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Capacitated Multi-Layer Network Design with Unsplittable Demands: Polyhedra and Branch-and-Cut

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Abstract: **We consider the *Capacitated Multi-Layer Network Design with Unsplittable demands* (CMLND-U) problem. Given a two-layer network and a set of traffic demands, this problem consists in installing minimum cost capacities on the upper layer so that each demand is routed along a unique ”virtual” path (even using a *unique* capacity on each link) in this layer, and each installed capacity is in turn associated a ”physical” path in the lower layer. This particular hierarchical and unsplittable requirement for routing arises in the design of optical networks, including optical OFDM based networks. In this paper, we give an ILP formulation to the CMLND-U problem and we take advantage of its sub-problems to provide a partial characterization of the CMLND-U polytope including several families of facets. Based on this polyhedral study, we develop a branch-and-cut algorithm for the problem and show its effectiveness though a set of experiments, conducted on random, realistic and real instances.**

Key words: Multi-layer network design, optical networks, polytope, facet, branch-and-cut

1 Introduction

User demand in traffic has increased significantly during the last decades. Nowadays telecommunication networks are already reaching their limits, and it is necessary to upgrade their transport capacity. Indeed, the arising of new services, mainly driven by internet applications and multimedia contents, requires more flexible and cost-effective network infrastructures. To overcome this explosive growth of traffic (estimated at 45 % per year in average (*23*)), telecommunication industry actors investigate new technologies that provide a solution to the increasing capacity requirements, as well as the flexibility needed to use smartly this capacity.

Telecommunication networks can be seen as an overlapping of multiple layers, upon which different services may be furnished. In particular, optical fibers networks consist of two layers : a physical layer and a virtual layer. The physical layer is based on optical fibers, while the virtual one supports the WDM (Wavelength Division Multiplexing) technology. Such a process is based on a set of devices referred to as multiplexers, interconnected by optical links, made of several wavelengths. Both layers are connected, as the wavelengths of the virtual layer use the optical fibers of the physical layer as a support to carry the customers traffic.

Although WDM technology is currently used to transport informations over long distances (metropolitan areas, submarine communication cables), with wavelength capacities of 2.5, 10 or 40Gb/s, it is not possible to reach similar distances with higher capacities. In fact, the existence of physical phenomena, also called transmission impairments (*13*), that affect the optical fibers, highlights the difficulty of setting up higher capacitated wavelengths on long distances. Recent innovations in optical fibers comunications concerning a new technology called Multi-band Orthogonal Frequency Division Multiplexing (OFDM) have shown very promising results, and should enable the transition of WDM-based infrastructures to high capacitated wavelengths (100 Gb/s and more) over long distances. OFDM is based on the division of each available wavelength into many subwavelengths, also called subbands, this is known as *Optical Multi-band OFDM network*.

Now consider an optical multi-band OFDM network that consists of an OFDM/WDM network over a fiber layer. The OFDM/WDM layer is called *virtual layer* and the fiber layer is called *physical layer* as well. The OFDM/WDM layer is composed of devices called *Reconfigurable*

Optical Add-Drop Multiplexers (ROADM), which are interconnected by virtual link. A virtual link may receive one or many OFDM subbands. Note that, although a subband is said to be installed over a virtual link, it is in fact generated by a pair of ROADMs at the extremities of the link. The physical layer is composed of several transmission nodes interconnected by physical links. Each physical link contains two optical fibers, so that the traffic can be transported in both directions. The physical and virtual layers are communicating via an interface referred to as OEO (Optical-Electrical-Optical) interface.

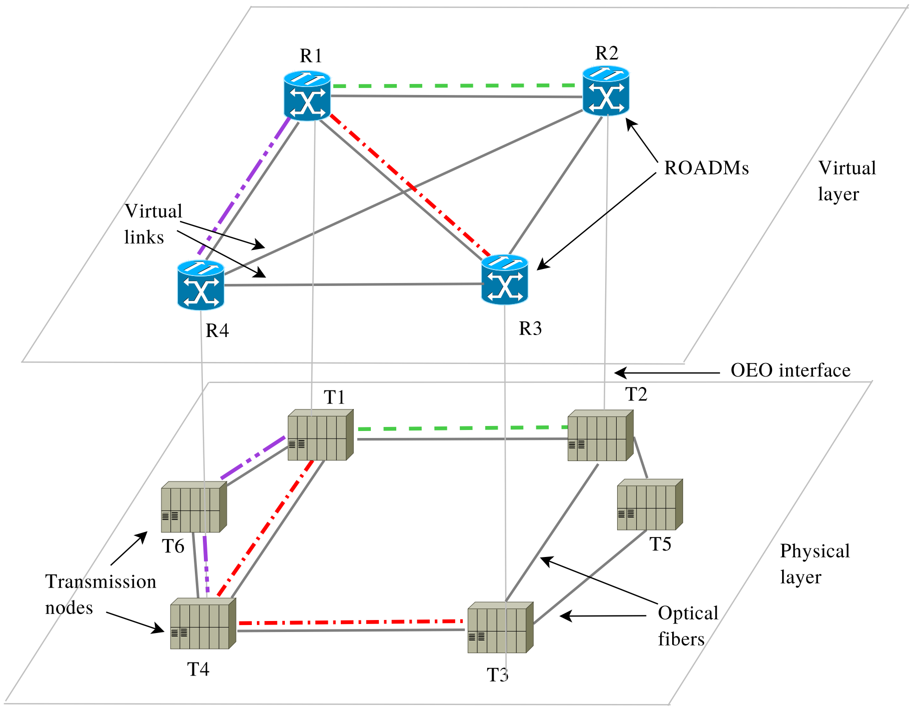


Figure 1: Example of multilayer network

Each ROADM in the virtual layer is associated with a transmission node in the physical layer. And every link in the virtual layer carries one or several subbands. We suppose that there exists a link between each pair of ROADMs in the virtual layer, as one or many subbands may eventually be installed between any pair of devices. Each subband installed over a virtual link is assigned a path in the physical layer. A link in the physical layer can be assigned to several different subbands. However, due to technical aspects of OFDM technology, a physical link can be assigned at most once to an installed subband. In practice, one or many ROADMs may be installed upon a transmission node. However, we assume that all the subbands installed over a virtual link are produced by a unique pair of ROADMs, set up on the extremities of this link. In addition, establishing a subband yields a certain cost, which is the cost of ROADMs that generate this subband. We assume that we have a traffic matrix, where each element is a point-to-point traffic demand that may correspond to a given service, internet application or a multimedia content. This traffic demand has a value that is an amount of informations measured in Mb/s or Gb/s.

Figure 1 shows a bilayer network. The virtual layer includes four ROADMs denoted R_1, R_2, R_3 and R_4, while physical layer contains six transmission nodes denoted T_1 to T_6. We can see that R_1, R_2, R_3 and R_4 are connected to T_1, T_2, T_3 and T_4 via OEO interfaces. In addition, there exists a link between each pair of installed ROADMs. Remark that nodes R_5 and R_6 have not been represented in the figure, as they do not carry any ROADM. Furthermore, three subbands are represented in the figure, respectively installed on the links (R_1, R_2), (R_1, R_3) and (R_1, R_4). The traffic using these virtual links is in fact transmitted through paths made of optical fibres in the physical layer. Indeed, the link (R_1, R_2) is associated with the path (T_1, T_2), while (R_1, R_3) is assigned the path (T_1, T_4), (T_4, T_3) and (R_1, R_4) is physically routed by (T_1, T_6), (T_6, T_4). It should be pointed out that there are two levels of routing in such networks. The traffic is routed using subbands installed on the virtual links, and the subbands themselves may be seen as demands for the physical layer. Thus, when given those two layers of network and a traffic matrix, one may determine the set of virtual links that will receive the subbands, and the set of physical links involved in the routing of those subbands, and establish the traffic commodities routing.

In this context, we are interested in a problem related to the design of OFDM/WDM networks. Thereby, assume that we are given an optical fiber layer, an OFDM/WDM layer and a traffic matrix. The *Capacitated Multi-Layer Network Design with Unsplittable Demands* (CMLND-U) problem consists in determining the number of subbands to be installed over the virtual links, and their physical path as well, so that the traffic can be routed on the virtual layer and the cost of the design is minimum. This work was initially motivated by a collaboration with Orange Labs, whose engineers are also interested in evaluating the performances of OFDM-based networks. For this reason, and throughout the paper, we will use this context to explain our model and the results we will provide.

Actually, the problem of designing layered networks have been studied first by (*15*). Authors wish to set up a set of virtual links referred to as "pipes" on the physical layer. They propose an integer linear programming formulation based on cut constraints for the problem. They study the associated polytope and provide several classes of valid inequalities that define facets under some conditions which are described. They also provide a cutting planes based algorithm embedding their theoretical results. Further works consider exact methods for different variants of the multilayer network design. In fact, in (*27*), Orlowski et *al.* propose a cutting planes approach for solving two-layer network design problems, using different MIP-based heuristic allowing to find good solutions early in the Branch-and-Cut tree. Belotti et *al.* (*7*) investigate the design of multilayer networks using MPLS technology. They propose a mathematical programming formulation based on paths, then apply a Lagrangian relaxation working with a column generation procedure to solve their model. We also cite a more recent work of Raghavan and Stanojević (*32*) that study the two-layer network design arising in WDM optical networks. They consider the non-splittable traffic demands and propose a path based formulation for the problem. They provide an exact Branch-and-Price algorithm which solves simultaneously the WDM topology design and the traffic routing subproblems. In (*28*), the authors address the problem of planning multilayer SDH/WDM networks. They consider the minimum cost installation of link and node hardware for both layers, under various practical constraints such as heterogeneity of traffic bit-rates, node capacities and survivability issues. They propose a mixed integer programming formulation and develop a Branch-and-Cut algorithm using non-trivial valid inequalities, from the single-layer network design problem, to solve it. In (*16*), the authors study the multi-layer network design problem. They propose a Branch-and-Cut algorithm to solve a capacity formulation based on the so-called metric inequalities, enhancing the results obtained in (*22*) for the same formulation. In (*24*), Mattia studies two versions of the two-layer network design problem. In particular, the author proposes capacity formulations for both versions and investigates the associated polyhedra. Some polyhedral results are provided for both versions of the problem, specifically proving that the so-called tight metric inequalities, introduced in (*5*), define all the facets of the considered polyhedra. The author shows how to extend these polyhedral results to an arbitrary number of layers. In (*12*), Borne et *al.* study the problem of designing an IP-over-WDM network with survivability against failures of the links. They conduce a polyhedral study of the problem, give several facet defining valid inequalities, and propose a Branch-and-Cut algorithm to solve the problem.

Our contribution

The capacitated design of single-layer networks has received a lot of attention in the literature, and a big amount of research has been conducted on the associated polyhedron. Yet the investigation of capacitated multilayer network design problems received only a limited attention, specifically in a polyhedral point of view. The objective of this paper is to investigate the CMLND-U problem within a polyhedral framework, and to provide an efficient Branch-and-Cut algorithm to solve it. In this context, we give an integer linear programming formulation for the problem and study the polyhedron associated with its solutions. We then introduce further classes of valid inequalities and study their facial structure. These inequalities are used within an efficient Branch-and-Cut algorithm for the CMLND-U problem.

The rest of the paper is organized as follows. In Section 2, we describe the CMLND-U problem in terms of graphs and give an ILP formulation to model it. In Section 3, we present the CMLND-U polyhedron and study its basic properties. We then introduce several classes of facet-defining valid inequalities. These results are used to devise a Branch-and-Cut algorithm which is described in Section 4. Several series of experiments are conducted and Section 5 is devoted to present a summary of the obtained numerical results. Finally, we give some concluding remarks in Section 6.

2 The capacitated multi-layer network design problem with unsplittable demands

2.1 Definition and notations

In terms of graphs, the CMLND-U problem can be presented as follows. We associate with the virtual layer, a directed graph $G_1 = (V_1, A_1)$. G_1 is a complete graph where V_1 is the set of nodes and A_1 the set of arcs. Each node $v \in V_1$ corresponds to a ROADM and each arc $e \in A_1$ corresponds to a virtual link between a pair of ROADMs. In addition, G_1 is a bi-directed graph, i.e. there exists two arcs $(u, v) \in A_1$ and $(v, u) \in A_1$, connecting each pair of nodes u and v of V_1. Consider the directed graph $G_2 = (V_2, A_2)$ that represents the physical layer of the optical network. V_2 denotes the set of nodes and A_2 is the set of arcs. Each node $v' \in V_2$ corresponds to a transmission node and each arc $a \in A_2$ corresponds to an optical fibre. Every node u in V_1 has its corresponding node u' in V_2. The graph G_2 is such that if there exists an arc (u', v') between two nodes u' and v' of V_2, then (v', u') is also in A_2. In this way, the link can be used in both directions between u' and v'.

Suppose that we have $n \in \mathbb{Z}^+$ available subbands. We denote by $W = \{1, 2, ..., n\}$, the set of indices associated with these subbands. Every subband $w \in W$ has a certain capacity C and a cost $c(w) > 0$. Moreover, a subband installed over an arc $e \in A_1$ can be seen as a copy of this arc. Each pair (e, w) such that w is installed over the arc $e = (u, v)$, is associated with a path in G_2 connecting nodes u' and v'. The same path in G_2 may be assigned to different subbands of W. Nevertheless, an arc $a \in A_2$ can be associated at most once with a given subband w. In other words, if the subband w is installed p times, $p \in \mathbb{Z}^+$, over different arcs $e_1, \dots, e_p$ of A_1, then the pairs (e_i, w), $i = 1, \dots, p$, have to be assigned p paths in G_2 that are arc-disjoint. This comes from an engineering restriction and will be called disjunction constraint. In addition to the design cost, we will also attribute a physical routing cost denoted $b^{ew}(a)$ for every arc a of A_2 involved in the routing of a pair (e, w) such that w is installed on e.

Now let K be a set of commodities in G_1. Each commodity $k \in K$ has an origin node $o_k \in V_1$, a destination node $d_k \in V_1$ and a traffic value $D_k > 0$. We suppose, that $D^k \leq C$, for all $k \in K$. Note that there might exist different commodities with the same origin and destination. A routing path in G_1 has to be assigned to each commodity $k \in K$ connecting its origin and its destination. Every section of a routing path uses the subbands installed over the arcs of A_1. Thereby, we will say that a pair (e, w), $e \in A_1$, $w \in W$ is used by a commodity k, if w is installed on e and (e, w) is involved in the routing of k. Furthermore, several commodities are allowed to use the same subband (e, w), if they fit in its capacity. However, one commodity can not be split into several subbands or several paths.

Definition 1. *Capacitated Multi-Layer Network Design with Unsplittable demands (CMLND-U)* *problem: Given two bi-directed graphs G_1 and G_2, a set of subbands W, the installation cost $c(w)$ for each subband w, and a set of commodities K, determine a set of subbands to be installed over the arcs of G_1 such that*

(i) the commodities can be routed in G_1 using these subbands,

(ii) paths in G_2, respecting the disjunction constraint, are associated with the installed subbands,

(iii) the total cost is minimum.

2.2 Integer linear programming formulation

Given a digraph G = (V, A) and a node set $T \subset V$, we denote by $\delta^+_{G(T)}$ (resp. $\delta^-_{G(T)}$), the set of arcs of A having their initial node (resp. terminal node) in T and their terminal node (resp. initial node) in $V \setminus T$, that is to say $\delta^+_{G(T)}$ = $\{a = (u, v) \in A$ with $u \in T$ and $v \notin T\}$.

Now we will present an integer linear programming formulation using three sets of variables. First, let the *design variables* $y \in \mathbb{R}^{A_1 \times W}$ be such that, for each arc $e \in A_1$ and for each subband $w \in W$, y_{ew} takes the value 1, if w is installed on e, and 0 otherwise. Let the *routing in G_2 variables* $z \in \mathbb{R}^{A_1 \times W \times A_2}$ be such that for each arc $e \in A_1$, for each subband $w \in W$ and for each arc $a \in A_2$, z_{ewa} takes the value 1 if a belongs to the path in G_2 associated with pair (e, w), and 0 otherwise. Finally, we denote by $x \in \mathbb{R}^{K \times A_1 \times W}$ the routing variables such that for each commodity $k \in K$, for each arc $e \in A_1$ and for each subband $w \in W$, x_{kew} takes the value 1 if k uses (e,w) for its *routing in G_1*, and 0 otherwise.

An instance of CMLND-U is defined by the quadruplet (G_1, G_2, K, C). Let $\mathcal{S}(G_1, G_2, K, C)$ denote the set of feasible solutions of the CMLND-U problem, associated with an instance (G_1, G_2, K, C). The CMLND-U problem is then equivalent to the following ILP:

$$\min \sum_{e\in A_1}\sum_{w\in W} c(w)y_{ew} + \sum_{e\in A_1}\sum_{w\in W}\sum_{a\in A_2} b^{ew}(a)z_{ewa} \tag{1}$$

$$\sum_{e\in\delta^+_{G_1}(T)}\sum_{w\in W} x_{kew} \geq 1, \qquad \begin{array}{l}\forall k\in K, \forall T\subset V_1,\\ \emptyset\neq T\neq V_1, o_k\in T, d_k\notin T,\end{array} \tag{2}$$

$$\sum_{k\in K} D^k x_{kew} \leq C y_{ew}, \qquad \forall e\in A_1, \forall w\in W, \tag{3}$$

$$\sum_{a\in\delta^+_{G_2}(T)} z_{ewa} \geq y_{ew}, \qquad \begin{array}{l}\forall e=(u,v)\in A_1, \forall w\in W,\\ \forall T\subset V_2, \emptyset\neq T\neq V_2, u'\in T, v'\notin T,\end{array} \tag{4}$$

$$\sum_{e\in A_1} z_{ewa} \leq 1, \qquad \forall w\in W, \forall a\in A_2, \tag{5}$$

$$x_{kew}\in\{0,1\}, \qquad \forall k\in K, \forall e\in A_1, \forall w\in W, \tag{6}$$

$$y_{ew}\in\{0,1\}, \qquad e\in A_1, \forall w\in W, \tag{7}$$

$$z_{ewa}\in\{0,1\}, \qquad \forall e\in A_1, \forall w\in W, \forall a\in A_2. \tag{8}$$

Inequalities (2) are the *cut constraints*. They will also be referred to as *connectivity constraints*. They ensure that a path in G_1 exists for each commodity k between nodes o_k and d_k. Inequalities (3) are the *capacity constraints* for each subband installed over an arc of G_1. They express the fact that the flow using the subband w on arc e does not exceed the capacity of w. They also ensure that the overall capacity installed on arc e is large enough to carry the traffic using e. Inequalities (4) are the *subband connectivity constraints*. They guarantee, for each pair (e,w) where w is installed on $e = (u,v)$, that a path in G_2 is associated with (e,w) between nodes u' and v'. Inequalities (5) are referred to as *disjunction constraint*. Finally, inequalities (6)-(8) are the *integrality constraints*.

3 Associated polyhedron and valid inequalities

In this section, we introduce and discuss the CMLND-U polytope, that is the convex hull of the solutions of problem (1)-(8). In what follows, we will assume that $G_2 = (V_2, A_2)$ is also a complete graph. This is a reasonable assumption, since the problem when G_2 is not complete can be reduced to the case when G_2 is complete by introducing dummy arcs with large costs. We also make the assumption that the number $|W|$ of available subbands is sufficiently large for allowing the routing of all commodities over a single arc $e \in A_1$, if this necessary. Note that such an assumption is resonable because the maximum number of subbands that can be potentially installed in practice is indeed large regarding to the number of commodities. Of course, the costs will prevent the installation of unnecessary subbands.

Given an instance of CMLND-U, defined by the quadruplet (G_1, G_2, K, C), we denote by $P(G_1, G_2, K, C)$ the convex hull of the incidence vectors of $\mathcal{S}(G_1, G_2, K, C)$, that is

$$P(G_1, G_2, K, C) := conv\{(x, y, z) \in \mathbb{R}^{K \times A_1 \times W} \times \mathbb{R}^{A_1 \times W} \times \mathbb{R}^{A_1 \times W \times A_2} :$$
$$(x, y, z) \quad \text{satisfies (2)} - \text{(8)}\}$$

In what follows, we will characterize the dimension of polytope $P(G_1, G_2, K, C)$ and investigate the facial aspect of inequalities (2)-(8).

Theorem 1. *$P(G_1, G_2, K, C)$ is full dimensional.*

Proof. See Appendix A. □

3.1 Capacitated Cutset Inequalities

Consider a partition of G_1 nodes in two subsets T and $\overline{T} = V_1 \setminus T$, we denote by $K(T)$ (respectively $K(\overline{T})$) the commodities of K having their origin and destination nodes in the subset T (respectively in $\overline{T}$), while the remaining subset of K will be denoted by $K^+(T)$ and $K^-(T)$. Note that $K^+(T)$ (respectively $K^-(T)$) is the subset of commodities having their origin node in T (respectively in $\overline{T}$) and their destination node in $\overline{T}$ (respectively in T). We will also denote by $D(K^+(T))$ the total traffic amount of $K^+(T)$. In other words, $D(K^+(T)) = \sum_{k \in K^+(T)} D^k$, and $D(K^-(T)) = \sum_{k \in K^-(T)} D^k$. We further denote by $BP(K^+(T))$ (resp. $BP(K^-(T))$) the smallest number of subbands required in $\delta^+_{G_1}(T)$ to route the commodities of $K^+(T)$. Actually, this value corresponds to the optimal solution of the *bin packing* problem with $K^+(T)$ (resp. $K^-(T)$) being the set of items to be packed and C the capacity of a bin. For example, if $K^+(T)$ is composed by 3 demands with 6 units of traffic and $C = 10$, then $BP(K^+(T)) = 3$. Again, this happens because the traffic of a commodity can not be split into distinct subbands, even if they are installed in the same arc.

Proposition 1. *Let $\emptyset \neq T \subsetneq V_1$, then the following inequality*

$$\sum_{e \in \delta^+_{G_1}(T)} \sum_{w \in W} y_{ew} \geqslant \lceil \frac{D(K^+(T))}{C} \rceil \tag{9}$$

is valid for $P(G_1, G_2, K, C)$.

Proof. The total capacity of the subbands installed over the cut must be greater than or equal to the traffic amount of the commodities going from T to $\overline{T} = V_1 \backslash T$ and using the arcs of that cut. Then, inequality

$$C \sum_{e \in \delta^+_{G_1}(T)} \sum_{w \in W} y_{ew} \geqslant D(K^+(T))$$

is clearly valid for $P(G_1, G_2, K, C)$. By dividing this by C and rounding up the right-hand side, we obtain inequality (9). □ □

Theorem 2. *Inequalities* (9) *define facets of* $P(G_1, G_2, K, C)$ *only if* $\lceil \frac{D(K^+(T))}{C} \rceil = BP(K^+(T))$.

Proof. Given two nodes u and v of V_1, we let $K(u, v)$ be the set of commodities such that $K(u, v) = \{k \in K : o_k = u, d_k = v\}$. Now suppose that $\lceil \frac{D(K(u,v))}{C} \rceil < BP(K(u, v))$. In this case, (9) can not be tight, since the commodities of $K(u, v)$ can not fit in $\lceil \frac{D(K(u,v))}{C} \rceil$ subbands, and thus (9) could not induce a proper face. □ □

Example 1. *For example, if* $K(u, v) = \{k_1, k_2, k_3\}$ *with* $D^{k_1} = D^{k_2} = D^{k_3} = 6$*, while* $C = 10$*. In this case, there is no solution of* $P(G_1, G_2, K, C)$ *such that* $\sum_{e \in \delta^+_{G_1}(T)} \sum_{w \in W} y_{ew} = \lceil \frac{K(u,v)}{C} \rceil = 2$*.*

Theorem 3. *Inequalities* (9) *define facets of* $P(G_1, G_2, K, C)$ *if the following conditions hold.*

(i) $\lceil \frac{D(K^+(T))}{C} \rceil = BP(K^+(T))$,

(ii) $BP(K^+(T) \cup \{k\}) = BP(K^+(T))$, *for all* $k \in K \setminus K^+(T)$,

(iii) for all $k' \in K^+(T)$*, there exists* $k'' \in K^+(T)$ *such that* $D^{k'} + D^{k''} \leqslant C$,

(iv) for all $k \in K^+(T)$, $BP(K^+(T) \setminus \{k\}) = BP(K^+(T))$ - 1.

Proof. See Appendix B. □

3.2 Flow Cutset inequalities

In what follows, we will describe a set of valid inequalities for $P(G_1, G_2, K, C)$ that are a generalization of the capacitated cutset inequalities (9). Similar inequalities have been introduced by (*14*) and were discussed in (*3*), (*10*) and (*31*) for network design problems where discrete modular capacities are installed on the arcs of the graph.

Proposition 2. *Consider a non empty subset of nodes* $T \subseteq V_1$ *and a partition* $F, \overline{F}$ *of the cut* $\delta^+_{G_1}(T)$ *induced by* T. *The following flow-cutset inequalities*

$$\sum_{w\in W}\sum_{e\in F} y_{ew} + \sum_{w\in W}\sum_{e\in \overline{F}}\sum_{k\in K^+(T)} x_{kew} \geqslant \lceil \frac{D(K^+(T))}{C} \rceil. \tag{10}$$

are valid for $P(G_1, G_2, K, C)$.

Proof. It is clear that the following inequalities

$$\sum_{w\in W}\sum_{e\in \delta^+_{G_1}(T)} x_{kew} \geqslant 1, \quad \text{for all } k \in K^+(T),$$

are valid for $P(G_1, G_2, K, C)$, as they express the connectivity constraints for the commodities of $K^+(T)$. Multiplying both sides of this inequality by D^k and summing over $K^+(T)$ yields

$$\sum_{w\in W}\sum_{e\in \delta^+_{G_1}(T)}\sum_{k\in K^+(T)} D^k x_{kew} \geqslant D(K^+(T)), \tag{11}$$

In addition, we have from the capacity constraints (3), restricted to the commodities of $K^+(T)$ and the arcs of F, that

$$\sum_{k\in K^+(T)} D^k x_{kew} - C y_{ew} \leqslant 0, \quad \text{for all } e \in F, w \in W.$$

By summing these inequalities, we obtain

$$\sum_{w\in W}\sum_{e\in F} C y_{ew} - \sum_{w\in W}\sum_{e\in F}\sum_{k\in K^+(T)} D^k x_{kew} \geqslant 0. \tag{12}$$

As $\delta^+_{G_1}(T) = F \cup \overline{F}$, by summing (11), (12), and dividing by C, we get

$$\sum_{w\in W}\sum_{e\in F} y_{ew} + \sum_{w\in W}\sum_{e\in \overline{F}}\sum_{k\in K^+(T)} \frac{D^k}{C} x_{kew} \geqslant \frac{D(K^+(T))}{C}, \tag{13}$$

Moreover, we have the following trivial inequality

$$\sum_{w\in W}\sum_{e\in \overline{F}}\sum_{k\in K^+(T)} (1 - \frac{D^k}{C}) x_{kew} \geqslant 0, \tag{14}$$

By summing (13), (14) and rouding up the right-hand side, we get (10) . □ □

Theorem 4. *A flow-cutset inequality* (10) *defines a facet of* $P(G_1, G_2, K, C)$, *different from* (9) *only if the following hold* *(i)* $F \neq \emptyset \neq \overline{F}$,

(ii) $D(K^+(T)) > C$ *and* $D(K^+(T))$ *is not a multiple of* C,

(iii) $\lceil \frac{D(K^+(T))}{C} \rceil = BP(K^+(T))$,

(iv) $BP(K^+(T)) < |K^+(T)|$,

(v) *there exists* $q \subsetneq K^+(T)$ *such that* $BP(K^+(T) \setminus q) \leqslant BP(K^+(T)) - |q|$,

(vi) $BP(K^+(T) \cup \{k\}) \leqslant BP(K^+(T))$, *for all* $k \in K \setminus K^+(T)$.

Proof. See Appendix C. □

Theorem 5. *A flow-cutset inequality* (10) *defines a facet of* $P(G_1, G_2, K, C)$, *different from* (9) *if the following conditions are satisfied*

(i) *conditions* (i) *to* (vi) *of Theorem 4,*

(ii) *if* $|F| = 1$, *for each* $k \in K^+(T)$, $BP(K^+(T) \setminus \{k\}) \leqslant BP(K^+(T)) - 1$,

(iii) *if* $|F| = 1$, *for each* $k \in K^+(T)$, *there exists* $k' \in K^+(T)$ *such that* $D^k + D^{k'} \leqslant C$.

Proof. Similar to proof of Theorem 3. □

3.3 Clique inequalities

In what follows, we will study an additional class of inequalities that are valid for $P(G_1, G_2, K, C)$. These inequalities are based on the so-called *clique inequalities* introduced by Padberg (*29*) in the context of the *stable set polytope* investigation. Similar inequalities have also been studied in (*6*) for the *Balanced Induced Subgraph problem*. More generally, clique inequalities arise in problems where *conflicts* may occur between objects (see (*11, 20*)). In order to identify these facet-defining inequalities, we will introduce first the concept of conflict graph.

Definition 2. *Given an instance of the CMLND-U problem, we consider a graph* $H = (V, E)$, *called conflict graph where each node of* V *is associated with a commodity in* K *and two commodities* k_1, k_2 *are connected by an edge* $k_1k_2 \in E$ *if and only if* k_1 *and* k_2 *cannot be packed in a subbband together. In other words, there exists an edge* k_1k_2 *in* E *if and only if* $D^{k_1} + D^{k_2} > C$.

A *clique* $\mathcal{C} \subseteq N$ in a graph is a set of nodes such that every two distinct nodes in $\mathcal{C}$ are adjacent. A clique $\mathcal{C}$ is said to be *maximal* if it is not strictly contained in a clique.

We have the following.

Proposition 3. *Let $\mathcal{C} \subseteq K$ be a clique in the conflict graph H, and $(e, w) \in A_1 \times W$, then the following clique-based inequality*

$$\sum_{k \in \mathcal{C}} x_{kew} - y_{ew} \leqslant 0, \tag{15}$$

is valid for $P(G_1, G_2, K, C)$.

Proof. It is clear that if a subband w is installed on e, then at most one commodity of $\mathcal{C}$ can be routed on e using w. If not, then x_{kew} = 0 for all $k \in \mathcal{C}$, and the constraint is trivially satisfied. □

3.4 Cover inequalities

A *cover* $\mathcal{I} \subseteq K$ is a subset of commodities such that $\sum_{k \in \mathcal{I}} D^k > C$. A cover is said to be *minimal* if it does not contain any cover as a subset.

Proposition 4. *Consider an arc $e \in A_1$, a subband $w \in W$ and a subset of commodities $\mathcal{I} \subseteq K$ defining a cover. Then, the following inequality*

$$\sum_{k \in \mathcal{I}} x_{kew} \leqslant (|\mathcal{I}|\text{-}1) y_{ew} \tag{16}$$

is valid for $P(G_1, G_2, K, C)$.

Proof. If y_{ew} = 0, then it is clear that no commodity can use e and w, that is to say x_{kew} = 0, for all $k \in K$, in particular for all $k \in \mathcal{I}$. Now suppose that y_{ew} = 1, and $\sum_{k \in \mathcal{I}} x_{kew} \geqslant (|\mathcal{I}|\text{-}1) y_{ew}$ + 1 = $|\mathcal{I}|$. This means that all the commodities of $\mathcal{I}$ use (e,w). In other words, x_{kew} = 1, for all $k \in \mathcal{I}$, but this is impossible since $\sum_{k \in \mathcal{I}} D^k > C$. □

Cover inequalities define facets under some conditions for the knapsack polytope (see (*26*, *33*)). They should also define facets for $P(G_1, G_2, K, C)$ polytope with appropriate additional conditions. Furthermore, facets based on covers and extensions of covers may be derived by using lifting procedure (see (*19*,*30*)).

3.5 Min set I inequalities

We introduce here a further class of valid inequalities induced by a subset of commodities for each arc. This class of inequalities has been described first in (*8*) for the unsplittable non-additive capacitated network design (UNACND) problem. They have been identified using the fact that the single arc UNACND problem reduces to the bin packing problem.

Proposition 5. *Given a subset $S \subseteq K$ of commodities and a non negative integer $p \in \mathbb{Z}^+$, inequalities*

$$\sum_{w \in W} \sum_{k \in S} x_{kew} \leqslant \sum_{w \in W} y_{ew} + p, \text{ for all } e \in A_1, \tag{17}$$

are valid for $P(G_1, G_2, K, C)$ if and only if $p \geqslant |S| - BP(S)$.

where $BP(S)$, likewise in inequalities (9) and (10), denotes the minimum number of subbands (with capacity C) necessary to pack the commodities in S.

Proof. The following inequalities

$$\sum_{k \in S} x_e^k \leqslant y_e + p, \text{ for all } e \in A_1, \tag{18}$$

Introduced in (*8*) for the *Unsplittable non-additive capacitated network design problem* are clearly valid for $P(G_1, G_2, K, C)$ if $p \geqslant |S| - BP(S)$. Indeed, by introducing new "aggregated" variables $x_e^k \in \{0, 1\}$, $\forall k \in K$, $\forall e \in A_1$, and $y_e \in \mathbb{Z}^+$, $\forall e \in A_1$, we can use the following transformation $x_e^k = \sum_{w \in W} x_{kew}$ and $y_e = \sum_{w \in W} y_{ew}$. This is possible since a commodity cannot be split over several subbands installed on the same arc $e \in A_1$. Thus, using the original variables to write (18) yields inequality (17). ☐

Theorem 6. *Inequality* (17) *defines a facet of $P(G_1, G_2, K, C)$ if and only if the following hold*

(i) p = $|S|$ - $BP(S)$,

(ii) $BP(S \cup \{s\})$ = $BP(S)$ = $|S|$ - p, *where s is the largest element in $K \setminus S$,*

(iii) $BP(S \setminus \{s\})$ = $BP(S)$ - *1* = $|S|$ - p - *1, where s is the smallest element in S.*

Proof. See Appendix D. ☐

3.6 Min set II inequalities

Likewise Min Set I, this class of inequalities has been presented first in (*8*) and originates from the study of the arc-set UNACND polyhedron.

Proposition 6. *Let S be a subset of K, and p and q, two non negative integer parameters such that $q \geqslant 2$. Then, the inequality*

$$\sum_{w \in W} \sum_{k \in S} x_{ew}^k \leqslant q \sum_{w \in W} y_{ew} + p, \quad \forall e \in A_1 \tag{19}$$

is valid for $P(G_1, G_2, K, C)$ if $p \geqslant (|S'| - qBP(S'))$, for all $S' \subseteq S$.

Proof. Similar to proof of Proposition 5. $\square$

Note that Proposition 5, Theorem 6 and Proposition 6 are adaptation of results in (*8*), where the facial structure of both Min Set I and Min Set II inequalities is investigated in details. The authors give necessary and sufficient conditions for these inequalities to define facets for the arc-set unsplittable non-additive capacitated network design polyhedron.

4 Branch-and-Cut algorithm

In this section we present a Branch-and-Cut algorithm for the CMLND-U problem. Our purpose is to substantiate the efficiency of the valid inequalities described in the previous section, and provide exact solutions for realistic and real instances of networks.

4.1 Overview

We describe the framework of our algorithm. Suppose that we are given two graphs $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$, that instantiate the virtual layer and the physical layer of the network, respectively. Also suppose given a set of commodities K where each commodity k is characterized by a pair $(o_k, d_k) \in V_1 \times V_1$ and a traffic value D^k. We consider a set W of available subbands having a capacity C. A cost vector $c \in R_+^{W \times A_1}$, is given as well.

To start the optimization, we set up the following restricted linear program.

$$
\begin{aligned}
& Min \sum_{e\in A_1}\sum_{w\in W} c(w)y^{ew} + \sum_{e\in A_1}\sum_{w\in W}\sum_{a\in A_2} z_a^{ew} && \\
& s.t: && \\
& \sum_{w\in W}\sum_{e\in\delta^+_{G_1}(s)} x^k_{ew} \geq 1, && \forall k\in K, s\in\{o_k, d_k\}, \\
& \sum_{k\in K} D^k x^k_{ew} \leq Cy^{ew}, && \forall e\in A_1, \forall w\in W, \\
& \sum_{e\in A_1} z_a^{ew} \leq 1, && \forall w\in W, \forall a\in A_2, \\
& 0\leq x^k_{ew}\leq 1, && \forall k\in K, e\in A_1, \\
& 0\leq y^{ew}\leq 1, && \forall w\in W, e\in A_1, \\
& 0\leq z_a^{ew}\leq 1, && \forall e\in A_1, \forall w\in W, \forall a\in A_2.
\end{aligned}
$$

We denote by $(\overline{x},\overline{y},\overline{z})$, $\overline{x}\in\mathbb{R}^{K\times W\times A_1}$, $\overline{y}\in\mathbb{R}^{W\times A_1}$, $\overline{z}\in\mathbb{R}^{W\times A_1\times A_2}$, the optimal solution of the restricted linear relaxation of the CMLND-U problem. This solution is feasible for the problem if $(\overline{x},\overline{y},\overline{z})$ is an integer vector that satisfies all the cut constraints of type (2) and (4). In most of the cases, the solution obtained this way is not feasible for the CMLND-U problem. We then manage to identify, at each iteration of the algorithm, valid inequalities that are violated by the solution of the current restricted linear program. This is referred to as the *separation problem*. Namely, given a class of valid inequalities, the separation problem is to check whether the solution $(\overline{x},\overline{y},\overline{z})$ meets all the inequalities of this class, and, if this is not the case, to find an inequality that is violated by $(\overline{x},\overline{y},\overline{z})$. The detected inequalities are then added to the current linear program, and such procedure is reiterated until no violated inequality can be identified. The algorithm uses then to branch over the fractional variables.

The Branch-and-Cut algorithm includes the inequalities described in the previous chapter, and their separations are accomplished in the following order

1. basic cut constraints ((2), (4))

2. min set I inequalities

3. capacitated cutset inequalities

4. flow-cutset inequalities
5. clique-based inequalities
6. cover inequalities
7. min set II inequalities

Observe that all the inequalities are global (i.e., valid for the whole Branch-and-Cut tree), and several inequalities may be added at each iteration. Furthermore, we move to the next class only if no violated inequalities of the current class are identified. Our strategy is to try to detect violated inequalities at each node of the Branch-and-Cut tree, in order to obtain the best possible lower bound by strengthening the linear relaxation, and thus limit the number of generated nodes.

In the sequel, we describe the separation procedures embedded in our algorithm. We use exact and heuristic algorithms as well, depending on the class of inequalities. Except for cut inequalities (4), all the separation routines are applied on the graph G_1.

4.2 Separation of basic Cut constraints

Algorithm 1: Separation of basic cut inequalities (2)

Data: a vector $(\overline{x}, \overline{y}, \overline{z})$
Result: a set $\mathcal{C}$I of cut inequalities (2) violated by $(\overline{x}, \overline{y}, \overline{z})$
for *each commodity* $k \in K$ **do**
  Associate a weight $c(e, w)$ = $\overline{x}^k_{ew}$ to each pair (e, w)$\in A_1 \times W$;
  Use Goldberg-Tarjan push relabel algorithm (*18*) to find the min cut separating o_k from d_k regarding to the assigned weights;
  Let $\delta^+(\overline{T})$ denote this cut, where $\overline{T} \subsetneq V_1$ ($\overline{T}$ contains o_k but does not contain d_k);
  if *cut inequality* (2) *induced by* $\delta^+(\overline{T})$ *is violated by* $(\overline{x}, \overline{y}, \overline{z})$ **then**
    add this inequality to $\mathcal{C}$I;
  return the identified cut inequalities $\mathcal{C}$I to be added to the current LP;

We used the implementantion of Goldberg and Tarjan algorithm for max flow/min cut available in LEMON GRAPH C++ library (*2*). It has a worst case complexity of $\mathcal{O}(n_1^2\sqrt{m_1})$ where n_1 and m_1 are the number of nodes and arcs of G_1, respectively. Therefore, the exact separation algorithm for cut constraints (2) runs in $\mathcal{O}(n_1^2 t\sqrt{m_1})$, where t = $|K|$.

For the cut constraints (4), we have to solve the separation problem that consists in computing for each pair $(e = (u, v), w) \in A_1 \times W$, such that $\overline{y}_{ew} > 0$, the minimum cut in G_2 separating

u' from v' considering $\overline{z}_{ewa}$ as the arc capacities. Using the same Goldberg and Tarjan min cut algorithm, the full exact separation has complexity $O(n_2^2 m_2 |W| \sqrt{m_2})$.

4.3 Separation of capacitated cutset inequalities

The separation problem associated with the cutset inequalities has been proven NP-hard in general (9). In our case, the separation problem related to capacitated cut-set inequalities (9) is also NP-hard. Therefore, we have developed two heuristics to separate these inequalities, one of which is based on the so-called n-cut heuristic, proposed by Bienstock et al. in (9) for the minimum cost capacity installation for multicommodity network flows. We adapt this heuristic in order to make it suitable for our problem.

This heuristic works as follows. For any commodity $k \in K$, we check whether there is a path in G_1 connecting nodes o_k and d_k, and using only pairs (e, w), $e \in A_1$, $w \in W$ with $\overline{y}^{ew} > 0$. Since this can be performed by any path finding algorithm, we use Dijkstra's algorithm. If such a path does not exist, then it is clear that a capacitated cutset inequality is violated. This inequality is induced by a subset of nodes T such that $o_k \in T$ and $d_k \notin T$. If a path between o_k and d_k is identified in G_1 for each commodity k, then we randomly pick a subset of nodes, say $T \subseteq V_1$, $0 \neq T \neq V_1$, and identify the subset of commodities P^+ having their origin node in T and their destination in $V_1 \setminus T$. After that, we compute the right-hand side, and we check if the constraint thus constructed is violated. Since we check the existence of a path for each commodity between its origin and its destination, the worst-case complexity of this procedure is $\mathcal{O}(|K|(m_1|W| + n_1 log(n_1)))$, where n_1 = $|V_1|$ and m_1 = $|A_1|$.

Algorithm 2: Separation of capacitated cutset inequalities (9)

Data: a vector $(\overline{x}, \overline{y}, \overline{z})$
Result: a set $\mathcal{CCS}$ of capacitated cutset inequalities (9) violated by $(\overline{x}, \overline{y}, \overline{z})$
Associate a weight $c(e, w)$ = $\overline{y}_{ew}$ to each pair (e, w)$\in A_1 \times W$;
for *each commodity* $k \in K$ **do**
  Check if there exists a path in G_1 from o_k to d_k using pairs (e, w) with $\overline{y}_{ew} > 0$;
  if *such path does not exist* **then**
    a capacitated cutset inequality induced by a subset T is violated and must be added to $\mathcal{CCS}$;

if *there is a path for each* k *between* o_k *and* d_k **then**
  Randomly pick a subset of nodes T in V_1;
  Construct the subset of commodities $K^+(T)$ such that $K^+(T)$ = $\{k \in K : o_k \in T$ and $d_k \notin T\}$ **if** $\sum_{e\in\delta^+(T)} \sum_{w\in W} \overline{y}_{ew} < \lceil \frac{D(K^+(T))}{C} \rceil$ **then**
    A violated capacitated cutset inequality is identified and must be added to $\mathcal{CCS}$

return the identified cut inequalities $\mathcal{CCS}$ to be added to the current LP;

In the second separation heuristic, we use Goldberg-Tarjan max-flow algorithm to find violated capacitated cut-set inequalities (9). We attribute to each pair $(e, w) \in A_1 \times W$ the capacity $\overline{y}^{ew}$, and determine for each $k \in K$ a minimum $o^k d^k$-dicut in G_1, say $\delta^+_{G_1}(T^*)$, with $T^* \subseteq V_1$. We then identify the subset of commodities $K^+(T) \subseteq K$ passing through this directed cut. We finally add inequality

$$\sum_{e\in\delta^+_{G_1}(T^*)} \sum_{w\in W} y_{ew} \geq \lceil \frac{D(K^+(T))}{C} \rceil,$$

in case it is violated. This procedure is based on max-flow computations, thus the worst case complexity is $\mathcal{O}(n_1^2 t \sqrt{m_1})$.

4.4 Separation of flow-cutset inequalities

We now discuss our separation procedure for the flow-cutset inequalities (10). Atamtürk shows in (*3*) that the separation problem associated with a more general form of flow-cutset inequalities is NP-hard even for one commodity. In case of a multiple commodity set, the complexity of simultaneously determining $K^+(T)$ and F is not known (*31*). As we do not know an efficient procedure to separate flow-cutset inequalities in general, we use here a simple heuristic based on Goldberg-Tarjan max-flow algorithm.

Algorithm 3: Separation of flow-cutset inequalities (10)

Data: a vector $(\overline{x}, \overline{y}, \overline{z})$
Result: a set $\mathcal{FCS}$ of flow-cutset inequalities (10) violated by $(\overline{x}, \overline{y}, \overline{z})$
for *each commodity* $k \in K$ **do**
  Associate a weight $c(e, w)$ = $\overline{x}^k_{ew}$ + $\overline{y}_{ew}$ to each pair (e, w)$\in A_1 \times W$;
  Use Goldberg-Tarjan push relabel algorithm (*18*) to find the min cut separating o_k from d_k regarding to the assigned weights;
  Let $\delta^+(\overline{T})$ denote this cut, where $\overline{T} \subsetneq V_1$ ($\overline{T}$ contains o_k but does not contain d_k);
  if *flow-cutset inequality* (10) *induced by* $\delta^+(\overline{T})$ *is violated by* $(\overline{x}, \overline{y}, \overline{z})$ **then**
    add this inequality to $\mathcal{FCS}$;

return the identified cut inequalities $\mathcal{FCS}$ to be added to the current LP;

The main idea consists in identifying, for each commodity the minimum cut separating its origin and its destination, then derivating the subset of commodities whose origin and destination nodes are separated by the same cut (see Algorithm 3. In other words, for each $k \in K$, we assign the capacity $\overline{y}_{ew} + \overline{x}_{kew}$ to each pair $(e, w) \in A_1 \times W$, and compute the minimum cut separating o^k from d^k in the graph G_1. Let $\delta^+_{G_1}(T^*)$, $T^* \subseteq V_1$, denote this cut. We then pick an arbitrary subset of arcs, say F^* of $\delta^+_{G_1}(T^*)$, such that $\emptyset \neq F^* \neq \delta^+_{G_1}(T^*)$, and we determine the subset of commodities $K^+(T^*) \subseteq K$ using $\delta^+_{G_1}(T^*)$. If $D(K^+(T))/C$ is not integer, we add the succeeding flow-cutset inequality

$$\sum_{e \in F^*} \sum_{w \in W} y_{ew} + \sum_{k \in K^+(T^*)} \sum_{e \in \overline{F}^*} \sum_{w \in W} x_{kew} \geq \lceil \frac{D(K^+(T^*))}{C} \rceil,$$

if it is violated by the current fractional solution $(\overline{x}, \overline{y}, \overline{z})$.

4.5 Separation of clique-based inequalities

Given a fractional solution $(\overline{x}, \overline{y}, \overline{z})$, and a pair $(\tilde{e}, \tilde{w}) \in A_1 \times W$, the separation problem associated with the clique-based inequalities (15) consists in identifying a clique $\mathcal{C}^*$ in the conflict graph H, such that

$$\sum_{k \in \mathcal{C}^*} \overline{x}_{kew} > \overline{y}_{ew},$$

if any. To do so, we use a greedy algorithm introduced by (*25*) for the independent set problem. This heuristic works as follows. We first construct the conflict graph H = (V, E) where each node $v \in V$ corresponds to a commodity in K and an edge $e \in E$ exists between two nodes $u, v \in V$ if D^u + $D^v > C$. For each pair $(e, w) \in A_1 \times W$, we assign a weight to each node v of V that is $\overline{x}^v_{ew}$, then we choose a node, say u, having the largest weight and we set $\mathcal{C}^* = \{u\}$. We then iteratively add to $\mathcal{C}^*$ the maximum weighted node of $V \setminus \mathcal{C}^*$ whenever it is neighbouring all the nodes of the current clique $\mathcal{C}^*$. We add the clique-based inequality induced by $\mathcal{C}^*$ if it is violated.

4.6 Separation of cover inequalities

We use a similar approach to identify violated cover inequalities (16) if any. Indeed, we put the largest weighted node u in $\mathcal{N}^*$, then we repeat the following operation

Let v be the maximum weighted node of $V \setminus \mathcal{N}^*$, then we simply insert v to $\mathcal{N}^*$ if $\mathcal{N}^* \cup \{v\}$ does not form a clique

until a cover is obtained ($\sum_{v \in \mathcal{N}^*} D^v > C$). Every node $v \in \mathcal{N}^*$ such that $\sum_{i \in \mathcal{N}^* \setminus \{v\}} D^i > C$ is deleted from the subset $\mathcal{N}^*$. Finally, we add the inequality

$$\sum_{k \in \mathcal{N}^*} x_{ew}^k \leq (|\mathcal{N}^*| - 1) y^{ew},$$

if it is violated. Note that there exists plenty more sophisticated algorithms to solve the separation problem associated with cover inequalities (see for example (*4, 21*) and references therein for the separation of cover inequalities), but our first idea was to take advantage from the separation performed for the clique-based inequalities and try to find subsets of commodities that form covers, if the heuristic fails to identify a clique. Besides, we consider only violated clique (respectively cover) inequalities where $|\mathcal{C}^*| \geq 3$ (respectively $|\mathcal{N}^*| \geq 3$) in our Branch-and-Cut algorithm.

4.7 Separation of Min Set I and Min Set II inequalities

Given a fractional solution $(\overline{x}, \overline{y}, \overline{z})$, deciding whether there exists a Min Set I (respectively Min Set II) inequality which is violated by $(\overline{x}, \overline{y}, \overline{z})$ is not an easy problem, since it requires solving the bin packing problem (which is NP-hard in general (*17*)). We use for the separation of these inequalities heuristic procedures inspired from those proposed in (*8*) and adapted for the CMLND-U problem. The idea, for both algorithms, is to identify for every arc $a \in A_1$ a subset of commodities S_a that may induce a violated Min Set I (respectively Min Set II) inequality. If such inequality exists, then it is added to the current LP.

5 Computational experiments

We have conducted several series of experiments to test the efficiency of our Branch-and-Cut algorithm. The purpose of this experimental study is first to give an insight of the effectiveness of the introduced valid inequalities in strengthening the linear relaxation of our model. Our second objective is to identify the classes of instances that are hard to solve in practice, regardless to their size.

The results shown in this chapter have been obtained by solving instances coming from real networks as well as realistic topologies. For all the instances, the graph G_1 representing the virtual layer is supposed to be complete. The cost induced by installing each subband is given by $c(w)$ = (1 + w)c, where w is the subband index and c is a fixed cost associated with the ROADMs generating the subband. This cost is justified by our wish to install the subbands progressively on one hand, and a sake of compliance with practical costs on the other hand. In other words, a subband w_i is not used before w_{i-1} is installed. We also take into account the length of routing path in G_2 associated with each installed subband. This length is given in terms of number of sections in the path. Note that we use the same objective function for both classes of instances.

The realistic instances come from SNDlib (*1*) library. The graph G_2 corresponds to the topology of each instance while G_1 is obtained by considering an edge between each pair of nodes (we add the complementary arcs so as to have a complete graph). Moreover, if the topology corresponds to a non directed graph, we replace each edge by two anti-parallel arcs in both G_2 and G_1. The number of available subbands per arc is set to $|W|$ = 5 for all the instances. Based on these topologies, we have considered two sub-classes of instances. The first one is obtained by using SNDlib topologies with randomly generated traffic commodities. We have tested 3 examples of each instance size and give the average of the results for these examples. The second sub-class uses SNDlib topologies and traffic matrices. We pick the K most important commodities for each topology and traffic matrix. We have considered the topologies *pdh*, *polska*, *nobel_us*, *atlanta*, *newyork*, *nobel_germany*, *geant*, *ta1*, *france*, and *india35*.

The real instances are derived from real network topologies provided by Orange (formerly France Télécom, the French historical telecommunication operator). Three topologies of real instances are considered here, all related to Bretagne area backhaul network. The traffic commodities, as well as the subbands capacities are also given by Orange. For each topology, we have considered three subband capacities C = 10 Gbit/s, C = 12.5 Gbit/s and C = 25 Gbit/s, so as to compare the performances of each type of OFDM multi-band solution. Our experimental results are reported in the tables of the following sections. The entries of the columns in these tables are:

$\|V_2\|$	:	number of nodes in G_2,
$\|A_2\|$	:	number of arcs,
$\|K\|$	:	number of commodities,
NcI	:	number of generated connectivity constraints,
NcII	:	number of generated subband connectivity constraints,
NMSI	:	number of Min Set I inequalities generated,
NCCS	:	number of capacitated cutset inequalities generated,
NFCS	:	number of flow-cutset inequalities generated,
NC	:	number of clique inequalities generated,
NCo	:	number of cover inequalities generated,
NMSII	:	number of Min Set II inequalities generated,
nodes	:	number of nodes in the Branch-and-Cut tree,
o/p	:	number of problem solved to optimality over number of tested instances (only for instances with randomly generated traffic),
Gap	:	the relative error between the best upper bound (optimal solution if the problem has been solved to optimality) and the lower bound obtained at the root node of the Branch-and-Cut tree (before branching),
TT	:	total CPU time in h:m:s,
TTsep	:	CPU time spent in performing the constraints separation, in seconds.

Note that in all the tables, the entries are reported in italic for the instances that could not be solved to optimality within the fixed CPU time limit.

5.1 Effectiveness of the constraints

Before giving the results of our experiments for the instances described above, we first propose to evaluate the impact of the valid inequalities that we use within the Branch-and-Cut algorithm. To this end, we show some numerical results obtained by considering, on one hand the basic cut formulation (2)-(8), and adding the valid inequalities on the other hand. We have tested our approach on a subset of instances whose topologies are pdh, polska, nobel_us, newyork and geant. We rely here on three criteria to make our comparison: the gap, the number of nodes in the Branch-andt-Cut tree, and the CPU time computation. The results are reported in Table 1.

Table 1 shows results obtained for graphs having up to 22 nodes, and 72 arcs. It clearly ap-

Table 1: The impact of adding valid inequalities

				Basic B&C			B&C with valid inequalities		
Instance	$\|V_2\|$	$\|A_2\|$	$\|K\|$	Gap	Nodes	TT	Gap	Nodes	TT
pdh	10	68	10	*25.64*	*34709*	18000	22.54	6	1593
pdh	10	68	12	17.54	2454	3573	4.63	104	4833
pdh	10	68	14	1.39	31	560	0.00	1	133
polska	12	36	10	*13.40*	34896	18000	3.95	56	2059
polska	12	36	12	*36.75*	30954	18000	*13.19*	3114	18000
polska	12	36	14	*34.48*	14983	18000	8.27	1115	13768
nobel_us	14	42	10	*36.89*	15682	18000	7.74	205	12653
nobel_us	14	42	12	*41.20*	5479	18000	7.77	322	12016
nobel_us	14	42	14	*42.95*	10937	18000	*28.33*	513	18000
newyork	16	98	10	13.87	44	306	3.22	11	6102
newyork	16	98	12	*33.09*	18179	18000	*11.65*	88	18000
newyork	16	98	14	14.97	7769	9942	0.00	1	1064
geant	22	72	10	*15.18*	24635	18000	3.57	17	8716
geant	22	72	12	*47.27*	20686	18000	*5.75*	2	18000
geant	22	72	14	*41.46*	17057	18000	*6.23*	14	18000

pears from this table that the formulation with valid inequalities performs much more better than the basic formulation on all the instances. In fact, we first notice that using valid inequalities enables solving some instances that could not be solved to optimality when considering the basic formulation. For example instance polska with 10 commodities could not solved to optimality within 5 hours when using the basic formulation whereas around 30 mns were enough to solve it to optimality with the new inequalities. Also we can see that both gap value and CPU time are smaller when adding the valid inequalities, for all the considered instances. Furthermore, observe that the number of nodes in the Branch-and-Cut tree decreases drastically when introducing valid inequalities. For example for instance geant_10, where the Branch-and-Cut algorithm for basic formulation explores no less than 24635 nodes, while this number drops to 17 nodes, by adding valid inequalities, and a similar improvment can be reported for pdh_10. All these observations lead us to conclude that using valid inequalities to strengthen the linear relaxation of (2)-(8) is a key issue to solve efficiently the CMLND-U problem. As we could see, this enabled to improve significantly the gap value, number of Branch-and-Cut tree as well as the time for computation.

Table 2 shows more accurately the gap evolution when adding the valid inequalities progressively. In fact, the column Gap(0) contains the gap value for basic formulation and Gap(6) contains the gap value when considering all the cuts. The remaining columns are intermediate gap value

Table 2: Effectiveness of the cuts - Gap evolution

Instance	$\lvert V_2\rvert$	$\lvert A_2\rvert$	$\lvert K\rvert$	Gap(0)	Gap(1)	Gap(2)	Gap(3)	Gap(4)	Gap(5)	Gap(6)
pdh	10	68	10	25.64	22.67	22.54	22.54	22.54	22.54	22.54
pdh	10	68	12	17.54	6.63	4.76	4.63	4.63	4.63	4.63
pdh	10	68	14	1.39	0.29	0.00	0.00	0.00	0.00	0.00
polska	12	36	10	13.40	4.03	3.95	3.95	3.95	3.95	3.95
polska	12	36	12	36.75	13.50	13.50	13.47	13.19	13.19	13.19
polska	12	36	14	34.48	15.63	9.60	8.27	8.27	8.27	8.27
nobel_us	14	42	10	36.89	16.00	8.84	8.16	7.74	7.74	7.74
nobel_us	14	42	12	41.20	41.13	41.13	8.10	7.77	7.77	7.77
nobel_us	14	42	14	42.95	33.82	32.41	28.33	28.33	28.33	28.33
newyork	16	98	10	13.87	4.27	3.27	3.27	3.27	3.22	3.22
newyork	16	98	12	33.09	24.08	11.97	11.65	11.65	11.65	11.65
newyork	16	98	14	14.97	1.35	0.00	0.00	0.00	0.00	0.00
geant	22	72	10	15.18	5.54	4.14	4.14	3.57	3.57	3.57
geant	22	72	12	47.27	16.96	8.71	8.46	8.46	5.85	5.75
geant	22	72	14	41.46	17.60	14.37	13.42	13.42	6.23	6.23

obtained by considering an additional family of valid inequalities. The constraints are separated in the order given in Section 4. Thus, Gap(1) corresponds to when adding Min set I inequalities, Gap(2) when adding capacitated cut inequalities and so on. It appears that the gap value decreases when adding valid inequalities. However, it seems that some inequalities are more efficient than others in strengthening the linear relaxation. In fact, the most significant improvement is observed when adding Min Set I inequalities (see columns Gap(0) and Gap(1)). Adding capacitated cutset and flow-cutset inequalities also allows to improve the gap value, while only a slight gain is notified when adding the remaining families of valid inequalities. In practice, their interest lies in the number of nodes in the Branch-and-Cut tree, which gets smaller as further families of valid inequalities are being separated.

5.2 Random instances

Our first series of experiments concerns the SNDlib topologies with randomly generated traffic commodities. The instances considered here have graphs with 10 up to 24 nodes and vary from sparse (like for polska) to highly meshed (like for ta1) topology. The number of commodities for each size of graphs ranges from 10 to 18 with values generated uniformly in the interval $]\epsilon C, C]$, with ϵ = 0.2 for these instances. For each instance size, we have generated 3 examples. The results are reported in Table 3.

Instance	$\vert V_2\vert$	$\vert A_2\vert$	$\vert K\vert$	Opt	NcI	NcII	NmsI	NCCS	NFCS	NC	NCo	NmsII	Nodes	Gap	TT
pdh	10	68	10	3/3	247.00	832.67	74.33	33.00	83.67	2.33	0.67	0.00	40.67	7.19	350.93
pdh	10	68	12	3/3	625.67	1504.00	150.67	72.33	1389.67	31.67	6.33	1.33	194.67	10.29	2244.68
pdh	10	68	14	3/3	450.33	1030.00	113.00	57.33	1031.33	11.00	6.33	0.00	63.33	5.95	1259.59
pdh	10	68	16	2/3	478.67	1047.33	133.00	23.00	7021.67	17.33	3.67	0.67	191.67	2.83	6061.61
pdh	10	68	18	0/3	3602.67	1703.33	124.67	40.67	11683.67	12.00	815.00	0.00	493.33	6.54	18000.00
polska	12	36	10	2/3	536.33	2782.33	185.33	61.33	6473.33	12.00	1.67	0.00	1263.67	9.08	6142.68
polska	12	36	12	3/3	1071.67	3042.00	197.33	81.33	5191.00	57.00	29.67	0.33	549.33	6.30	6224.30
polska	12	36	14	2/3	1019.67	3896.33	259.67	73.67	7919.00	31.33	42.67	0.00	918.33	11.53	9440.76
polska	12	36	16	2/3	955.33	3484.33	166.33	62.33	7991.67	30.67	15.33	0.33	852.33	13.81	10248.26
polska	12	36	18	2/3	1234.67	3881.33	286.00	46.00	7527.67	78.33	14.67	0.33	1098.00	8.77	12166.49
nobel_us	14	42	10	2/3	1504.67	5064.67	214.00	120.67	4303.33	30.67	34.67	0.67	1388.67	19.05	14031.90
nobel_us	14	42	12	2/3	1529.00	4793.00	294.33	93.67	4377.67	76.00	1.67	2.33	1350.67	14.83	15391.00
nobel_us	14	42	14	0/3	1543.33	4569.33	256.67	116.33	8392.33	59.00	19.33	1.00	877.33	26.14	18000.00
nobel_us	14	42	16	1/3	1755.33	5120.33	301.67	105.33	4693.33	114.33	0.67	0.00	828.33	16.35	17840.40
nobel_us	14	42	18	1/3	1440.67	3590.33	318.33	55.00	8551.00	78.67	8.33	0.00	211.33	13.53	15614.03
newyork	16	98	10	2/3	664.00	2763.67	186.33	43.67	1554.67	9.67	3.33	0.67	315.00	14.74	8385.19
newyork	16	98	12	1/3	1102.33	4453.67	217.00	104.00	915.33	52.00	8.00	0.00	672.33	24.31	13484.55
newyork	16	98	14	2/3	1721.00	4043.33	245.33	109.00	1822.67	38.67	4.33	0.00	308.67	15.01	17578.47
newyork	16	98	16	2/3	911.67	3468.00	170.67	107.67	1484.33	16.67	8.33	0.00	291.00	10.37	14519.00
newyork	16	98	18	1/3	1428.33	3492.00	264.33	105.00	1924.00	48.00	6.33	0.33	330.00	6.35	16097.23
ta1	24	102	10	2/3	169.00	3551.67	19.33	32.67	82.33	0.33	0.00	0.00	2.33	7.10	810.45
ta1	24	102	12	2/3	2.67	287.67	5.33	0.00	0.00	0.00	0.00	0.00	15.00	20.26	357.88
ta1	24	102	14	0/3	779.89	6955.22	148.11	166.00	199.00	19.00	4.67	0.00	146.00	14.74	18000.00
ta1	24	102	16	0/3	249.33	3414.00	23.33	33.00	93.00	4.00	0.33	0.00	515.00	21.16	18000.00
ta1	24	102	18	0/3	996.33	10244.33	157.67	150.33	1187.33	45.33	1.67	0.00	185.67	15.51	18000.00

Table 3: Branch-and-Cut results for SNDlib instances with randomly generated traffic

It appears from this table that 20 over 25 groups of instances have been solved to optimality within the fixed time limit. Observe that no more than 4 groups of instances among those solved to optimality have a gap value greater than 20%. Table 3 also shows that the difficulty of solving an instance is not only related to its size, but also to the size of the commodities. For example, instances polska with 12 commodities are solved to optimality within the time limit, while 2 over 3 instances polska with 10 commodities are solved to optimality. Even though the instances of the second group are larger in size, they are solved more easily. Moreover, it should be emphasized again that in practice, the subbands installed on the arcs of G_1 are considered as additional commodities. Indeed, since two levels of routing must be performed, there are $|K| + |W|(n_1(n_1 - 1))$ commodities, where n_1 = $|V_1|$, $|K|$ being the traffic demands and $|W|(n_1(n_1 - 1))$ the number of subbands that can be installed in G_1. Remark also that an important number of Min Set I, capacitated cutset and flows-cutset inequalities are being generated along the Branch-and-Cut tree, which means that they are helpful for solving the problem. However, the number of clique and cover inequalities separated is smaller. This can be explained by the fact that each arc of G_1 potentially induces the same cliques (respectively cover subsets), since it depends on the commodities size. Thus, if all the commodities are ”small” regarding to the capacity of a subband, then clique and eventually cover inequalities are unlikely to appear.

5.3 Realistic instances

The second series of experiments that we have conducted concerns SNDlib instances with realistic traffic commodities. We have considered instances with graphs having 10 to 35 nodes. 11 instances have been tested, and 6 instances among them have been solved to optimality within the time limit. The remaining instances, often having more than 14 commodities, could not reach the optimal solution after 5 hours of computation. Also we can see in Table 4 that, for the instances that could be solved to optimality, the gap value does not exceed 20% and the number of nodes in the Branch-and-Cut tree remains reasonable (less than 300 for most of the instances). All the

Instance	$\|V_2\|$	$\|A_2\|$	$\|K\|$	NcI	NcII	NmsI	NCCS	NFCS	NC	NCo	NmsII	Nodes	Gap	TT	TTsep
pdh	10	68	10	5861	1082	2000	8	217	0	3	0	2	13.40	4500	340
pdh	10	68	14	1494	1738	647	24	2004	0	0	0	177	2.50	4939	1312
pdh	10	68	20	3641	1840	2000	6	164	0	1	0	733	*7.40*	18000	122
nobel_us	14	42	10	1759	5245	180	147	8795	41	56	0	67	17.74	1239	654
nobel_us	14	42	14	1706	4514	298	109	7611	123	0	1	210	5.67	1443	740
nobel_us	14	42	20	1528	9023	263	66	9378	57	2	0	126	19.37	12343	4528
newyork	16	98	10	658	3041	108	60	368	7	0	0	281	9.58	9081	6686
newyork	16	98	14	1047	2708	233	96	871	7	0	1	226	*14.70*	18000	12322
newyork	16	98	20	1328	10245	261	103	11389	5	0	0	264	*27.94*	18000	3171
france	25	90	10	1118	4079	0	119	16	3	0	0	139	*11.24*	18000	5507
india	35	160	8	1146	5074	0	143	23	3	0	0	2	*42.34*	18000	5386

Table 4: Branch-and-Cut results for SNDlib instances with realistic traffic

instances for which the algorithm provided an optimal solution have been solved in less than 3 hours. Similarly to random instances, some instances may be more difficult to solve than other instances, even larger in size. Besides, it should be pointed out that the CPU time spent by the algorithm in performing separation of valid inequalities can be important. In fact, we noticed that this time could reach more than 50% of the total CPU time of computation (see for example instances newyork with 10 and 12 commodities). More precisely, we noticed that the separation procedure for generating flow-cutset inequalities is the most time consuming routine.

5.4 Real instances

The tested real instances have graphs with 9 to 45 nodes and a number of commodities that varies between 15 and 30 for the smaller instances. In particular, we have considered $|W|$ = 4 for all the instances, and three possible subband capacities, namely C = 10 Gbit/s, 12.5 Gbit/s and 25 Gbit/s. Table 5 shows the results obtained for two over the three families of instances considered. We further give and example of solution obtained when solving an instance with 45 nodes and 10 commodities.

Instance	$\lvert V_2\rvert$	$\lvert A_2\rvert$	$\lvert K\rvert$	NMSI	NCCS	NFCS	NC	NCo	NMSII	Nodes	Gap	TT
bretagne_10	9	20	15	36	39	597	0	2	0	42	29.49	107
bretagne_10	9	20	20	36	38	353	8	6	0	24	35.34	104
bretagne_10	9	20	30	30	23	2341	0	0	0	4	*33.77*	18000
bretagne_12	9	20	15	24	24	951	0	4	1	22	44.53	76
bretagne_12	9	20	20	42	41	98	0	11	0	22	48.63	46
bretagne_12	9	20	30	24	35	443	0	2	0	28	*47.68*	18000
bretagne_25	9	20	15	49	74	652	0	11	0	249	28.33	1036
bretagne_25	9	20	20	73	44	821	5	6	0	327	33.33	1100
bretagne_25	9	20	30	112	81	789	11	4	0	23509	*39.93*	18000
bretagne_10	22	52	15	21	8	347	0	0	0	38	41.30	1242
bretagne_10	22	52	20	21	4	5	0	0	0	14	36.96	247
bretagne_10	22	52	30	28	28	1076	1	0	0	16	*47.71*	18000
bretagne_12	22	52	15	6	0	3	0	5	0	38	43.76	209
bretagne_12	22	52	20	36	17	3825	1	13	2	60	44.78	*18000*
bretagne_12	22	52	30	26	33	1483	0	0	0	16	37.96	*18000*
bretagne_25	22	52	5	511	419	4465	21	14	3	7149	*31.00*	18000
bretagne_25	22	52	10	9	22	3308	0	0	4	273	*49.00*	18000
bretagne_25	22	52	15	38	12	9825	0	0	21	876	*53.00*	18000

Table 5: Branch-and-Cut results for real instances

It appears from Table 5 that 9 instances among the 18 instances tested were solved to optimality within the CPU time limit. An optimal solution could be obtained within one hour for all the solved instances. Several observations can be made based on these results. First concerning instances Bretagne with 9 nodes, we can see that we get better results when using a larger subband capacity

C. This is due to the topology of these instances which is quite sparse compared to the original network with 45 nodes. Basically, finding a feasible routing for the commodities by using less subbands is a challenging task because of the graph topology. Indeed, the disjunction constraints make difficult to reuse the same paths in G_2 for the installed subbands. Besides, when C = 25 Gbit/s, commodities are more likely to be packed in the same subbands, which makes easier to find a good solution within the fixed time limit. We also notice from Table 5 that results for instances with 22 nodes gets better when C = 10 Gbit/s. In fact, since the graph holds more nodes and arcs, it offers more possible paths, and hence more routing alternatives for both commodities and subbands. Finally, we notice that an important number of cover inequalities are generated for these instances. In fact, the traffic commodities here are relatively small and tend to be uniform in size. Cover inequalities are then more expected to appear than clique based inequalities.

6 Concluding remarks

In this paper we have studied a multilayer network design problem with specific requirements arising in optical OFDM networks. We have proposed a cut-based ILP formulation for the problem and studied its basic polyhedral properties. Exploiting the underlying sub-problems, we have proposed several families of valid inequalities and investigated their facial structure. These inequalities have been embedded within a branch-and-cut algorithm to solve the problem.

The focus made on the hierarchical routing and the traffic indivisibility increases the computational challenges and makes the problem further difficult to solve. The proposed valid inequalities have helped a lot in strengthening the linear relaxation of the problem, yet the associated separation procedures are still time-consuming and could be enhanced. In addition, derivating good upper bounds from the fractional solutions is a non-trivial task but would hopefully allow to better manage the size of the branch-and-cut tree. Finally, we expect that further valid inequalities, involving the demands structure (traffic amount compared to the capacity value) would be very interesting and increase the efficiency of the algorithm, especially on real instances where a wide disparity may exist among the demands.

Appendices

Definition 3. *A solution S of the CMLND-U problem is given by two subsets of arcs F_1, F_2 of A_1 (with F_2 eventually empty), $|K|$ subsets of arcs $\mathcal{C}_1$, ..., $\mathcal{C}_k$, of A_1, a subset of subbands $\overline{W}$ of W installed on the arcs of $F_1 \cup F_2$, a subset of arcs Δ of A_2, and $|A_1| \times |W|$ subsets of arcs Δ_{ew}, $e \in A_1$, $w \in W$, of A_2 in such a way that*

(i) *at least one subband is installed on each arc of $F_1 \cup F_2$,*

(ii) $F_1 = \bigcup_{k \in K} \mathcal{C}_k$,

(iii) $\mathcal{C}_k$, $k \in K$, *contains a path between o_k and d_k,*

(iv) $\Delta = \bigcup_{e \in F_1 \cup F_2, w \in \overline{W}} \Delta_{ew}$,

(v) *with every arc $e = (u, v) \in F_1 \cup F_2$ and $w \in \overline{W}$, one can associate an arc subset Δ_{ew} (which may be empty), in such a way that if w is installed on e, then Δ_{ew} contains a path, say $P_{ew} \subseteq \Delta_{ew}$ between u' and v',*

(vi) *for every $w \in \overline{W}$, any arc of Δ belongs to at most one path P_{ew}, for $e \in F_1 \cup F_2$.*

We will denote by Γ the pairs (e, w) such that $e \in (F_1 \cup F_2)$ and $w \in \overline{W}$ such that w is installed on e. We then define the solution S by $S = (F_1, F_2, \Delta, \overline{W})$. The incidence vector of S, $(x^S, y^S, z^S) \in \mathbb{R}^{K \times A_1 \times W} \times \mathbb{R}^{A_1 \times W} \times \mathbb{R}^{A_1 \times W \times A_2}$, will be given by:

$$x^S_{kew} = \begin{cases} 1, & \text{if } e \in \mathcal{C}_k \text{ and } (e, w) \in \Gamma, \\ 0, & \text{otherwise.} \end{cases}$$

$$y^S_{ew} = \begin{cases} 1, & \text{if } w \in \overline{W}, e \in F_1 \cup F_2 \text{ and } (e, w) \in \Gamma, \\ 0, & \text{otherwise.} \end{cases}$$

$$z^S_{ew}(a) = \begin{cases} 1, & \text{if } a \in \Delta_{ew}, \\ 0, & \text{otherwise.} \end{cases}$$

A Sketch of the proof for Theorem 1

Assume that $P(G_1, G_2, K, C)$ is contained in a hyperplane given by the linear equation

$$\alpha x + \beta y + \gamma z = \delta \tag{20}$$

where $\alpha = (\alpha^k_{ew}, k \in K, e \in A_1, w \in W) \in \mathbb{R}^{K\times A_1\times W}$, $\beta = (\beta^{ew}, e \in A_1, w \in W) \in \mathbb{R}^{A_1\times W}$, $\gamma = (\gamma^{ew}_a, e \in A_1, w \in W, a \in A_2) \in \mathbb{R}^{A_1\times W\times A_2}$ and $\delta \in \mathbb{R}$. We will show that α=0, β=0, γ=0. Therefore $P(G_1, G_2, K, C)$ can not be included in a hyperplane, thus implying that $P(G_1, G_2, K, C)$ is full dimensional. To this end, let us first construct a solution S^0 = $(F^0_1, F^0_2, \Delta^0, W^0)$ of the problem.

For each commodity $k \in K$, we consider a path in G_1 between its origin and destination nodes, consisting of arc (o_k, d_k). This is possible since G_1 is complete. We install over this arc one subband. In other words, every subband is assigned at most to one commodity. Note that every arc (u, v) receives as much subbands as there are demands going from u to v. All the installed subbands are supposed to be different. After that, we associate with each subband, installed over (o_k, d_k), $k \in K$, a path in G_2 consisting of the arc (o'_k, d'_k). Again, this is possible since G_2 is also a complete graph. Let $S^0 = (F^0_1, F^0_2, \Delta^0, W^0)$, be the solution given by $F^0_1 = \{(o_k, d_k), k \in K\}$, F^0_2 = $\emptyset$, $\Delta^0 = \{(o'_k, d'_k), k \in K\}$, where Δ^0_{ew} = $\{(o'_k, d'_k)\}$ if e = (o_k, d_k) and w the subband installed on e, and $\Delta^0_{ew} = \emptyset$ otherwise. W^0 denotes the subset of $|K|$ different subbands installed on the arcs of F^0_1. Now consider a pair $(e, w) \in A_1 \times W$. Let S^1 = $(F^1_1, F^1_2, \Delta^1, W^1)$ be a solution obtained from S^0 by adding an arc $f \in A_2$ to Δ^0_{ew}, while the other elements of S^0 remain the same. (In other words, S^1 is such that F^1_1 = F^0_1, F^1_2 = F^0_2, Δ^1 = $\Delta^0 \cup \{f\}$, and W^1 = W^0.) It is clear that both, S^0 and S^1 are feasible for the problem and, in consequence, their incidence vectors both satisfy equality (20). This implies that γ^{ew}_f = 0. As f, e and w are chosen arbitrarily in A_2, A_1 and W, respectively, we obtain that γ^{ew}_f = 0, for all f in A_2, $e \in A_1$ and $w \in W$. By similar arguments, we can show that the remaining coefficients of the hyperplane (20) are equal to zero. $\square$

B Proof of Theorem 3

Suppose that conditions (i) to (iv) of Theorem 3 are fulfilled. Let us denote by $\alpha x + \beta y + \gamma z \geq \delta$ inequality (9). Let $\tilde{\mathcal{F}}$ denote the face induced by inequality (9). Then,

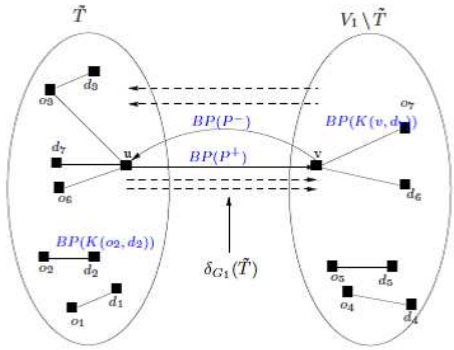
$$\tilde{\mathcal{F}} = \{(x, y, z) \in P(G_1, G_2, K, C) : \sum_{e \in \delta^+_{G_1}(\tilde{T})} \sum_{w \in W} y_{ew} = \lceil \frac{D(P^+)}{C} \rceil \}.$$

We first show that $\tilde{\mathcal{F}}$ is a proper face of $P(G_1, G_2, K, C)$. To do this, let us construct a feasible solution S^0 that satisfies (9) with equality.

For each pair of nodes $(s, t) \in \tilde{T}$ (respectively in $V_1 \setminus \tilde{T}$), we install $BP(K(s, t))$ different subbands on the arc (s, t) of A_1. Notice that if there is no commodity $k \in K$ such that $o_k = s$ and $d_k = t$ we do not use (s, t) in the solution. Moreover, each commodity k in $K(\tilde{T})$ (respectively in $K(V_1 \setminus \tilde{T})$) is associated with path $\{(o_k, d_k)\}$ and the subband w_k. Note that, in this solution, a subband w_k may be associated with more than one commodity (see Figure B). Now, we choose a node $u \in \tilde{T}$ and a node $v \in V_1 \setminus \tilde{T}$. Observe that $(u, v) \in \delta^+_{G_1}(\tilde{T})$. We then install on the arc (o_k, u) (respectively (v, d_k)) of A_1, $BP(K(o_k, u))$ (respectively $BP(K(v, d_k))$) new subbands of W, while (u, v) receives $BP(K(u, v))$ new subbands. Note that (u, v) is the only arc of the cut $\delta^+_{G_1}(\tilde{T})$ that is used in this solution. We do the same operation for the commodities of $K^-(\tilde{T})$. Furthermore, we associate with each pair (e, w) such that w is installed on $e = (i, j)$ the arc (i', j') of A_2. This is possible since G_2 is a complete graph. Notice that, in this solution, each commodity $k \in K(\tilde{T})$ (respectively $k \in K(V_1 \setminus \tilde{T})$) uses the subband w_k on path $\{e_k\}$, $e_k = (o_k, d_k)$ for its routing, while the commodities of $K^+(\tilde{T})$ have a path of length at most three $\{(o_i, u), (u, v), (v, d_i)\}$, $i \in K^+(\tilde{T})$. The node u (respectively v) can obviously be equal to some o_i (respectively d_i), $i \in K^+(\tilde{T})$. Moreover, in this solution, every commodity of K uses at least one subband for its routing, and we assume that all the set up subbands are different so that the disjunction constraints (5) are satisfied. Also note that many commodities may share the same subband, however, as $BP(K(s, t))$ subbands are installed on each pair of nodes $s, t \in \tilde{T}$ and ($V_1 \setminus \tilde{T}$), we ensure that the capacity constraints (3) are satisfied.

In this solution, a path in G_1 is assigned to each commodity of K. Moreover, a path is also associated with every pair (e, w) such that w is installed on e. Furthermore, both capacity

Figure 3 Solution S^0



constraints (3) and disjunction constraints (5), are satisfied, as enough different subbands are installed on each arc used in the solution. It is not hard to see that S^0 induces a feasible solution of $P(G_1, G_2, K, C)$ whose incidence vector satisfies $\alpha x + \beta y + \gamma z \geq \delta$ with equality. Hence, $\tilde{\mathcal{F}} \neq \emptyset$. As $P(G_1, G_2, K, C) \setminus F \neq \emptyset$, we have that $\tilde{F}$ is a proper face of $P(G_1, G_2, K, C)$.

Now suppose that there exists a facet defining inequality $\lambda x + \mu y + \nu z \geq \xi$ such that

$$\tilde{\mathcal{F}} \subseteq \mathcal{F} = \{(x, y, z) \in P(G_1, G_2, K, C) : \lambda x + \mu y + \nu z = \xi\}.$$

We will show that there exists a scalar $\rho \in \mathbb{R}$ such that $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$.

Let us first show that $\nu_a^{ew} = 0$, for all $e \in A_1$, $w \in W$ and $a \in A_2$. Consider an arc $a \in A_2 \setminus \Delta^0$, and a pair $(e^*, w^*) \in A_1 \times W$. Clearly, the solution $S^1 = (F_1^0, F_2^0, \Delta^1 = \Delta^0 \cup \{a\}, W^0)$, where $\Delta^1_{e_i w_i} = \Delta^0_{e_i w_i}$, if $(e_i, w_i) = (e^*, w^*)$ and $\Delta^1_{e^* w^*} = \Delta^0_{e^* w^*} \cup \{a\}$ is a solution of the $P(G_1, G_2, K, C)$. Moreover, its incidence vector satisfies $\alpha x + \beta y + \gamma z \geq \delta$ with equality. It then follows that $\nu_a^{e^* w^*} = 0$, and therefore

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, a \in A_2. \tag{21}$$

By similar arguments, we can easily show that

$$\begin{cases} \mu^{ew} = 0, & \text{for all } e \in A_1 \setminus \delta^+_{G_1}(\tilde{T}), w \in W \\ \lambda^k_{ew} = 0, & \text{for all } k \in K, e \in A_1 \setminus \delta^+_{G_1}(\tilde{T}), w \in W. \end{cases} \tag{22}$$

Now consider a commodity $k^* \in K$. We will show that the coefficients λ related to the commodities of K and arcs of $\delta^+_{G_1}(\tilde{T})$ are equal to zero. We distinguish two cases.

Case 1. $k^* \in K \setminus K^+(\tilde{T})$

Thus k^* is not supposed to pass through the cut for its routing. Consider an arc e^* of $\delta^+_{G_1}(\tilde{T})$ and a subband w^* of W. We will assume that $e^* = (u, v)$ and that w^* is installed on e^*. Consider the solution S^2, obtained from S^0 by associating e^* to k^* in addition to its routing. In other words, $S^2 = (F_1^0 \cup \{e^*\}, F_2^0, \Delta^0, W^0)$ and $\mathcal{C}_{k^*}^2 = \mathcal{C}_{k^*}^0 \cup \{e^*\}$. Condition (ii) makes the solution feasible for the problem, as it allows capacity constraints to be satisfied. Thus, both solutions S^0 and S^2 are feasible and belong to $\tilde{\mathcal{F}}$ as well. Hence, $\lambda_{e^*w^*}^{k^*} = 0$. As k^*, e^* and w^* are arbitrary in $K \setminus K^+(\tilde{T})$, $\delta_{G_1}^+(\tilde{T})$ and W, respectively, we obtain that

$$\lambda_{ew}^k = 0, \text{ for all } k \in K \setminus K^+(\tilde{T}), e \in \delta_{G_1}^+(\tilde{T}), w \in W. \tag{23}$$

Case 2. $k^* \in K^+(\tilde{T})$

Let k be some commodity of $K^+(\tilde{T})$ such that $D^{k^*} + D^k \leq C$. Such a commodity exists because of condition (iii). Let $e^* = (s,t)$ be an arc of $\delta_{G_1}^+(\tilde{T})$ and let w^* be one of the subbands installed on (u,v). Recall that $BP(K^+(\tilde{T}))$ different subbands have been installed on (u,v). We will construct a solution S^3 obtained from S^0 by moving w^* from arc (u,v) to arc (s,t), and associating with $((s,t),w^*)$ the path $\{(s',t')\}$ in G_2. In this solution, we will also replace (u,v) in the routing path of k^* by $\{(u,s), e^*, (t,v)\}$, where (u,s) and (t,v) are two arcs of $A_1 \setminus \delta_{G_1}^+(\tilde{T})$. (u,s) and (t,v) also receive the subband w^* and are assigned the paths $\{(u',s')\}$ and $\{(t',v')\}$, in G_2, respectively. Since we know by condition (iv), the capacity constraints (3) are satisfied. It follows that S^3 is feasible. Now let us derive a solution S^4 which slightly differs from S^3 in that we associate (s,t) to k in addition to its routing. Again, this is possible thanks to condition (iii). This variation in the solution yields $x_{k(s,t)w^*}^{S^4} = 1$ while $x_{k(s,t)w^*}^{S^3} = 0$. Solution S^4 is clearly feasible, and both incidence vectors $(x^{S^3}, y^{S^3}, z^{S^3})$ and $(x^{S^4}, y^{S^4}, z^{S^4})$ are in $\tilde{\mathcal{F}}$, and then, also in $\mathcal{F}$. Thus, we obtain that $\lambda_{(s,t)w^*}^k = 0$. Since k^*, e, and w are chosen arbitrarily in $K^+(\tilde{T})$, $\delta_{G_1}^+(\tilde{T})$ and W, respectively, we get

$$\lambda_{ew}^k = 0, \text{ for all } k \in K^+(\tilde{T}), e \in \delta_{G_1}^+(\tilde{T}), w \in W. \tag{24}$$

And, by (22), (23) and (24), we finally obtain

$$\lambda_{ew}^k = 0, \text{ for all } k \in K, e \in A_1, w \in W. \tag{25}$$

Now, we still have to show that all the coefficients μ^{ew} are the same for the arcs of the cut $\delta_{G_1}^+(\tilde{T})$.

Indeed, let $e^* = (s,t)$ be an arc of $\delta^+_{G_1}(\tilde{T})$, different from (u,v). Recall that $BP(K^+(\tilde{T}))$ different subbands are installed over the arc (u,v). Let $\overline{w}$ be one of these subbands. Consider the solution S^5 where we replace the pair $((u,v),\overline{w})$ in S^0 by $((u,s),\overline{w})$, $((s,t),\overline{w})$ and $((t,v),\overline{w})$, with (u,s), $(t,v) \in A_1 \setminus (F_1^0 \cup F_2^0)$. By comparing solutions S^0 and S^5 and by (21) and (22), we obtain that $\mu^{(u,v)\overline{w}} = \mu^{(s,t)\overline{w}}$. Since (s,t) is arbitrary in $\delta^+_{G_1}(\tilde{T})$, we get

$$\mu^{ew} = \begin{cases} \rho, & \text{for some } \rho \in \mathbb{R}^*, \text{ for all } e \in \delta^+_{G_1}(\tilde{T}), w \in W, \\ 0, & \text{otherwise.} \end{cases} \tag{26}$$

Hence, from (21)-(26), $(\alpha,\beta,\gamma) = \rho(\lambda,\mu,\nu)$, and the proof is complete. □

C Proof of Theorem 4

Let T be a subset of nodes of V_1 and $\overline{T} = V_1 \setminus T$. Consider the cut $\delta^+_{G_1}(T)$ induced by T, and let F, $\overline{F}$ be a partition of $\delta^+_{G_1}(T)$. Now consider the flow-cutset inequality induced by T and F

$$\sum_{e\in F}\sum_{w\in W} y_{ew} + \sum_{e\in\overline{F}}\sum_{w\in W}\sum_{k\in K^+(T)} x_{kew} \geqslant \lceil \frac{D(K^+(T))}{C} \rceil.$$

(i) If $\overline{F} = \emptyset$, then (10) is equivalent to

$$\sum_{w\in W}\sum_{e\in F} y_{ew} \geqslant \lceil \frac{D(K^+(T))}{C} \rceil,$$

which reduces to the cutset inequality (9) when $\lceil \frac{D(K^+(T))}{C} \rceil = BP(K^+(T))$. Hence, (10) cannot be a facet of $P(G_1,G_2,K,C)$ different from (9). If $F = \emptyset$, then $\overline{F} = \delta^+_{G_1}(T)$ and (10) is equivalent to

$$\sum_{e\in\delta^+_{G_1}(T)}\sum_{w\in W}\sum_{k\in K^+(T)} x_{kew} \geqslant \lceil \frac{D(K^+(T))}{C} \rceil, \tag{27}$$

which implies that the number of commodities allowed to use the cut $\delta^+_{G_1}(T)$ is greater than or equal to $\lceil \frac{D(K^+(T))}{C} \rceil$. Note that $\lceil \frac{D(K^+(T))}{C} \rceil \leqslant |K^+(T)|$, as $D^k \leqslant C$, for all $k \in K^+(T)$. Thus, inequality (27) is dominated by inequality

$$\sum_{k\in K^+(T)}\sum_{e\in\delta^+_{G_1}(T)}\sum_{w\in W} x_{kew} \geqslant |K^+(T)|,$$

which is the sum of the connectivity constraints (2) over the commodities of $K^+(T)$. Thus, (10) cannot define a facet for $P(G_1,G_2,K,C)$.

(ii) Now if $D(K^+(T)) < C$. Then (10) is equivalent to

$$\sum_{w \in W} \sum_{e \in F} y_{ew} + \sum_{w \in W} \sum_{e \in \overline{F}} \sum_{k \in K^+(T)} x_{kew} \geqslant 0,$$

which is nothing but a linear combination of trivial inequalities $y_{ew} \geqslant 0$, and $x_{kew} \geqslant 0$, summed up over the subsets F, W and $K^+(T)$, $\overline{F}$, W, respectively.

(iii) If $D(K^+(T))/C$ is integer, then (10) can be obtained from inequalities (2), (3) and the trivial constraints $x_{kew} \geqslant 0$.

(iv) Suppose that $\lceil \frac{D(K^+(T))}{C} \rceil < BP(K^+(T))$. Then inequality (10) does not induce a proper face, as it may be empty.

(v) Now assume that $BP(K^+(T)) = |K^+(T)|$. Then, inequality (10) is equivalent to the following expression

$$\sum_{e \in F} \sum_{w \in W} y_{ew} + \sum_{k \in K^+(T)} \sum_{e \in \overline{F}} \sum_{w \in W} x_{kew} \geqslant BP(K^+(T)) = |K^+(T)|,$$

which is a linear combination of inequalities (27) and trivial constraints $y_{ew} \geqslant 0$ summed up over F and W. Hence, (10) cannot define a facet.

(vi) Suppose that condition (vi) is not verified, that is to say $BP(K^+(T) \setminus q) \geqslant BP(K^+(T)) - q + 1$ for all $q \subsetneq K^+(T)$. Then we cannot find a solution with x_{kew} = 1, for some commodity $k \in K^+(T)$, $e \in \overline{F}$ and $w \in W$. In other words, the face induced by (10) is included in

$$\overline{\mathcal{F}} = \{(x, y, z) \in P(G_1, G_2, K, C) : x_{kew} = 0, \text{ for } k \in q, e \in \overline{F}, w \in W\},$$

and then, (10) cannot define a facet. Now if there exists a commodity k in $K \setminus P^+$ such that $BP(P^+ \setminus \{k\}) \geqslant BP(P^+) + 1$. Then it is not possible to identify a solution of the problem with x_{kew} = 1, for $e \in F \cup \overline{F}$, $w \in W$. Also in this case, the face induced by (10) is included in

$$\overline{\mathcal{F}} = \{(x, y, z) \in P(G_1, G_2, K, C) : x_{kew} = 0, \text{ for } k \in q, e \in \overline{F}, w \in W\},$$

and then, (10) cannot define a facet.

D Proof of Theorem 6

Necessity

(i) The validity condition for inequalities (17) states that $p \geqslant |S|$ - $BP(S)$. It is clear that any value of $p > |S| - BP(S)$ would induce a valid inequality which is redundant regarding to

$$\sum_{w \in W} \sum_{k \in S} x_{kew} \leq \sum_{w \in W} y_{ew} + (|S| - BP(S)), \text{ for all } e \in A_1,$$

and thus, that could not define a facet of $P(G_1, G_2, K, C)$.

(ii) Now let us denote by s the largest element of $K \setminus S$ such that $BP(S \cup \{s\})$ = $BP(S)$ + 1. Then, inequality (17) with respect to $S \cup \{s\}$ can be written as

$$\sum_{w \in W} \sum_{k \in S \sup\{s\}} x_{kew} \leq \sum_{w \in W} y_{ew} + |S| + 1 - BP(S \cup \{s\}) = \sum_{w \in W} y_{ew} + p.$$

However, (ii) dominates (17). Thus the latter cannot define a facet of $P(G_1, G_2, K, C)$.

(iii) Now let s' denote the smallest element of S, and suppose that $BP(S \setminus \{s'\})$ = $BP(S)$ = $|S|$ - p. Then, inequality (17) with respect to $S \setminus \{s'\}$ can be written as

$$\sum_{w \in W} \sum_{k \in S \setminus \{s'\}} \leqslant \sum_{w \in W} y_{ew} + |S| - 1 - BP(S \setminus \{s'\}) = \sum_{w \in W} y_{ew} + p - 1. \tag{28}$$

Note that inequalities (17) can be obtained as a linear combination of (28) and $\sum_{w \in W} x_{s'ew} \leqslant 1$, which is valid (by definition of the CMLND-U problem).

Sufficiency

Suppose that conditions (i) to (iii) of Thorem 6 are fulfilled. Let us denote by $\alpha x + \beta y + \gamma z \geqslant \delta$ the inequality (17) induced by $\tilde{S}$ and $\tilde{e}$ = (u, v), and let $\mathcal{F}$ be the face defined as follows

$$\tilde{\mathcal{F}} = \{(x, y, z) \in P(G_1, G_2, K, C) : \sum_{k \in \tilde{S}} \sum_{w \in W} x_{k\tilde{e}w} = \sum_{w \in W} y_{\tilde{e}w} + p\}.$$

We first show that $\tilde{\mathcal{F}}$ is a proper face of $P(G_1, G_2, K, C)$. To this end, we construct a feasible solution S^1, whose incidence vector belongs to $\tilde{\mathcal{F}}$.

The solution S^1 is obtained from the solution S^0 introduced in the proof of Theorem 1 as follows. We install a set of $BP(\tilde{S})$ non previously used and different subbands $\tilde{W} \subseteq W$ over the arc $\tilde{e}$ and we add the pairs $(\tilde{e}, w)$, $w \in \tilde{W}$ to S^0. In other words, we let $y^{S^1}_{\tilde{e}w} = y^{S^0}_{\tilde{e}w} + 1$, for all $w \in \tilde{W}$. We then associate to every pair $(\tilde{e}, w)$, $w \in \tilde{W}$ of the solution a path in G_2 that is the arc $\tilde{a} = (u', v') \in A_2$. This is possible since the subbands of $\tilde{W}$ are newly installed and are not assigned physical paths in the solution S^0. The disjunction constraints are therefore satisfied. Now let us associate with the commodities of $\tilde{S}$ the pairs $(\tilde{e}, w)$, $w \in \tilde{W}$, in addition to their initial routing paths. Such assignment is possible since there are enough subbands installed on $\tilde{e}$ so that the capacity constraint (3) for every pair $(\tilde{e}, w)$ such that $w \in W$ is installed on $\tilde{e}$ is satisfied. It is clear that the solution S^1 is feasible and $(x^{S^1}, y^{S^1}, z^{S^1})$ belongs to $\tilde{\mathcal{F}}$. Hence, $\tilde{\mathcal{F}} \neq \emptyset$, and therefore, is a proper face of $P(G_1, G_2, K, C)$.

Now suppose that there exists a facet-defining inequality $\lambda x + \mu y + \nu z \geqslant \xi$, such that

$$\tilde{\mathcal{F}} \subseteq \mathcal{F} = \{(x, y, z) \in P(G_1, G_2, K, C) : \lambda x + \mu y + \nu z = \xi\}.$$

We will show that there exists a scalar $\rho \in \mathbb{R}$ such that $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$.

Consider a pair $(e, w) \in A_1 \times W$. Let S^2 be the solution obtained from S^1 by adding an arc $f \in A_2$ to Δ^1_{ew}. We suppose without loss of generality that $f = (u', v') \notin \Delta^1_{ew}$ (if this is not true, then we can add two arcs (u', s), (s, v'), $s \in V_2$ to the current solution, and remove f). In other words, S^2 is such that $F^2_1 = F^1_1$, $F^2_2 = F^1_2$, $\Delta^2 = \Delta^1 \cup \{f\}$ and $W^2 = W^1$. It is clear that S^2 is feasible and its incidence vector belongs to $\tilde{\mathcal{F}}$. Comparing both incidence vectors of S^1 and S^2 implies that $\nu^{ew}_f = 0$. As f, e and w are chosen arbitrarily in A_2, A_1 and W, we obtain that $\nu^{ew}_a = 0$, for all $e \in A_1$, $w \in W$ and $a \in A_2$. By similar arguments, we can show that $\mu_{ew} = 0$, for all $e \in A_1 \setminus \{\tilde{e}\}$, $w \in W$ and $\lambda^k_{ew} = 0$, for all $k \in K \setminus \tilde{S}$, $e \in A_1 \setminus \{\tilde{e}\}$ and $w \in W$.

It is easy to show that $\lambda^k_{ew} = 0$, for any commodity $k \in \tilde{S}$, $e \in A_1 \setminus \{\tilde{e}\}$, $w \in W$, by constructing further feasible solutions where k is shifted to a routing path that avoids using arc $\tilde{e}$. Besides, if we pick a commodity $\tilde{k}$ in $K \setminus \tilde{S}$, and a subband installed on $\tilde{e}$, say $\tilde{w}$, then we can easily see that a solution obtained by associating the pair $(\tilde{e}, \tilde{w})$ to $\tilde{k}$ in addition to its routing path remains feasible and its incidence vector belongs to both $\tilde{\mathcal{F}}$ and $\mathcal{F}$ (by condition (ii) of Theorem 6). This observation implies that $\lambda^k_{\tilde{e}w} = 0$, for all $k \in K \setminus \tilde{S}$, $w \in W$.

Now let $\hat{k}$ be a commodity arbitrarily choosen in $\tilde{S}$, and let $\hat{w}$ be the subband of W such that ($\tilde{e}$, $\hat{w}$) is associated with the routing of $\hat{k}$. We will construct a solution S^* from S^0 such that all the commodities of $\tilde{S} \setminus \{\hat{k}\}$ use $\tilde{W} \setminus \{\hat{w}\}$ for their routing and $\hat{w}$ is completely devoted to the commodity $\hat{k}$. Such a solution is feasible and its incidence vector belongs to $\tilde{\mathcal{F}}$ as well as $\mathcal{F}$ thanks to condition (iii) of Theorem 6.

Now if we remove some commodity, say $\hat{k}$, of $\tilde{S}$ from $\tilde{e}$ and shift it to a routing path that does no longer use $\tilde{e}$, then by condition (iii) of Theorem 6, setting $\sum_{w \in W} y_{\tilde{e}w}$ to BP($\tilde{S}$) - 1 keeps the new solution feasible. Moreover, its incidence vector belongs to $\tilde{\mathcal{F}}$ and then to $\mathcal{F}$. As a consequence, we obtain, by comparing both solutions, that $\lambda^{\hat{k}}_{\tilde{e}\hat{w}} = \mu_{\tilde{e}\hat{w}}$. Since, $\tilde{k}$ and $\tilde{w}$ are arbitrarily chosen in $\tilde{S}$ and W, respectively, we obtain that

$$\mu^{\tilde{e}w} = \rho, \qquad \text{for all } w \in W, \tag{29}$$

$$\lambda^k_{\tilde{e}w} = \rho, \text{ for all } k \in \tilde{S}, w \in W, \tag{30}$$

where $\rho \in \mathbb{R}$. From (29) and (30), we obtain that $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$, and the proof is complete.

References

1. http://sndlib.zib.de/home.action.
2. Lemon graph. https://lemon.cs.elte.hu/trac/lemon.
3. A. Atamtürk. On capacitated network design cut-set polyhedra. *Math. Program.*, 92(3):425–437, 2002.
4. A. Atamtürk. Cover and pack inequalities for (mixed) integer programming. *Annals of Operations Research*, 139(1):21–38, 2005.
5. P. Avella, S. Mattia, and A. Sassano. Metric inequalities and the network loading problem. *Discrete Optimization*, 4(1):103–114, 2007.
6. F. Barahona and A. R. Mahjoub. Facets of the balanced (acyclic) induced subgraph polytope. *Mathematical Programming*, 45(1-3):21–33, 1989.
7. P. Belotti, A. Capone, G. Carello, and F. Malucelli. Multi-layer mpls network design: The impact of statistical multiplexing. *Computer Networks*, 52(6):1291 – 1307, 2008.

8. A. Benhamiche, A. R. Mahjoub, N. Perrot, and E. Uchoa. Unsplittable non-additive capacitated network design using set functions polyhedra. *Computers & Operations Research*, 66:105 – 115, 2016.

9. D. Bienstock, S. Chopra, O. Günlük, and C-Y. Tsai. Minimum cost capacity installation for multicommodity network flows. *Mathematical Programming*, 81:177–199, 1995.

10. D. Bienstock and O. Günlük. Capacitated network design - polyhedral structure and computation. *INFORMS Journal on Computing*, 8:243–259, 1994.

11. N. Boland, A. Bley, C. Fricke, G. Froyland, and R. Sotirov. Clique-based facets for the precedence constrained knapsack problem. *Mathematical Programming*, 133(1-2):481–511, 2012.

12. S. Borne, E. Gourdin, B. Liau, and A. R. Mahjoub. Design of survivable IP-over-Optical networks. *Annals of Operations Research*, 146(1):41–73, 2006.

13. F. Buchali, R. Dischler, and X. Liu. Optical ofdm: A promising high-speed optical transport technology. *Bell Labs Technical Journal*, 14(1):125–146, 2009.

14. S. Chopra, I. Gilboa, and S.Trilochan Sastry. Source sink flows with capacity installation in batches. *Discrete Applied Mathematics*, 85(3):165 – 192, 1998.

15. G. Dahl, A. Martin, and M. Stoer. Routing through virtual paths in layered telecommunication networks. *Operations Research*, 47(5):pp. 693–702, 1999.

16. B. Fortz and Michael Poss. An improved Benders decomposition applied to a multi-layer network design problem. *Operations Research Letters*, 37(5):359–364, 2009.

17. M. R. Garey and D. S. Johnson. *Computers and Intractability: A Guide to the Theory of NP-Completeness*. W.H. Freeman, San Francisco, 1979.

18. A. V. Goldberg and R. E. Tarjan. A new approach to the maximum-flow problem. *Journal of the Association for Computing Machinery 35*, pages 921–940, 1988.

19. Ralph E. Gomory. Some polyhedra related to combinatorial problems. *Linear Algebra and its Applications*, 2(4):451 – 558, 1969.

20. K. L. Hoffman and M. W. Padberg. Solving airline crew scheduling problems by branch-and-cut. *Management Science*, 39(6):657–682, June 1993.

21. K. Kaparis and A. N. Letchford. Separation algorithms for 0-1 knapsack polytopes. *Mathematical Programming*, 124(1-2):69–91, 2010.

22. A. Knippel and B. Lardeux. The multi-layered network design problem. *European Journal of Operational Research*, 183(1):87 – 99, 2007.

23. E. Le Rouzic. Network evolution and the impact in core networks. In *Optical Communication (ECOC), 2010 36th European Conference and Exhibition on*, pages 1–8, 2010.

24. S. Mattia. A polyhedral study of the capacity formulation of the multilayer network design problem. *Networks*, 62(1):17–26, 2013.

25. G. L. Nemhauser and G. Sigismondi. A Strong Cutting Plane/Branch-and-Bound Algorithm for Node Packing. *Journal of the Operational Research Society*, 43(5):443–457, 1992.

26. G. L. Nemhauser and L. A. Wolsey. *Integer and combinatorial optimization*. Wiley-Interscience, New York, 1988.

27. S. Orlowski, A. M. C. A. Koster, C. Raack, and R. Wessäly. Two-layer network design by branch-and-cut featuring mip-based heuristics. In *Proceedings of the INOC 2007 also ZIB Report ZR-06-47*, Spa, Belgium, 2007.

28. S. Orlowski, C. Raack, A. M. C. A. Koster, G. Baier, T. Engel, and P. Belotti. Branch-and-cut techniques for solving realistic two-layer network design problems. In *Graphs and Algorithms in Communication Networks*, pages 95–118. Springer Berlin Heidelberg, 2010.

29. M. W. Padberg. On the facial structure of set packing polyhedra. *Mathematical Programming*, 5(1):199–215, 1973.

30. M. W. Padberg. A note on 0-1 programming. *Operations Research*, 23:833–837, 1979.

31. C. Raack, A. M. C. A. Koster, S. Orlowski, and R. Wessäly. On cut-based inequalities for capacitated network design polyhedra. *Networks*, 57(2):141–156, 2011.

32. S. Raghavan and D. Stanojević. Branch and price for WDM optical networks with no bifurcation of flow. *INFORMS Journal on Computing*, 23(1):56–74, 2011.

33. R. Weismantel. On the 0/1 knapsack polytope. *Mathematical Programming*, 77(3):49–68, 1997.