dominated by b_2.

c) Add a new profile between C_2 and C_3.

Consider $B_0 = \{b_0\}$, $B_1 = \{b_1\}$, $B_2 = \{b_{2,1}, b_{2,2}\}$, and $B_3 = \{b_3\}$. Please note that the new limiting profile $b_{2,2}$ is indifferent to $b_{2,1}$ and dominates b_1. Note also that $b_{2,2}$ is preferred to b_1.

Table 1c. Performances of limiting profiles

	g_1	g_2	g_3	g_4
b_0	1	1	1	1
b_1	2	2	2	2
$b_{2,1}=b_2$	3	3	3	3
$\boldsymbol{b_{2,2}}$	**4**	**3**	**3**	**2**
b_3	6	6	6	6

The λ-relations among the action and the single limiting profiles and the sets of limiting profiles are shown in Table 2c.

Table 2c. λ-preference relations

	$B_0=\{b_0\}$	$B_1=\{b_1\}$	$b_{2,1}$	$b_{2,2}$	$B_2=\{b_{2,1}, b_{2,2}\}$	$B_3=\{b_3\}$
x	P	R	R	P^{-1}	P^{-1}	P^{-1}

Note: $xP^{-1}B_k \equiv B_k P x$

Adding the new profile leads to:

- Assigning x to the same category (C_1) as in case a) by the pseudo-conjunctive procedure.
- Assigning x to C_2 by the pseudo-disjunctive procedure (instead of C_3 as in case a));

d) Add two new limiting profiles $b_{1,2}$ and $b_{2,2}$.

Consider $B_0 = \{b_0\}$, $B_1 = \{b_{1,1}, b_{1,2}\}$, $B_2 = \{b_{2,1}, b_{2,2}\}$, and $B_3 = \{b_3\}$.

Table 1d. Performances of limiting profiles

	g_1	g_2	g_3	g_4
b_0	1	1	1	1
$b_{1,1}=b_1$	2	2	2	2
$\boldsymbol{b_{1,2}}$	**3.2**	**2.8**	**1**	**1**
$b_{2,1}=b_2$	3	3	3	3
$\boldsymbol{b_{2,2}}$	**4**	**3**	**3**	**2**
b_3	6	6	6	6

Note that $b_{1,2}$ is dominated by $b_{2,2}$, restarting fulfilment of the primal dominance-based separability condition. Hence, the set B fulfils both the dominance-based separability condition and the preference-based separability condition. Thus, the hyper- separability condition (Condition 2.iii) is fulfilled too.

The λ-relations among the action and the single limiting profiles and the sets of limiting profiles are shown in Table 2d.

Table 2d. λ-preference relations

	$B_0=\{b_0\}$	$b_{1,1}$	$b_{1,2}$	$b_{2,1}$	$b_{2,2}$	$B_1=\{b_{1,1}, b_{1,2}\}$	$B_2=\{b_{2,1}, b_{2,2}\}$	$B_3=\{b_3\}$
x	P	R	P	R	P^{-1}	P	P^{-1}	P^{-1}

Note: $xP^{-1}B_k \equiv B_kPx$

The action is assigned to C_2 by the two procedures.

4.2 Evaluating the capacity to provide appropriate assignments

In this example we consider a universe of actions also evaluated on four criteria, g_1, g_2, g_3, g_4, with performances within the value range [0.5, 7.5] (preference increasing with criterion values). All criteria will play the same role in the decision aiding process.

Let us consider a firm, for instance, a bank, which frequently needs to assign actions (credit demand applications) to two categories C_1 (files to be rejected) and C_2 (files to be accepted). Suppose this firm can *a posteriori* examine a set of actions, which have been accepted, to see if accepting them was justified. Table 3 in Appendix B presents this set of actions: Those wrongly accepted (that should have been rejected) are in category C_1; and the others are in category C_2.

The firm wants to build a tool allowing for the assignment of every new action in the most appropriate way possible in one of the two categories. For such a purpose the firm wants to build an ELECTRE TRI-nB model, which when applied to the set of actions in Table 3 makes the assignment as good as possible in the appropriate category.

The analyst and the DM considered the following set of parameters:

- Weights: w_i= 0.25, i=1,…,4;
- Veto thresholds: v_i = 3.4, i=1,…,4;
- Indifference thresholds: q_i= 0.15, i=1,…,4;
- Preference thresholds: p_i= 0.7, i=1,…,4;
- Credibility level: λ = 0.7;
- $g(b_0)$= (0,0,0,0); $g(b_2)$= (8.5,8.5,8.5,8.5).

In a first step the analyst and the DM considered only a single limiting profile: $g(b_1)$= (4,4,4,4). With this profile the number of incomparabilities is 60. For each action in Table 3, its assignment is compared with the assignments provided by ELECTRE TRI-B. The accuracy (frequency of coincidence of both assignments) was calculated. The results are shown in Table 4.

Table 4. Accuracy of the results of ELECTRE TRI-B

Procedure	Accuracy
Pseudo-conjunctive	0.84
Pseudo-disjunctive	0.66

In a second step, the analyst and the DM wanted to improve the performance of the tool by successively increasing the number of limiting profiles and reducing the number of incomparabilities $xR(\lambda)B_1$. For this purpose they examined Table 3 and also took into account the role played by each criterion in promoting the success of an action. Three cases were considered:

1. $B_1 = \{b_{1,1}, b_{1,2}\}$,
   where $g(b_{1,1})$ =(4,4,4,4), $g(b_{1,2})$=(6,2,4,4);

2. $B_1 = \{b_{1,1}, b_{1,2}, b_{1,3}\}$,
   where $g(b_{1,1})$ = (4,4,4,4), $g(b_{1,2})$ =(6,2,4,4), $g(b_{1,3})$ =(2,4,6,4);

3. $B_1 = \{b_{1,1}, b_{1,2}, b_{1,3}, b_{1,4}\}$,
   where $g(b_{1,1})$ = (4,4,4,4), $g(b_{1,2})$ =(6,2,4,4), $g(b_{1,3})$ =(2,4,6,4), $g(b_{1,4})$ = (4,4,2,6).

Note that the hyper-separability condition (Condition 2.iii) is fulfilled. We should note that we cannot have a weaker separability condition since we only have two categories, and the ideal and the anti-ideal profiles always fulfill dominance and preference conditions with profiles in B_1.

Table 5. Accuracy of the results obtained with ELECTRE TRI-nB

Set of limiting profiles	Accuracy (pseudo-conjunctive)	Accuracy (pseudo-disjunctive)	Number of incomparabilities
$B_1 = \{b_{1,1}, b_{1,2}\}$	0.89	0.67	30
$B_1 = \{b_{1,1}, b_{1,2}, b_{1,3}\}$	0.92	0.69	24
$B_1 = \{b_{1,1}, b_{1,2}, b_{1,3}, b_{1,4}\}$	0.95	0.71	20

Table 5 shows that ELECTRE TRI-nB allows for better results than ELECTRE TRI-B. These results are particularly good for the pseudo-conjunctive procedure. It also shows that the quality of the results provided by the two procedures is not the same. This is not surprising since the two procedures are based on two different logics; one is less demanding than the other. When an action is not assigned to the same category by the two procedures, the pseudo-disjunctive procedure always assigns the action to a better category than the category provided by the pseudo-conjunctive procedure.

The pseudo-conjunctive procedure suggests an inappropriate assignment when I) *not*($xS(\lambda)B_1$) but its appropriate assignment is C_2; and II) $xS(\lambda)B_1$, whereas its appropriate assignment is C_1. The first (respectively, second) situation is called here Error 1-c (respectively, Error 2-c). When incomparability is reduced by enhancement of the boundary B_1, the error 1-c is diminished. Besides, the errors associated with the pseudo-disjunctive procedure come from I) $B_1P(\lambda)x$ but its appropriate assignment is C_2 (Error 1-d); and II) *not*($B_1P(\lambda)x$), whereas its appropriate assignment is C_1 (Error 2-d). This error diminishes when the reduction of incomparability becomes *not*($B_1P(\lambda)x$) into $B_1P(\lambda)x$. The calculation of these errors and the incomparability situations among actions and boundary are shown in Table 6. In this table $b_{1,j}$ (j= 2,3,4) is a permutation of the action (6, 2, 4, 4) and $b_{1,j} \neq b_{1,i}$, $j \neq i$. Take into account that $b_{1,j}I(\lambda)b_{1,i}$ or $b_{1,j}R(\lambda)b_{1,i}$(j=1,…,4, i=1,…,4), then the elements $b_{1,j}$

(j=2,3,4) satisfy the conditions needed to be members of the boundary. Each boundary was replicated for every possible permutation of $b_{1,j}$.

Table 6. Analysis of errors (pseudo-conjunctive rule)

Boundary	Average number of incomparabilities	Ave. number of Errors 1-c	Ave. number of Errors 2-c
{(4,4,4,4)}	60	13	3
{(4,4,4,4), $b_{1,2}$}	29.56	6.96	2.32
{(4,4,4,4), $b_{1,2}$, $b_{1,3}$}	24.33	5.08	2.68
{(4,4,4,4), $b_{1,2}$, $b_{1,3}$, $b_{1,4}$}	20.20	3.76	2.27

Table 7. Analysis of errors (pseudo-disjunctive rule)

Boundary	Average number of incomparabilities	Ave. number of Errors 1-d	Ave. number of Errors 2-d
{(4,4,4,4)}	60	1	33
{(4,4,4,4), $b_{1,2}$}	29.56	0.43	25.36
{(4,4,4,4), $b_{1,2}$, $b_{1,3}$}	24.33	0.34	22.27
{(4,4,4,4), $b_{1,2}$, $b_{1,3}$, $b_{1,4}$}	20.20	0.34	18.83

The enhancement of the boundary reduces the number of actions that are incomparable with it. As a consequence, Error 1-c is strongly diminished. Error 2-d decreases too although less noticeably.

4.3 Performances analysis of ELECTRE TRI-nB

The aim of this example is to analyze the ELECTRE TRI-nB's performance in a more complex sorting problem in which 10 criteria are considered and 100 actions are classified in four possible ordered categories. We shall use an outranking model similar to Example 2:

- Weights: w_i= 0.1, i=1,…,10;
- Veto thresholds: v_i = 3.4, i=1,…,10;
- Indifference thresholds: q_i= 0.15, i=1,…,10;

- Preference thresholds: p_i= 0.7, i=1,…,10;
- Credibility level: λ = 0.7;
- $g(b_0)$= (0,0,0,0,0,0,0,0,0,0); $g(b_4)$= (8.5,8.5,8.5,8.5,8.5, 8.5,8.5,8.5,8.5,8.5).

The set of actions is given in Table 8 in Appendix C.

In a first step, only a single profile is considered in each boundary, as in ELECTRE TRI-B. Thus, $g(b_{1,1})$ = (2,2,2,2,2,2,2,2,2,2); $g(b_{2,1})$ = (4,4,4,4,4,4,4,4,4,4); $g(b_{3,1})$ = (6,6.6,6,6,6,6,6,6,6). These profiles also fulfill the preference-based separability condition (Condition 2.ii). With these profiles the pseudo-conjunctive and pseudo-disjunctive assignment of the actions are shown in the second column of Table 9 Appendix C.

In Columns 3-6, Table 9 provides the assignment results when the limiting boundaries were enhanced from $card(B_k)$=1 to $card(B_k)$=4, for k=1,2,3. The limiting profiles $b_{k,j}$ were generated through random variations of some criterion values $g_i(b_{k,1})$ in the interval [$-2p_i$, $2p_i$]. Each $b_{k,j}$ was generated fulfilling λ-indifference with $b_{k,1}$ and Condition 1. In this first experiment the preference-based separability condition was imposed too. Some complementary results are shown in Table 10.

Table 10. Other results under Condition 2.ii

	$card(B_k)$=1	$card(B_k)$=2	$card(B_k)$=3	$card(B_k)$=4
Number of actions for which C_{pc}=C_{pd}	20	28	35	44
Number of actions for which incomparability reduces with respect to $card(B_k)$=1	-	19	28	36
Number of actions for which incomparability increases with $card(B_k)$	-	0	0	0
Total number of boundaries incomparable with actions	97	77	66	56

Table 11 provides results of a similar experiment in which Condition 2.ii is replaced by the dominance-based separability condition (Condition 2.i).

Table 11. Results under Condition 2.i

	card(B_k)=1	*card(B_k)*=2	*card(B_k)*=3	*card(B_k)*=4
Number of actions for which $C_{pc}=C_{pd}$	20	31	36	43
Number of actions for which incomparability reduces with respect to *card(B_k)*=1	-	21	28	36
Number of actions for which incomparability increases with *card(B_k)*	-	0	0	0
Total number of boundaries incomparable with actions	97	75	66	57

When the hyper-separability condition (Condition 2.iii) is imposed on the set of limiting profiles, we obtained the results given in Table 12.

Table 12. Results under Condition 2.iii

	card(B_k)=1	*card(B_k)*=2	*card(B_k)*=3	*card(B_k)*=4
Number of actions for which $C_{pc}=C_{pd}$	20	30	41	48
Number of actions for which incomparability reduces with respect to *card(B_k)*=1	-	20	34	41
Number of actions for which incomparability increases with *card(B_k)*	-	0	0	0
Total number of boundaries incomparable with actions	97	77	60	52

The different experiments show a robust reduction of incomparability, thus achieving better characterization of the boundaries. The results under the hyper-separability condition (combining Conditions 2.i and 2.ii) reduce the number of incomparabilities. Although more experiments are needed, this preliminary conclusion is consistent with fulfillment of the structural requirements of ELECTRE TRI-nB.

5. Concluding remarks

Inspired by the success of ELECTRE TRI-B, we presented in this paper a new multiple criteria sorting method, called ELECTRE TRI-nB, which gives new possibilities to the DM for characterizing the limiting boundaries between adjacent categories. When each boundary is characterized by a single limiting profile, ELECTRE TRI-nB does not differ from ELECTRE TRI-B.

The new method is especially recommended when a single limiting profile is not sufficient for good characterization of its associated boundary. In ELECTRE TRI-nB, each limiting profile added to the description of a boundary is a new piece of information that leads to improved characterization of that boundary. The main contribution of ELECTRE TRI-nB with regard to ELECTRE TRI-B comes principally from the fact that it enriches how the actions to be assigned compare to the limiting profiles (in particular, by reducing the number of incomparabilities). In each decision problem, starting with a single (or a few) profile(s) and using a sample of the universe to be classified, the analyst can calculate the magnitude of the incomparability actions-boundaries as in Example 3. On this basis and taking into account the cognitive effort and the availability of the decision-maker, the analyst can make a decision about whether more profiles are necessary.

The examples show how ELECTRE TRI-nB works, and why it can suggest more appropriate assignments when the boundaries are enhanced with additional limiting profiles. The process of enhancing the limiting boundaries should be a result of co-constructive work between the DM and the analyst.

It was proved in the paper that the ELECTRE TRI-nB fulfills some fundamental properties: uniqueness, independence, conformity, monotonicity, homogeneity and stability.

There are several possible avenues for future research, for instance:

- To conduct more extensive testing of the effectiveness of such a variant on actual and test data and set guidelines for specification of the profiles (*e.g.*, their number).
- To investigate whether the dual extension of ELECTRE TRI-B proposed by Bouyssou and Marchant (2015) fulfills the fundamental properties of ELECTRE TRI-nB. The extension can be stated as follows: for the dual of the pseudo-disjunctive procedure (i. Compare x to B_k for $k=M,\ldots,1$; ii. Let B_k be the first profile such that $xP(\lambda)B_k$; iii. Assign x to category C_k.) and for the dual of the pseudo-conjunctive procedure (i. Compare x to B_k for $k=1,2,\ldots,M$; ii. Let B_{k-1} be the first profile such that $B_{k-1}S(\lambda)x$; iii. Assign x to category C_k.).

- To make a conjoint elicitation of the preference parameters (e.g., weights, veto thresholds, and/or limiting profiles) and the lambda cutting level through an inference preference-disaggregation approach in the same kind of philosophy as the one proposed by Mousseau and Słowiński (1998).
- As a complement to the previous line of research, develop and/or implement procedures to deal with “inconsistent” judgments in the sense proposed by Mousseau *et al.* (2003).
- To develop procedures for robustness concerns in several directions and in Dias *et al.*(2002), Greco *et al*. (2008, 2011), Kadziński *et al*. (2015), and Tervonen *et al*. (2009).
- To consider the model with constraints in the size of categories and in DIS-CARD method by Kadziński and Słowiński (2013).
- To improve the process by considering a kind of hierarchy process for the modeling of the criteria tree as in the recent work by Corrente *et al.* (2016).

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Appendix A (Proofs of Theorem 1)

i. (*Uniqueness*).

The proof is trivial given the way the procedures were designed (see Definitions 3 and 4).

ii. (*Independence*).

The proof is trivial given the way the assignments are made.

iii. (*Conformity*).

(a) Consider the pseudo-conjunctive procedure under the primal dominance-based separability condition (Condition 2.i primal). We have $xS(\lambda)B_k$ and from Proposition 1.i x is assigned to category C_f with $f \geq k+1$. Since we have $B_{k'}P(\lambda)x$, then we cannot have $xS(\lambda)B_h$ for $h \geq k'$ (from Definition 2.i and Proposition 1.ii). Thus x is assigned to category C_f with $f \leq k'$. For the pseudo-disjunctive procedure $B_{k'}P(\lambda)x$ and Proposition 1.ii imply x is assigned to category C_f with $f \leq k'$. $xS(\lambda)B_k$ and Proposition 1.ii imply $B_hP(\lambda)x$ can fulfill *only* for $h \geq k+1$. Thus f has to be greater than or equal to $k+1$. This completes the proof.

(b) Consider the pseudo-conjunctive procedure under the primal and dual preference-based separability condition (Condition 2.ii). $b_{k,j}S(\lambda)B_k$ from Condition 1.ii and Definition 2.i. For $h>k$ *not*($b_{k,j}S(\lambda)B_h$) from Proposition 1.vi. For $h<k$ *not*($B_hP(\lambda)b_{k,j}$) from Definition 2.iv and Condition 2.ii (dual). Hence $b_{k,j}$ is assigned to C_{k+1}. Now consider the pseudo-disjunctive procedure. $B_{k+1}P(\lambda)b_{k,j}$ from Proposition 1.vi. For $h \leq k$ *not*($B_hP(\lambda)b_{k,j}$) from Definition 2.iv and Conditions 1.ii and 1.iii. For $h>k+1$ it follows from Proposition 1.vi that *not*($b_{k,j}S(\lambda)B_h$). Hence $b_{k,j}$ is assigned to C_{k+1}.

iv. (*Monotonicity*).

Consider the pseudo-conjunctive procedure. $y \in C_k \Rightarrow yS(\lambda)B_{k-1}$ and *not*($B_hP(\lambda)y$) for $h<k-1$. Since xDy then $xS(\lambda)B_{k-1}$ (Remark 2.vi). Suppose there is $h < k-1$ such that $B_hP(\lambda)x$. From Remark 2.v this implies $B_hP(\lambda)y$, which contradicts the hypothesis. Then, x is assigned to $C_{k'}$. with $k' \geq k$. The proof for the pseudo-disjunctive procedure is similar.

v. (*Homogeneity*).

The proof is trivial.

vi. (*Stability*)

Consider the pseudo-conjunctive procedure.

a. Merging two consecutive categories

   i. Let x denote an action previously assigned to category C_h, for $h> k+1$ (*i.e.*, $h-1>k$). For such an action x, since we have both, $not(xS(\lambda)B_h)$ and $xS(\lambda)B_{h-1}$, removing set B_k does not change the functioning of the algorithm; action x is thus assigned in the same way as before the merging operation.

   ii. Let x denote an action previously assigned to category C_k or C_{k+1}. If before the merging operation x is assigned to C_{k+1}, then we have $xS(\lambda)B_k$; this implies, from Proposition 1.iii, that $xS(\lambda)B_{k-1}$. After removing set B_k, x is assigned to the new category. If before the merging operation x is assigned to C_k, then we have $xS(\lambda)B_{k-1}$. Removing the set B_k, does not make any change; x is assigned to the new category.

   iii. Let x denote an action previously assigned to category C_h, for $h < k$. For such an x, it is obvious that removing B_k has no impact on the functioning of the algorithm; x is thus assigned in the same way as before the merging operation.

b. Splitting a single category in two new ones.

   i. Let x denote an action previously assigned to category C_h, for $h> k$. For such an x, the functioning of the algorithm could be modified by a splitting operation only when adding B'_k leads to $B'_kP(\lambda)x$. This is not possible. Suppose that it is the case, then it implies, from Proposition 1.ii, that $not(xS(\lambda)B'_l)$, for $l>k$. This contradicts that $xS(\lambda)B'_{k+1}$, which comes from $xS(\lambda)B_k$ and Definition 5.(b).

   ii. Let x denote an action previously assigned to category C_k. For such an x, we have both, $not(xS(\lambda)B_k)$ and $xS(\lambda)B_{k-1}$. After adding B'_k, we have both, $not(xS(\lambda)B'_{k+1})$ and $xS(\lambda)B'_{k-1}$ (from Definition 5.(b)). If $xS(\lambda)B'_k$, then x is assigned to C'_{k+1}. Otherwise, x is assigned to C'_k.

   iii. Let x denote an action previously assigned to category C_h, for $h < k$. For such an x, we cannot have $xS(\lambda)B'_k$. This is impossible from Proposition

1.iii and $not(xS(\lambda)B_h)$. In such conditions, the splitting operation does not lead to any modification of the algorithm.

vii. Consider the pseudo-disjunctive procedure.

The proof is based on the same type of reasoning, but now supported by Propositions 1.iv and 1.i instead of Propositions 1.iii and 1.ii.

Appendix B (Data set for Example 2)

Table 3. The set of assignment examples

Action	g_1	g_2	g_3	g_4	Category
1	5.3	3.3	4.7	2.2	1
2	2.6	5.3	4.4	6.1	2
3	4.7	0.6	3.9	1.0	1
4	3.5	5.2	3.0	4.4	1
5	3.6	2.7	0.8	2.1	1
6	0.8	4.0	4.0	5.8	1
7	0.9	6.3	4.5	4.3	1
8	2.8	7.1	6.7	0.5	1
9	2.9	4.0	2.2	2.4	1
10	6.2	3.4	1.0	3.4	1
11	3.5	1.8	1.2	3.8	1
12	6.5	3.7	5.1	6.9	2
13	6.0	5.5	1.5	3.6	1
14	6.3	2.3	6.3	6.6	2
15	1.2	0.7	3.5	3.4	1
16	4.6	7.1	3.5	4.4	2
17	0.9	2.5	6.3	6.6	1
18	5.4	6.8	6.8	5.8	2
19	1.1	0.5	2.1	4.4	1
20	1.1	4.1	3.9	4.0	1
21	2.0	2.2	7.1	0.8	1
22	4.1	3.3	7.0	2.1	1
23	3.4	7.3	2.4	0.6	1
24	4.2	2.7	4.5	4.6	1
25	4.7	0.6	1.0	7.0	1
26	6.9	7.3	5.3	4.8	2
27	4.7	4.2	1.7	5.4	2
28	0.8	2.5	1.8	6.7	1
29	1.6	5.8	4.4	5.2	2
30	5.9	3.8	4.1	1.8	1
31	0.9	6.0	6.3	4.7	2
32	5.6	0.6	6.1	2.8	1
33	0.7	6.7	2.3	4.4	1
34	3.9	7.1	1.7	1.1	1
35	1.2	7.3	6.0	1.5	1
36	6.7	4.7	5.1	0.7	1
37	7.3	1.9	2.7	5.7	2
38	2.6	3.7	4.4	3.0	1
39	2.2	3.2	4.5	7.3	2
40	0.6	3.2	7.0	5.1	1
41	2.3	6.2	2.1	3.1	1
42	3.1	0.7	3.7	3.8	1
43	4.8	6.5	2.2	1.3	1
44	1.0	6.8	1.6	0.8	1
45	1.2	1.1	3.4	3.2	1
46	4.4	7.3	3.1	3.4	2
47	7.3	7.2	0.6	7.4	2
48	7.2	4.4	2.4	6.4	2
49	3.1	3.9	1.9	5.7	1
50	1.4	5.1	6.4	5.8	2
51	1.5	1.1	6.6	2.0	1
52	4.7	5.0	2.3	5.4	2
53	3.0	5.2	5.5	4.2	2
54	2.3	1.1	7.2	2.1	1
55	5.1	4.6	6.3	4.8	2
56	1.6	1.1	3.7	1.5	1
57	4.6	5.1	4.1	2.8	2
58	7.0	3.0	1.1	1.0	1
59	0.9	4.6	2.5	5.2	1
60	6.5	1.7	7.4	2.0	1
61	6.3	2.8	5.8	5.5	2
62	3.4	2.8	7.1	1.1	1
63	4.3	5.8	2.8	2.7	1
64	3.8	5.9	3.7	0.9	1
65	0.9	7.3	0.6	7.4	1
66	7.1	1.2	5.3	4.9	2
67	2.6	7.3	6.9	1.6	2
68	1.4	4.2	0.5	4.6	1
69	6.5	5.8	2.6	2.5	2
70	5.4	6.5	7.4	2.2	2
71	2.2	7.0	4.5	5.5	2
72	2.8	5.1	3.2	3.2	1
73	2.2	3.2	7.5	1.8	1
74	1.3	5.2	3.6	3.4	1
75	2.3	3.0	1.9	0.6	1
76	6.8	1.9	4.7	5.8	2
77	4.6	4.1	5.1	2.5	2
78	7.5	2.4	1.6	2.2	1
79	2.0	5.6	7.2	4.3	2
80	3.1	2.8	4.3	4.8	1
81	5	3.9	3.5	3.7	1
82	6.3	6.6	6.7	1.1	2
83	6.5	1.1	1.2	3.1	1
84	6.9	2.8	5.7	4.0	2
85	3.8	0.7	3.3	3.8	1
86	2.6	4.4	5.4	1.4	1
87	6.9	2.4	2.6	6.8	2
88	4.8	6.4	1.5	7.2	2
89	7.5	4.5	3.5	6.3	2
90	1.0	7.0	7.0	4.3	2
91	4.9	0.7	6.9	4.3	1
92	7.3	2.5	5.2	3.6	2
93	2.8	5.3	6.8	4.9	2
94	2.3	2.1	3.2	6.0	1
95	4.1	5.2	5.3	1.5	2
96	4.1	6.3	5.4	1.5	2
97	0.7	5.7	7.3	5.6	2
98	5.2	4.1	2.4	7.0	2
99	1.7	6.2	3.8	1.5	1
100	1.2	5.9	4.6	3.5	1

Appendix C (Data set and results for Example 3)

Table 8. The set of assignment examples

Action	g_1	g_2	g_3	g_4	g_5	g_6	g_7	g_8	g_9	g_{10}
1	3.1	4.3	4.2	2.0	1.4	6.4	2.4	4.5	4.8	2.7
2	4.7	7.2	2.6	4.6	1.2	2.8	7.4	1.0	6.9	5.8
3	1.7	6.5	1.7	2.7	0.8	2.1	3.4	1.2	1.3	2.1
4	5.7	1.2	3.3	2.4	7.1	4.1	1.4	2.0	5.5	3.0
5	1.6	2.7	2.8	1.0	4.1	3.5	0.7	4.0	1.3	7.1
6	2.3	2.5	6.1	3.6	2.1	3.7	2.5	5.0	4.5	0.6
7	4.0	2.7	5.7	6.8	1.9	5.3	0.8	7.1	4.2	3.1
8	2.6	2.6	5.3	4.9	3.1	1.9	5.3	3.3	2.8	6.1
9	7.3	4.7	5.5	3.2	5.1	4.5	6.4	4.4	6.4	0.7
10	4.5	7.2	2.9	2.7	3.9	1.7	0.5	4.1	1.3	1.6
11	4.1	0.8	3.7	1.9	5.3	3.7	0.7	7.4	6.5	3.0
12	3.4	6.3	4.5	5.7	2.0	6.4	7.1	5.2	3.4	5.9
13	5.4	4.8	3.0	5.2	7.0	6.4	3.7	4.4	7.4	4.5
14	2.8	4.0	4.9	6.0	2.8	7.0	2.2	3.0	4.2	5.6
15	2.8	4.4	1.8	4.2	7.0	0.6	3.1	6.6	5.3	3.4
16	2.4	0.6	0.6	4.9	2.6	4.5	3.8	5.9	1.4	1.1
17	2.9	3.7	4.6	4.7	6.6	6.9	1.5	5.7	6.7	2.6
18	3.8	2.1	6.5	5.1	5.8	6.1	5.2	5.8	2.5	7.3
19	1.7	4.4	4.8	1.8	6.1	6.9	5.8	6.8	5.3	4.1
20	7.4	5.1	4.6	1.9	6.6	3.8	1.3	4.9	6.3	4.9
21	7.0	2.6	6.5	3.4	4.6	2.1	6.3	6.6	7.4	1.3
22	6.4	5.9	2.5	3.7	7.2	1.1	0.6	2.8	4.8	2.7
23	6.4	4.7	7.2	0.9	6.1	3.6	1.5	4.2	1.1	7.3
24	1.6	4.9	2.4	4.9	0.8	6.5	3.9	6.6	3.0	1.1
25	7.4	1.9	6.5	2.4	2.5	3.5	7.4	6.9	5.9	4.7
26	2.1	2.1	6.3	6.1	2.5	2.2	2.3	3.5	3.3	7.2
27	3.3	1.8	4.7	2.6	6.3	5.0	6.0	2.7	4.2	1.5
28	0.6	1.5	7.3	6.6	0.8	6.7	2.6	5.0	6.1	1.0
29	6.6	0.6	7.0	2.7	6.2	2.0	4.5	5.4	2.3	7.3
30	2.5	2.5	6.0	6.6	4.6	4.7	1.5	3.1	4.2	2.5
31	1.5	4.4	3.5	1.2	0.8	1.1	4.8	3.0	5.6	3.3
32	0.8	4.7	0.8	7.3	7.0	3.9	1.7	1.3	1.8	0.9
33	5.5	3.7	2.9	3.9	7.2	7.0	5.5	1.2	7.0	6.6
34	3.2	5.3	0.8	3.5	3.4	1.2	4.1	0.7	3.7	6.6
35	3.5	4.0	1.2	3.8	1.1	5.0	7.1	6.7	5.9	5.8
36	7.1	1.2	6.3	2.5	4.6	6.0	6.3	7.0	4.1	5.8
37	6.1	6.8	3.6	6.5	7.2	6.5	4.5	1.2	6.7	5.0
38	7.3	7.1	1.5	1.0	0.7	6.4	5.5	4.6	5.6	1.3
39	2.9	5.1	1.9	6.1	4.4	3.4	4.6	3.3	7.3	1.2
40	1.2	2.4	1.6	6.5	7.2	1.8	5.5	2.6	5.9	0.9
41	3.4	5.7	5.5	2.2	4.3	2.5	2.5	5.8	5.2	2.3
42	1.7	3.3	4.9	2.1	6.6	1.9	5.5	3.7	2.6	2.0
43	3.9	3.7	2.4	5.6	5.3	1.5	7.0	0.6	4.7	2.2
44	2.2	2.8	1.4	3.3	1.8	1.1	1.9	4.1	3.3	4.7
45	4.5	6.2	2.9	6.8	5.2	4.0	1.8	7.2	2.8	3.9
46	6.3	3.0	4.4	1.2	1.1	6.6	6.6	0.6	6.7	3.8
47	6.6	1.4	3.4	0.6	4.2	4.7	1.2	3.0	5.6	4.0
48	7.2	2.6	7.0	6.9	2.0	2.0	3.4	0.6	6.0	2.5
49	1.3	4.8	5.0	5.3	2.8	2.9	4.4	1.9	7.4	3.6
50	2.5	6.5	1.9	5.4	3.9	2.9	6.9	4.5	5.4	5.1
51	5.3	2.4	4.5	2.1	1.8	3.3	3.7	2.0	3.4	2.2
52	4.0	1.6	3.8	5.8	6.4	3.5	1.2	0.6	4.9	1.1
53	1.1	4.3	4.4	2.4	2.2	0.8	4.8	1.6	4.8	7.1
54	3.5	7.0	2.0	0.6	6.0	3.2	3.4	2.1	4.8	3.7
55	1.2	5.6	2.1	1.9	3.9	5.4	4.9	2.0	5.5	6.6
56	2.6	3.4	3.3	6.5	5.3	5.0	4.2	7.0	3.5	1.5
57	6.6	6.5	5.3	5.4	3.9	3.8	5.5	4.1	2.7	7.1
58	7.2	0.8	5.2	6.2	2.1	6.0	4.1	3.8	0.6	2.0
59	2.9	7.0	4.9	3.1	3.4	7.0	0.6	7.0	6.5	0.9
60	5.4	2.9	4.2	3.1	0.8	0.6	3.8	5.8	4.2	3.3
61	5.3	1.3	3.6	7.4	7.0	2.6	5.9	0.9	3.2	3.2
62	2.4	5.6	2.7	4.2	5.6	2.9	1.1	5.7	2.5	4.4
63	3.5	4.7	6.8	7.2	7.3	4.4	4.7	0.9	7.1	1.4
64	3.8	2.2	6.5	4.2	6.5	3.3	6.3	4.8	3.7	2.0
65	4.9	3.0	4.5	4.4	4.0	2.6	6.8	4.6	5.2	6.2
66	1.5	1.1	3.3	5.2	5.2	0.6	6.5	2.4	5.3	6.0
67	0.6	1.6	5.1	3.9	2.6	4.1	6.7	1.4	5.8	7.3
68	7.3	0.5	7.2	1.7	4.4	3.7	1.1	1.1	5.2	5.8

69	6.8	3.5	3.8	7.0	5.6	5.8	4.3	4.5	0.7	2.1
70	7.4	5.2	0.6	5.1	1.6	2.7	6.1	0.8	1.0	1.7
71	5.0	5.1	1.7	4.7	6.3	5.6	0.9	7.0	6.2	2.9
72	2.6	2.9	5.9	5.9	6.8	3.9	4.2	3.6	5.3	1.8
73	2.6	5.3	6.4	2.6	7.2	0.5	2.2	3.1	5.2	2.6
74	4.3	2.7	7.2	5.5	4.2	3.4	3.7	1.9	2.9	6.8
75	4.3	5.0	6.6	7.0	0.7	3.2	0.8	1.8	3.7	5.7
76	3.1	5.7	0.8	2.0	5.3	4.9	6.4	6.9	0.6	4.0
77	2.1	4.4	3.6	6.2	6.8	7.3	2.1	3.0	1.7	1.8
78	6.7	2.9	3.7	3.1	2.4	3.9	3.2	2.8	5.2	6.4
79	5.3	5.1	4.6	5.7	3.9	2.4	3.1	2.8	6.2	3.2
80	3.7	0.8	4.4	6.8	3.8	3.7	3.9	2.7	3.6	2.5
81	4.1	7.1	4.9	7.2	2.7	4.2	1.0	5.4	3.8	3.0
82	4.3	1.7	0.6	5.8	6.8	4.0	5.1	2.4	3.6	3.7
83	2.4	6.8	1.4	6.3	6.1	4.6	2.5	6.8	6.9	5.6
84	6.2	0.8	5.1	3.6	0.5	4.7	4.6	5.4	7.0	1.0
85	0.8	1.1	2.1	0.9	6.4	5.8	1.7	1.3	5.0	4.8
86	4.5	6.9	4.1	2.8	5.7	7.0	6.9	5.1	3.7	3.6
87	7.5	2.4	3.9	2.5	2.8	1.3	6.7	7.0	6.2	3.5
88	4.8	6.5	4.1	3.7	4.2	7.3	2.0	2.8	1.1	6.5
89	7.1	2.5	3.3	1.0	4.8	5.8	4.9	1.5	3.4	1.1
90	4.6	0.8	2.9	5.4	2.8	5.3	6.2	6.3	2.1	2.3
91	2.3	6.3	5.6	3.3	6.9	6.7	7.5	5.8	2.0	1.1
92	2.1	5.9	7.5	4.9	3.8	2.1	3.3	1.2	3.1	3.5
93	1.7	4.6	3.8	1.5	2.5	3.4	3.6	5.6	2.2	5.2
94	4.7	1.4	1.4	2.8	4.2	5.2	6.4	4.1	3.4	0.8
95	2.1	5.1	3.6	6.4	6.9	6.9	1.0	7.0	4.8	3.6
96	3.0	3.4	5.1	3.7	4.4	7.1	6.6	0.5	2.6	5.7
97	2.6	6.8	6.6	0.8	6.5	0.6	5.5	5.4	1.6	1.4
98	3.0	0.5	3.4	6.1	6.4	7.1	2.7	4.3	6.6	7.1
99	7.4	6.5	2.9	5.0	2.7	4.2	2.0	6.1	4.2	4.1
100	4.3	3.6	0.7	0.8	3.9	6.8	0.9	6.3	4.6	6.3

Table 9. The results for Example 3 under Condition 2.ii

Action	card(B_k)=1 (C_{pc}, C_{pd})	card(B_k)=2 (C_{pc}, C_{pd})	card(B_k)=3 (C_{pc}, C_{pd})	card(B_k)=4 (C_{pc}, C_{pd})
1	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)
2	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
3	(C_1, C_2)	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)
4	(C_2, C_3)	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)
5	(C_1, C_2)	(C_1, C_2)	(C_2, C_2)	(C_2, C_2)
6	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)
7	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
8	(C_2, C_3)	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)
9	(C_2, C_3)	(C_2, C_3)	(C_3, C_3)	(C_3, C_3)
10	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
11	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
12	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)
13	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)
14	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
15	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
16	(C_1, C_2)	(C_1, C_2)	(C_2, C_2)	(C_2, C_2)
17	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
18	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)
19	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)
20	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)
21	(C_2, C_4)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
22	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
23	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
24	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
25	(C_2, C_4)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
26	(C_2, C_3)	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)
27	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
28	(C_1, C_3)	(C_1, C_3)	(C_2, C_3)	(C_2, C_3)
29	(C_2, C_4)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
30	(C_2, C_3)	(C_2, C_3)	(C_2, C_2)	(C_2, C_2)
31	(C_1, C_2)	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)
32	(C_1, C_3)	(C_1, C_2)	(C_1, C_2)	(C_1, C_2)
33	(C_3, C_4)	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)
34	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)
35	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
36	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)
37	(C_3, C_4)	(C_3, C_4)	(C_3, C_4)	(C_3, C_3)
38	(C_1, C_3)	(C_1, C_3)	(C_2, C_3)	(C_2, C_3)
39	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
40	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_2)
41	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
42	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
43	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
44	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)
45	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)
46	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
47	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
48	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
49	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
50	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)
51	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)
52	(C_1, C_2)	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)
53	(C_2, C_3)	(C_2, C_3)	(C_2, C3)	(C_2, C_3)
54	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
55	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
56	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
57	(C3, C3)	(C3, C3)	(C3, C3)	(C3, C3)
58	(C2, C3)	(C2, C3)	(C2, C3)	(C2, C3)
59	(C2, C4)	(C2, C4)	(C2, C3)	(C2, C3)
60	(C2, C2)	(C2, C2)	(C2, C2)	(C2, C2)
61	(C2, C3)	(C2, C3)	(C2, C3)	(C2, C3)
62	(C2, C3)	(C2, C2)	(C2, C2)	(C2, C2)
63	(C2, C4)	(C2, C3)	(C2, C3)	(C2, C3)
64	(C2, C3)	(C2, C3)	(C2, C3)	(C3, C3)
65	(C3, C3)	(C3, C3)	(C3, C3)	(C3, C3)
66	(C2, C3)	(C2, C2)	(C2, C2)	(C2, C2)
67	(C2, C3)	(C2, C3)	(C2, C3)	(C2, C3)
68	(C1, C3)	(C2, C3)	(C2, C3)	(C2, C3)
69	(C2, C3)	(C2, C3)	(C2, C3)	(C2, C3)
70	(C1, C3)	(C1, C3)	(C2, C3)	(C2, C3)
71	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
72	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
73	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
74	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
75	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)

76	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
77	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
78	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
79	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
80	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)
81	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
82	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
83	(C_2, C_4)	(C_2, C_4)	(C_2, C_4)	(C_3, C_3)
84	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
85	(C_1, C_3)	(C_1, C_2)	(C_1, C_2)	(C_2, C_2)
86	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)
87	(C_2, C_4)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
88	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
89	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
90	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
91	(C_2, C_4)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
92	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
93	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)	(C_2, C_2)
94	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_2)
95	(C_2, C_4)	(C_2, C_3)	(C_3, C_3)	(C_3, C_3)
96	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
97	(C_1, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)
98	(C_2, C_4)	(C_2, C_4)	(C_3, C_3)	(C_3, C_3)
99	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)	(C_3, C_3)
100	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)	(C_2, C_3)