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23 JAN. 1987

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Explicative Models in Multicriteria Preference Analysis (*)

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N° 5 - 1976

Mai 1976

- * Prepared for EURO II, the second European congress on Operations Research, Stockholm, November 29 - December 1, 1976.
- * Research supported by COREF, 8 rue Domat, 75006 PARIS and DGRST (France) project number 75-7-0230 (action concertée Informatique en Sciences Humaines)

ABSTRACT

Most of the work which has been done in multicriteria decision-making deals with a normative or prescriptive attitude and involves therefore the problem of assessing a model of a global preference (value or utility function, outranking relation, a model used in an interactive approach, ...). The problem, then is how to aggregate completely, partially or locally the criteria.

But in a lot of problems, it is useful to adopt a descriptive or even explicative attitude in analyzing a multicriteria preference.

The problem is then to explain a wholistic preference whose model can be a binary relation (complete or partial) by the assessment of an analytic global preference, whose model can be a multicriteria one. If in the first attitude, the decision maker has to give his weights or trade-offs as an input of a model in order to build his global preference, in the second attitude, he gives his wholistic preference (through a binary relation) as an input of a model and obtains his weights or trade-offs as an output of the model. This second attitude has involved by the past the use of multiple linear regression. But in such a model, both, the wholistic preference and the criteria have to be quantitative variables. We present in this paper two models where the wholistic preference to explain is a binary relation. In the first model the explicative criteria are quantitative variables and the multicriteria model is a value function. In the second model, the explicative criteria are ordinal and the multicriteria model is a binary relation obtained by the aggregation of the ordinal criteria.

criterion g_i . Now assume that the wholistic observed preference can be known through a quantitative variable $y(a)$, then, the set of weights (trade-offs) $\{P_i\}$ can be estimated in using linear regression such that $y(a)$ is as closed as possible as $V(a)$. Different monotone transformations can be tested in order to choose those giving the best correlation between $y(a)$ and $V(a)$.

- But in a lot of cases, at least as far as preferences are concerned, the wholistic preference can be known only through an ordinal variable and more generally through a binary relation for which cases of non comparability or even intransitivity might occur. It is the purpose of this paper to present some explicative models of binary relations. In section 1, I present models based on additive value functions and in section 2 a model based on an ordinal aggregation of criteria. Section 3 is a short discussion on this problem formulation.

1 - EXPLICATIVE MODELS BASED ON VALUE FUNCTIONS

As in the linear regression model, assume in this section that the explicative model is an additive value function $V(a) = \sum_{i=1}^n P_i f_i [g_i(a)]$.

But now assume that the wholistic observed preference can be known only through a binary relation $R = (P, I)$ defined on $A \times A$ where P , the antisymmetric part of R represents strict preference and I , the symmetric part of R the indifference. R might be not complete (cases of non comparability) and intransitive. If R is a complete weak-order then we are in the special case of explaining an ordinal variable or ordinal wholistic judgement. In the two models presented here, the problem is

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Constraint (3) prevents to obtain the solution $P_i = 0$, for all i . Having used this model, I would like to present here some difficulties one might encounter when using it.

- 1) Using a weighted sum of the g_i , one has to assume that these criteria are graduations, that is to say that if $g_i(a) - g_i(b) = g_i(c) - g_i(d)$, the superiority of a over b has to be judged to be the same as the one of c over d . It is then important to be able to introduce monotone transformation f_i of the g_i .
- 2) If the number of criteria is high compared to the number of actions $|A|$, there might exist an infinite number of solution $\{P_i\}$ giving all the minimum value $F = 0$, some of the set being completely different. This is a great inconvenient since we search for the P_i and not for the F .
- 3) Only the antisymmetric part of R is taken into account. An indifference I observed in R is not taken into account in the error function.
- 4) The constraint (3) might appear arbitrary. I think that in most applications, one might suggest a more meaningful constraint.

Anyway, this model is most interesting, and one can easily avoid such difficulties in giving him some modifications.

1.2. Some extension of the ORDREG model

First as it is necessary to assume that $f_i[g_i(a)]$ is a graduation (see remark 1) here above), it is necessary to introduce the possibility to transform the criteria $g_i(a)$ into graduation in using for instance some monotone function $f_i(g_i)$.

Second in order to avoid the constraint (3). I suggest the following procedure. Assume now that $f_i(g_i)$ is a graduation, then any linear

In the error function F , one can choose the relative importance of a and b by considering the relative importance given to the fact of respecting condition (4) and condition (5). If this importance is the same, one can choose for instance $a = 1$, $b = 1/2$. In other case, one could choose a more complex linear form for F , in order for instance to give more importance to some couples (a_k, a_1) . If we impose condition (5), that is to say if $b > 0$, as it has been already said, the minimum of F will be strictly positif in most real cases and the solution $\{P_i\}$ will most probably be unique.

The output of such a model can be, besides the minimum of F , the set $\{P_i\}$ and the weak order $R' = (P', I')$ corresponding to the value function $V(a)$:

$$V(a_k) > V(a_1) \iff (a_k, a_1) \in P'$$

$$V(a_k) = V(a_1) \iff (a_k, a_1) \in I'$$

It is then possible to compare the initial relation R and the weak order R' with the symmetric difference or even better with the following numbers (or percentages) of transformation of pairs of $A \times A$

$n(P, P')$ number of strict preferences being the same

$n(I, I')$ number of indifferences being the same

$k(P, P')$ number of strict preferences of R being inversed in R'

$n(P, I')$ number of strict preferences in R being transformed into indifferences in R'

$n(I, P')$ number of indifferences in R being transformed into strict preferences in R'

$n(R - PUI, P')$ number of non compared pairs being transformed into strict preferences in R'

$n(R - PUI, I')$ number of non compared pairs being transformed into indifferences in R'

If we consider a threshold α ($1/2 \leq \alpha < 1$), then the aggregation rule gives a complete relation $R_\alpha = (P_\alpha, I_\alpha)$ defined by :

$$(6) \quad (a_k, a_l) \in P_\alpha \iff g_{kl} > \alpha \iff g_{lk} < 1 - \alpha$$

$$(7) \quad (a_k, a_l) \in I_\alpha \iff 1 - \alpha \leq g_{kl} \leq \alpha$$

For $\alpha = 1/2$, this method is the majority rule.

For $\alpha = 1 - \varepsilon$ (ε very small), this method is the unanimity rule (defined on strict preference).

2.2. The corresponding explicative model

Let $R = (P, I)$ a given binary relation to be explained with the help of n binary relation $\{r^1, \dots, r^n\}$ and the aggregation rule given in paragraph 2.1.

A set of weights $\underline{P} = \{P_1, \dots, P_n\}$ is an exact solution to the problem if the following conditions are satisfied.

$$(8) \quad \begin{cases} (a_k, a_l) \in P \implies g_{kl} > \alpha \iff g_{lk} < 1 - \alpha \\ (a_k, a_l) \in I \implies 1 - \alpha \leq g_{kl} \leq \alpha \iff g_{kl} \leq \alpha \text{ and } g_{lk} \leq \alpha \end{cases}$$

Two cases must be considered since there might exist or not, exact solution to this problem (*).

1st case - There exist no exact solution $\underline{P}$ satisfying conditions (8).

Let us introduce the new variables defined by :

(*) The same attitude could be observed when using the ORDREG model.

This case occurs when the linear program here above give a solution $\underline{P}$ with the value 0 for F .

In this case it is most important to characterize as well as possible the set of the feasible solution $\underline{P}$. One way to characterize such a set is to give solutions $\underline{P}$ as different as possible and satisfying to all the conditions (8). These conditions can be expressed under the following form :

$$(10) \left\{ \begin{array}{l} (a_k, a_l) \in P \Rightarrow \sum_{i=1}^n P_i r_{kl}^i > \alpha \\ (a_k, a_l) \in I \Rightarrow \sum_{i=1}^n P_i r_{kl}^i \leq \alpha \text{ and } \sum_{i=1}^n P_i r_{lk}^i \leq \alpha \end{array} \right.$$

The set of these linear constraints define in R^n a polyedron since there exist at least one solution $\underline{P}$ which satisfies all the constraints. This polyedron is the set $\mathcal{P}$ of all the feasible solution $\underline{P}$.

In order to caraterize this set $\mathcal{P}$, I suggest the following procedure :

- Let us find the solutions $\underline{P}$ maximizing and minimizing each of the n weight separatly. One has to solve $2n$ linear programs defined by :

$$\begin{array}{ll} \cdot \sum P_i = 1 & \cdot \sum P_i = 1 \\ \cdot \text{constraints (10)} & \cdot \text{constraints (10)} \\ \cdot \text{Max } P_i, i = 1, n & \cdot \text{Min } P_i, i = 1, n \end{array}$$

- If the number of explicative binary relations n is to high, it would be to heavy to solve $2n$ linear program, one can then maximize and minimize some linear form of the weight $\{P_i\}$. One can for instance make a clustering of the explicative relation and if we have m clusters, define m linear form, sum of the corresponding weights.

Assume that the nuisance (noise, air pollution) is created only by traffic. This nuisance will affect a certain number of people, depending how far they are from the road. Let then d be the distance of a point in the nuisance area from the road. How can we aggregate the two sub-dimensions traffic T and distance d into one dimension of nuisance $g(T,d)$. One solution consist in using the following procedure which use the model presented in paragraph 1.2.

Let A be a set of given couples (T,d) $a_1 = (T_1, d_1), \dots, a_k = (T_k, d_k), \dots$. Ask a set of people to compare in term of nuisance different couples of A (with the help of magnetic tapes for instance). After such an experimental study, one could obtain a binary relation R on $A \times A$ which could be explained with variables T and d . One could test different functional forms as :

$$g(a) = g(T,d) = T e^{-\alpha d}$$

An additive form could be

$$V(a) = \text{Log} [g(a)] = \text{Log } T - \alpha d$$

The adapted model of type of the paragraph 1.2. would be here :

$$\text{Min } F = a \sum_{(a_k, a_1) \in P} z_{k1} + b \sum_{(a_k, a_1) \in I} (z_{k1} + z_{1k})$$

$$V(a_k) - V(a_1) + z_{1k} - z_{k1} = 0 \text{ for all } (a_k, a_1) \in R$$

$$z_{k1} \geq 0 \text{ for all } (a_k, a_1) \in R.$$

Having estimated the parameter α through such a model, a criterion could be assessed in using this nuisance scale $g(T,d) = T e^{-\alpha d}$ and in considering as an indicator of modulation (see Roy (1975-1)) the number of people $N(T,d)$ affected by traffic T and located at a distance d from the road.

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