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ON THE PROBLEM OF WEIGHTS IN
MULTIPLE CRITERIA DECISION MAKING
(THE NONCOMPENSATORY APPROACH) (*)

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CONTENTS

	<u>Pages</u>
Résumé	1
Abstract	1
1. Introduction	2
2. Notations and preliminary definitions	2
3. The compensatory and noncompensatory approaches to multiple criteria decision problems	3
4. How to define "more important than" ?	5
5. How to construct $\gg$?	7
6. Theoretical approach to the problem of additive weights : Representation theorems	9
7. TACTIC	13
References	16

A PROPOS DU PROBLEME DES POIDS
EN ANALYSE MULTICRITERE
(L'APPROCHE NON COMPENSATOIRE)

RESUME

Ce cahier montre comment il est possible de définir la notion d'"importance relative des attributs" dans le cadre d'une approche non compensatoire des problèmes de décision multicritères. La question des poids apparaît alors comme un problème de représentation numérique de relations binaires. Divers résultats théoriques sont présentés à ce sujet ainsi que les grandes lignes d'une méthode d'aide à la décision (TACTIC) se fondant sur les notions introduites dans ce cahier.

ON THE PROBLEM OF WEIGHTS
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ABSTRACT

This paper shows how it is possible to define the notion of "relative importance of attributes" within the framework of the noncompensatory approach to multiple criteria decision problems. The problem of weights then appears as a problem of functional representation of relations. We state some theoretical results about that problem and give some indications concerning a practical decision-aid method (TACTIC) based on the ideas introduced in the paper.

1. INTRODUCTION

Roughly speaking, the problem of multiple criteria decision making consists in constructing a global preference relation on a set of alternatives evaluated on several attributes taking into account the decision-maker's personality. In order to do that, we have in particular to obtain from the decision-maker inter-criteria information and to model it in a mathematical form. This paper is devoted to the study of this topic. It is organized as follows. In section 2, we introduce some notations and preliminary definitions. In section 3, we present the two main aggregation logics which can be used in multiple criteria decision-making : the compensatory approach and the noncompensatory approach. Only within the framework of the latter one can the notion of relative importance of attributes be clearly defined. In section 4, we propose a few possible definitions of this notion. Sections 5 and 6 are devoted to the problem of weights. Finally, in section 7, we briefly present a decision-aid method (TACTIC) based on the ideas introduced in the paper.

2. NOTATIONS AND PRELIMINARY DEFINITIONS

The following notations will be used : $\mathbb{R}$ = set of real numbers, $\mathbb{R}^+ = \{ r \in \mathbb{R} | r \geq 0 \}$, $\mathbb{Q}$ = set of rational numbers, $\mathbb{Q}_1 = \{ q \in \mathbb{Q} | q \geq 1 \}$, $\mathbb{N} = \{ 0, 1, 2, \dots, n, \dots \}$.

Throughout the paper, we shall let $\mathcal{A}$ denote a set of alternatives evaluated on a set $\Omega = \{ 1, 2, \dots, m \}$ (with $m \in \mathbb{N}$ and $m \geq 2$) of attributes in a given decision problem.

Definition 2.1 An attribute i , $i \in \Omega$, consists of a set X_i of at least two elements expressing different levels of some underlying dimension, and of a total strict order P_i on X_i (i.e. an irreflexive, transitive and weakly connected binary relation on X_i) modelling the decision-maker's preferences among the levels of X_i .

Definition 2.2 We shall note S the set of all pairs of disjoint sets of attributes. $S = \{(A, B) \mid A, B \subseteq \Omega \text{ and } A \cap B = \emptyset\}$.

We shall also use the following notations. $\mathcal{P}(\Omega)$ = set of all subsets of Ω , $\prod_{i=1}^m X_i$ = cartesian product $X_1 \times X_2 \times \dots \times X_m$, $x = (x_1, \dots, x_m)$ and $y = (y_1, \dots, y_m)$ = elements of $\prod_{i=1}^m X_i$.

3. THE COMPENSATORY AND NONCOMPENSATORY APPROACHES TO MULTIPLE CRITERIA DECISION PROBLEMS

As mentioned in the introduction, the problem of multiple criteria decision making consists for us in constructing a global preference relation $\succ$ on $\mathcal{P}(\Omega)$ (i.e. an asymmetric binary relation on $\mathcal{P}(\Omega)$) taking into account the decision-maker's personality. We insist on the term "constructing" because we completely agree with Roy's and Bouyssou's views - see for instance Bouyssou (1984) - that the analyst's problem is not to describe basic attitudes but to structure the decision-maker's preferences on the basis of conventions he agrees with.

The first and most important convention to be discussed with the decision-maker certainly is the aggregation logic which will be used.

A possible way is to adopt a compensatory aggregation logic. This approach is based on the idea that, to compare two alternatives $a, b \in \mathcal{A}$, we have to consider the differences between the evaluations of a and b on each attribute and to declare that " $a \succ b$ " iff the differences favourable to a do more than compensate those favourable to b . Practically, to use this approach, we have to determine, for each $i \in \Omega$, a mapping $\Psi_i : X_i \rightarrow \mathbb{R}$ which provides an interval scale of measurement - see Roberts (1979), ch.2 - and to assess scaling constants in order to precise how the compensation must be accomplished - given the scales Ψ_i - between the different attributes. On this topic we refer to Keeney and Raiffa (1979) and to Vansnick (1984) but we want here to emphasize that the scaling constants which appear in the compensatory approach depend on the scales Ψ_i ; thus they do NOT characterize the intrinsic relative importance of attributes. Let us also point out that this approach can be related to Borda's method in social choice theory - see Borda (1781) 4. An example will help to understand this statement. Consider an election with three candidates C_1 , C_2 and C_3 and five voters, and suppose that, for three voters, C_1 is preferred to C_2 who is preferred to C_3 , and for two voters, C_2 is preferred to C_3 who is preferred to C_1 . In Borda's method, the global comparison between two candidates, say C_1 and C_2 , is achieved by determining the differences between the ranks of these two candidates for each voter and by comparing the sum of the differences in favour of one candidate to the sum of those favourable to the other one; here, we have : "sum of the differences in favour of C_1 = $3 \times (2-1) = 3$ ", "sum of the differences in favour of C_2 = $2 \times (3-1) = 4$ ", so that, for Borda, C_2 is globally preferred to C_1 . Borda proposed his method in 1781. Four years later, Condorcet proposed another method, basically different - see Condorcet (1785) 5. In Condorcet's method, the global comparison between C_1 and C_2 is achieved by determining the set of voters who prefer C_1 to C_2 and the set of voters who prefer C_2 to C_1 and by looking which one is the most important; here, three voters prefer C_1 to C_2 and two voters prefer C_2 to C_1 , so that, for Condorcet,

C_1 is globally preferred to C_2 . In a multiple criteria decision problem, it is possible to propose to the decision-maker to construct a global preference relation $\succ$ on $\mathcal{O}$ by using an aggregation logic related to Condorcet's idea. This is what we shall call the noncompensatory approach to multiple criteria decision problems. Briefly, it consists in :

- defining what is meant by "A is more important than B" (notation : $A \gg B$), where $(A,B) \in S$,
- obtaining from the decision-maker information concerning the relative importance of some subsets of Ω (from a practical point of view, it is impossible to ask this information for all the elements of S),
- using a judicious method to extrapolate the information elicited from the decision-maker so as to can say, for each $(A,B) \in S$, if it is considered that $A \gg B$, $B \gg A$ or $[\text{Not } (A \gg B) \text{ and Not } (B \gg A)]$ (notation : $A \approx B$); taking into account the definition of "more important than" it is then possible to construct $\succ$ on $\mathcal{O}$.

The next two sections are devoted to the study of these points.

4. HOW TO DEFINE "MORE IMPORTANT THAN" ?

We can imagine three definitions of different degrees of generality for "A is more important than B", where $(A,B) \in S$. In order to be clearly understood, we propose hereafter these three definitions rather than only the most general one.

Definition 4.1 $A \gg B$ (for the decision-maker) iff $\forall x, y \in \prod_{i=1}^m X_i$ such that $x_i \succ_i y_i \forall i \in A$ and $x_j = y_j \forall j \in \Omega \setminus (A \cup B)$, x is globally preferred to y (by the decision-maker).

Let us point out that the idea of noncompensation between attributes clearly appears in this definition : it is impossible to compensate " $x_i \succ_i y_i \forall i \in A$ " by anything on the attributes of B .

Definition 4.1 is simple and can directly be applied because it uses, for each attribute i , the total strict order P_i which is known. This order models the preferences that the decision-maker has, independently of any real context, towards the levels of X_i . However the decision-maker may think that, in the case of a noncompensatory approach, some differences of levels are not relevant. To take into account this possibility, it is interesting to introduce the following definition.

Definition 4.2 $A \gg B$ (for the decision-maker) iff $\forall x, y \in \prod_{i=1}^m X_i$ such that $x_i \succ_i y_i \quad \forall i \in A$ and $[\text{Not } (x_j \succ_j y_j) \text{ and } \text{Not } (y_j \succ_j x_j)] \quad \forall j \in \Omega \setminus (A \cup B)$, x is globally preferred to y (by the decision-maker), where for each $i \in \Omega$, $\succ_i$ is a binary relation on X_i included in P_i which must be determined with the decision-maker.

We propose to qualify as "noncompensatory" the decision-aid methods developed on the basis of this definition. An example of such a method is the "lexicographical ordering" (often applied with $\succ_i = P_i$). For a theoretical study of noncompensatory preference structures, we refer to Fishburn (1976) and to Bouyssou and Vansnick (1984).

It might happen that some decision-maker agrees on the principle of a noncompensatory approach but does not like the "absolute" character of the previous definition. It is the reason why we propose a third definition which, in a sense, allows to restrict the "domain of noncompensation".

Definition 4.3 $A \gg B$ (for the decision-maker) iff $\forall x, y \in \prod_{i=1}^m X_i$ such that $x_i \succ_i y_i \quad \forall i \in A$, $[\text{Not } (x_j \succ_j y_j) \text{ and } \text{Not } (y_j \succ_j x_j)] \quad \forall j \in \Omega \setminus (A \cup B)$ and $\text{Not } (y_k \succ_k x_k) \quad \forall k \in B$, x is globally preferred to y (by the decision-maker), where, for each $i \in \Omega$, $\succ_i$ and $\succsim_i$ are two binary relations on X_i with $\succ_i \subset \succsim_i \subset P_i$, which must be determined with the decision-maker.

In this definition, we still have the noncompensation idea but it is limited by the introduction of the condition "Not $(y_k \succ_k x_k) \quad \forall k \in B$ "; this condition expresses the idea that, for each attribute $k \in B$, y_k must be not too much preferred to x_k . We propose to call "noncompensatory methods with veto" the decision-aid methods developed on the basis of definition 4.3. An example of such a method (TACTIC) is presented in section 7. For a theoretical study of their underlying preference structures, we refer to Bouyssou and Vansnick (1984).

The use of a noncompensatory method with veto requires to define the binary relations $\succ_i$ and V_i on each X_i , $i \in \Omega$, and to obtain information concerning the relative importance of attributes. Let us point out that these points are very interconnected and that it is difficult to present a general procedure to question the decision-maker about them because it depends on practical details of the considered method. For instance, the elements (A, B) of S , about which it is interesting to ask the decision-maker in order to know whether, for him, $A \gg B$, $B \gg A$ or $A \approx B$, depend on the particular procedure which will be used to extrapolate this information. In section 7, we give some indications concerning this problem in the case of the method TACTIC.

5. HOW TO CONSTRUCT $\gg$?

The actual problem in the noncompensatory approach to multiple criteria decision problems is to construct the binary relation $\gg$ from the partial information given by the decision-maker concerning the relative importance of the attributes. Let us first note that this information can always be presented by defining two disjoint subsets of S , $S_{\gg}$ and $S_{\approx}$, such that:

$(A, B) \in S_{\gg}$ iff we can conclude from the discussion with the decision-maker that for him $A \gg B$, and $(A, B) \in S_{\approx}$ iff we can conclude from the discussion with the decision-maker that, for him, $A \approx B$. Normally, these subsets are such that, $\forall A, B \in \mathcal{P}(\Omega)$:

$$(5.1) (A, B) \in S_{\gg} \Rightarrow (B, A) \notin S_{\gg}$$

$$(5.2) (A, B) \in S_{\approx} \Rightarrow (B, A) \in S_{\approx}$$

$$(5.3) A \neq \emptyset \Rightarrow (A, \emptyset) \in S_{\gg}$$

$$(5.4) (\emptyset, \emptyset) \in S_{\approx}$$

If not so, the analyst should rethink the problem with the decision-maker.

A general idea for solving the extrapolation problem consists in looking for a mapping $w : \mathcal{P}(\Omega) \rightarrow \mathbb{R}$ and an asymmetric binary relation $\mathcal{R}$ on $\mathbb{R}$ such that, $\forall A, B \in \mathcal{P}(\Omega)$:

$$\begin{cases} (A, B) \in S_{\gg} \Rightarrow w(A) \mathcal{R} w(B) \\ (A, B) \in S_{\approx} \Rightarrow \text{Not } (w(A) \mathcal{R} w(B)) \end{cases}$$

and in considering that, $\forall (A, B) \in S$:

$$A \gg B \text{ iff } w(A) \mathcal{R} w(B) .$$

An important particular case arises when it is imposed that w also satisfies the following property :

$$\forall (A, B) \in S : w(A \cup B) = w(A) + w(B) .$$

In that case, $w(\emptyset) = 0$ and the mapping w is entirely determined by the knowledge of $w(\{1\})$, $w(\{2\})$, ... $w(\{m\})$. These m numbers can be considered as a set of weights representing the relative importance of attributes for the decision-maker given the obtained information; moreover, we know how to use these weights : additively and with the relation $\mathcal{R}$.

Among all the relations we can take for $\mathcal{R}$, we shall hereafter study the asymmetric binary relations $>_{\epsilon, \rho}$ defined on $\mathbb{R}$ by : $\forall r_1, r_2 \in \mathbb{R}$, $r_1 >_{\epsilon, \rho} r_2$ iff $r_1 > \rho \cdot r_2 + \epsilon$ with $\epsilon \in \mathbb{R}^+$ and $\rho \in \mathbb{Q}_1$. As we shall see, these relations are specially interesting both from a theoretical point of view (see section 6) and from a practical point of view (see section 7).

6. THEORETICAL APPROACH TO THE PROBLEM OF ADDITIVE WEIGHTS: REPRESENTATION THEOREMS

In our presentation, the problem of weights appears as a problem of functional representation of relations and is thus relevant to measurement theory - see Krantz et al. (1971) and Roberts (1979). This section is devoted to the statement of two representation theorems which give necessary and sufficient conditions for the existence of additive weights in the cases where $\mathcal{R} = \succ_{o,\rho}$ and $\mathcal{R} = \succ_{\varepsilon,\rho}$.

Throughout this section, A, B, C, D, E, F, G and H (with or without index) will represent any elements of $\mathcal{P}(\Omega)$ and, for each $A \in \mathcal{P}(\Omega)$, $\mathcal{M}(A)$ will denote the $1 \times m$ - matrix $(\delta_1 \ \delta_2 \ \dots \ \delta_m)$ defined by $\delta_i = 1$ iff $i \in A$ and $\delta_i = 0$ otherwise for each $i \in \Omega$.

Theorem 6.1 (case $\mathcal{R} = \succ_{o,\rho}$)

Let $R_{\gg}$ and $R_{\approx}$ be two nonempty binary relations on $\mathcal{P}(\Omega)$ and ρ a given element in $\mathbb{Q}_1$.

(6.1) There exist $w_1, w_2, \dots, w_m \in \mathbb{R}$ such that, $\forall A, B \in \mathcal{P}(\Omega)$:

$$AR_{\gg}B \Rightarrow w(A) \succ_{\rho} w(B)$$

$$\text{and } AR_{\approx}B \Rightarrow w(A) \leq_{\rho} w(B)$$

$$\text{where } w(A) = \sum_{i \in A} w_i \quad (w(\emptyset) = 0)$$

iff

(6.2) $\forall k, n \in \mathbb{N}$,

$$\sum_{i=0}^n \mathcal{M}(A_i) + \rho \cdot \sum_{o < j < k+1} \mathcal{M}(C_j) \neq \rho \cdot \sum_{i=0}^n \mathcal{M}(B_i) + \sum_{o < j < k+1} \mathcal{M}(D_j)$$

whenever $\begin{cases} A_i R_{\gg} B_i \quad \forall i \in \{0, 1, \dots, n\} & \text{and} \\ D_j R_{\approx} C_j \quad \forall j \in \mathbb{N} \text{ such that } o < j < k+1. \end{cases}$

The relations $R_{\gg}$ and $R_{\approx}$ which appear in this theorem clearly correspond to the sets $S_{\gg}$ and $S_{\approx}$ previously introduced; let us point out that, if we suppose that

(6.3) $AR_{\gg} \emptyset$ for each $A \in \mathcal{P}(\Omega)$ such that $A \neq \emptyset$, which corresponds to property (5.3) of $S_{\gg}$, then (6.1) implies that $w_i > 0$ for each $i \in \Omega$. If they exist, the weights are thus strictly positive in the case where $\mathcal{R} = \succ_{\varepsilon, \rho}$.

Theorem 6.2 (case $\mathcal{R} = \succ_{\varepsilon, \rho}$)

Let $R_{\gg}$ and $R_{\approx}$ be two nonempty binary relations on $\mathcal{P}(\Omega)$ and ρ a given element in $\mathbb{Q}_1$.

(6.4) There exist $w_1, w_2, \dots, w_m, \varepsilon \in \mathbb{R}$ such that, $\forall A, B \in \mathcal{P}(\Omega)$:

$$\begin{aligned} AR_{\gg} B &\Rightarrow w(A) > \rho \cdot w(B) + \varepsilon \\ \text{and } AR_{\approx} B &\Rightarrow w(A) \leq \rho \cdot w(B) + \varepsilon \\ \text{where } w(A) &= \sum_{i \in A} w_i \quad (w(\emptyset) = 0) \end{aligned}$$

iff

(6.5) $\forall n \in \mathbb{N}$,

$$\sum_{i=0}^n m(A_i) + \rho \cdot \sum_{i=0}^n m(C_i) \neq \rho \cdot \sum_{i=0}^n m(B_i) + \sum_{i=0}^n m(D_i)$$

whenever $A_i R_{\gg} B_i$ and $D_i R_{\approx} C_i \quad \forall i \in \{0, 1, \dots, n\}$.

Let us point out that, if $R_{\gg}$ and $R_{\approx}$ satisfy (6.3) and $\emptyset R_{\approx} \emptyset$, which corresponds to properties (5.3) and (5.4) of $S_{\gg}$ and $S_{\approx}$, then (6.4) implies that $\varepsilon \geq 0$ and that $w_i > \varepsilon$ for each $i \in \Omega$. If they exist, the weights are thus strictly positive in the case where $\mathcal{R} = \succ_{\varepsilon, \rho}$.

The proofs of theorems 6.1 and 6.2 are similar. Both are based on one of the various theorems of the alternative - see Mangasarian (1969), ch.2 - :

Motzkin's transposition theorem

Let $I \in \mathbb{R}^{i \times h}$, $J \in \mathbb{R}^{j \times h}$, $K \in \mathbb{R}^{k \times h}$ be given matrices with I nonempty.

There is $Z \in \mathbb{R}^{h \times 1}$ verifying at the same time

$$I.Z > 0_{1 \times 1}, \quad J.Z \geq 0_{j \times 1} \quad \text{and} \quad K.Z = 0_{k \times 1}$$

or

There are $\lambda \in \mathbb{R}^{1 \times i}$, $\mu \in \mathbb{R}^{1 \times j}$ and $\tau \in \mathbb{R}^{1 \times k}$ verifying at the same time $\lambda.I + \mu.J + \tau.K = 0_{1 \times h}$, $\lambda \geq 0_{1 \times i}$, $\mu \geq 0_{1 \times j}$ and $\lambda \neq 0_{1 \times i}$

but never both.

We shall give here only the most difficult proof : that of theorem 6.2. Let us first observe that (6.4) can be stated as :

$$(6.4 \text{ bis}) \quad \text{There exist } W = \begin{bmatrix} w_1 \\ \vdots \\ w_m \end{bmatrix} \in \mathbb{R}^{(m+1) \times 1} \text{ such that, } \forall A, B \in \mathcal{P}(\Omega):$$

$$\begin{cases} AR \gg B \Rightarrow (\mathcal{M}(A) \ 0).W > (\rho.\mathcal{M}(B) \ 1).W \\ AR \approx B \Rightarrow (\mathcal{M}(A) \ 0).W \leq (\rho.\mathcal{M}(B) \ 1).W \end{cases}$$

where, $\forall r \in \mathbb{R}$ and $\forall 1 \times m$ -matrix L , $(L \ r)$ represents the $1 \times (m+1)$ -matrix the first m elements of which are those of L and the last element of which is r .

Proof of theorem 6.2

• (6.4 bis) $\Rightarrow$ (6.5) Let $n \in \mathbb{N}$, $A_i R \gg B_i$ and $D_i R \approx C_i$ $\forall i \in \{0, 1, \dots, n\}$. By assumption, there is $W \in \mathbb{R}^{(m+1) \times 1}$ such that, $\forall i \in \{0, 1, \dots, n\}$, $(\mathcal{M}(A_i) \ 0).W > (\rho.\mathcal{M}(B_i) \ 1).W$ and $(\mathcal{M}(D_i) \ 0).W \leq (\rho.\mathcal{M}(C_i) \ 1).W$.

After summation, we have

$$\left(\sum_{i=0}^n \mathcal{M}(A_i) + \rho \cdot \sum_{i=0}^n \mathcal{M}(C_i) \right) (n+1).W > \left(\rho \cdot \sum_{i=0}^n \mathcal{M}(B_i) + \sum_{i=0}^n \mathcal{M}(D_i) \right) (n+1).W.$$

This implies

$$\sum_{i=0}^n \mathcal{M}(A_i) + \rho \cdot \sum_{i=0}^n \mathcal{M}(C_i) \neq \rho \cdot \sum_{i=0}^n \mathcal{M}(B_i) + \sum_{i=0}^n \mathcal{M}(D_i).$$

• (6.5) $\Rightarrow$ (6.4 bis). We shall work by contraposition and prove that Not (6.4 bis) $\Rightarrow$ Not (6.5)

Let $\{(E_i, F_i) | i = 1, 2, \dots, M\}$ be the set of elements of $\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)$ such that $E_i R_{\gg} F_i$ and $\{(G_j, H_j) | j = 1, 2, \dots, N\}$ the set of elements of $\mathcal{P}(\Omega) \times \mathcal{P}(\Omega)$ such that $G_j R_{\approx} H_j$ (the relations $R_{\gg}$ and $R_{\approx}$ being nonempty, we have $M, N \geq 1$). By assumption (Not (6.4 bis)), there is no $W \in \mathbb{R}^{(m+1) \times 1}$ such that,

$$\begin{cases} \forall 1 \leq i \leq M : (\mathcal{M}(E_i) - \rho \cdot \mathcal{M}(F_i) - 1) \cdot W > 0 \\ \forall 1 \leq j \leq N : (\rho \cdot \mathcal{M}(H_j) - \mathcal{M}(G_j) - 1) \cdot W \geq 0 \end{cases}$$

Therefore, according to Motzkin's transposition theorem, there are $\lambda_1, \dots, \lambda_M, \mu_1, \dots, \mu_N \in \mathbb{R}^+$ with $(\lambda_1 \dots \lambda_M) \neq 0_{1 \times M}$ such that

$$\sum_{i=1}^M \lambda_i \cdot (\mathcal{M}(E_i) - \rho \cdot \mathcal{M}(F_i) - 1) + \sum_{j=1}^N \mu_j \cdot (\rho \cdot \mathcal{M}(H_j) - \mathcal{M}(G_j) - 1) = 0_{1 \times (m+1)}.$$

As the elements of $(\mathcal{M}(E_i) - \rho \cdot \mathcal{M}(F_i) - 1)$ and $(\rho \cdot \mathcal{M}(H_j) - \mathcal{M}(G_j) - 1)$ are rational numbers, this implies that there are $\lambda_1^*, \dots, \lambda_M^*, \mu_1^*, \dots, \mu_N^* \in \mathbb{N}$ such that :

$$(6.6) \quad \sum_{i=1}^M \lambda_i^* \cdot \mathcal{M}(E_i) + \rho \cdot \sum_{j=1}^N \mu_j^* \cdot \mathcal{M}(H_j) = \rho \cdot \sum_{i=1}^M \lambda_i^* \cdot \mathcal{M}(F_i) + \sum_{j=1}^N \mu_j^* \cdot \mathcal{M}(G_j)$$

$$(6.7) \quad \sum_{j=1}^N \mu_j^* = \sum_{i=1}^M \lambda_i^* \in \mathbb{N} \setminus \{0\}.$$

This proves that (6.5) does not hold. Indeed, if we let :

$$1) \quad n = \sum_{i=1}^M \lambda_i^* - 1 = \sum_{j=1}^N \mu_j^* - 1,$$

$$2) \quad \text{for } i = 1, 2, \dots, M \text{ and } h = 0, 1, \dots, n, \quad A_h = E_i \text{ and } B_h = F_i$$

$$\text{if } \sum_{k=0}^{i-1} \lambda_k^* < h+1 \text{ and } h+1 \leq \sum_{k=0}^i \lambda_k^*$$

where, by convention, $\lambda_0^* = 0$,

$$3) \quad \text{for } j = 1, 2, \dots, N \text{ and } h = 0, 1, \dots, n, \quad C_h = H_j \text{ and } D_h = G_j$$

$$\text{if } \sum_{k=0}^{j-1} \mu_k^* < h+1 \text{ and } h+1 \leq \sum_{k=0}^j \mu_k^*$$

where, by convention, $\mu_0^* = 0$,

then - $n \in \mathbb{N}$ by (6.7)

- $A_h R_{\gg} B_h$ and $D_h R_{\approx} C_h \quad \forall h \in \{0, 1, \dots, n\}$ by definition

- (6.6) can be written

$$\sum_{h=0}^n \mathbb{M}(A_h) + \rho \cdot \sum_{h=0}^n \mathbb{M}(C_h) = \rho \cdot \sum_{h=0}^n \mathbb{M}(B_h) + \sum_{h=0}^n \mathbb{M}(D_h)$$

and the proof is complete.

Remark. When $\rho = 1$ and that $R_{\gg}$ and $R_{\approx}$ respectively are the asymmetric part and the symmetric part of a complete binary relation R on $\mathcal{P}(\Omega)$ ($\forall A, B \in \mathcal{P}(\Omega)$, ARB or BRA), the conditions (6.2) and (6.5) particularize into conditions introduced by Scott (1964), Fishburn (1969) and Domotor and Stelzer (1971) in theorems devoted to qualitative probability on finite sets. That isn't to be wondered at because our problem is very similar to the problem of representation of comparative probabilities - see Fine (1973).

7. TACTIC

TACTIC is a noncompensatory decision-aid method with veto using additive weights and the relations $\succ_{o,\rho}$ ($\rho \geq 1$). Its name comes from "Treatment of the Alternatives aCcording To the Importance of Criteria".

It consists of four parts :

- 1) a methodological part presenting a procedure for questioning the decision-maker,
- 2) an algorithm which determines a set of weights and a value for the parameter ρ from the information obtained in part 1 concerning the relative importance of attributes,

- 3) an algorithm which constructs a global preference relation $\succ$ on the set $\mathcal{A}$ of alternatives, taking into account the first two parts,
- 4) a procedure for representing this global preference relation by a graph easily understandable by the decision-maker.

The aim of the first part is to determine :

- (7.1) for each attribute $i \in \Omega$:
 - a measurable value function $\psi_i : X_i \rightarrow \mathbb{R}$ - see Sarin (1983) and Vansnick (1984) - ,
 - two constant thresholds α_i and β_i belonging to $\mathbb{R}^+$ (with $0 \leq \alpha_i < \beta_i$) for defining $\succ_i$ and V_i by : $\forall x_i, y_i \in X_i$,
 $x_i \succ_i y_i$ iff $\psi_i(x_i) > \psi_i(y_i) + \alpha_i$
 $x_i V_i y_i$ iff $\psi_i(x_i) \geq \psi_i(y_i) + \beta_i$,
 - in the uncertainty case, a utility function $u_i : X_i \rightarrow \mathbb{R}$ of the form $u_i = g_i \circ \psi_i$ where $g_i : \mathbb{R} \rightarrow \mathbb{R} : r \rightarrow g_i(r)$ is either linear or exponential in r - see Dyer and Sarin (1982) and Krzysztofowicz (1983) - ,
- (7.2) a semiorder T on Ω such that, $\forall i, j \in \Omega$,
 $i T j$ iff $\{i\} \gg \{j\}$,
- (7.3) which minimum gathering of not very important attributes is necessary to obtain a set of attributes more important than the most important attribute.

The questions to put to the decision-maker in order to obtain this information are strongly interconnected but very simple ; they make an intensive use of standard actions clearly defined.

In the second part, we simultaneously determine a set of weights $\{w_1, w_2, \dots, w_m\}$ and a value for the parameter ρ from (7.2) and (7.3). Of course, this can be done in many different ways. Our approach is based on two principles: to respect into all the details the information given by the decision-maker and to be careful concerning the extrapolation.

The weights can be obtained with the help of linear programming but we also developed a particular algorithm to get them which takes into account the special structure of the problem - see Pirlot and Vansnick (1984) .

In order to present the third part, we have to introduce some notations for representing the evaluations of the alternatives according to the m attributes. We shall denote the mappings which model these evaluations by $f_i : \mathcal{B} \rightarrow X_i$ in the certainty case and by $\gamma_i : \mathcal{B} \rightarrow \Theta_{X_i}$ in the uncertainty case, where $i = 1, 2, \dots, m$ and Θ_{X_i} represents the set of probability measures on the σ -algebra generated by the intervals of X_i .

In the third part, we construct a global preference relation $\succ$ on $\mathcal{B}$ from (7.1) and the results of the second part. $\forall a, b \in \mathcal{B}$, we declare that " $a \succ b$ " iff two conditions are satisfied.

$$\text{Condition 1} \quad \sum_{i \in P(a,b)} w_i > \alpha \cdot \sum_{j \in P(b,a)} w_j$$

where $P(a,b) = \{k \in \Omega \mid \psi_k(f_i(a)) - \psi_k(f_i(b)) > \alpha_k \text{ (certainty case) or}$

$$g_k^{-1} [E(u_k, \gamma_k(a))] - g_k^{-1} [E(u_k, \gamma_k(b))] > \alpha_k \text{ (uncertainty case)}\}.$$

$$\text{Condition 2} \quad \forall k \in P(b,a) :$$

$$- \psi_k(f_i(b)) - \psi_k(f_i(a)) < \beta_k \text{ in the certainty case}$$

$$- g_k^{-1} [E(u_k, \gamma_k(b))] - g_k^{-1} [E(u_k, \gamma_k(a))] < \beta_k \text{ in the uncertainty case.}$$

We can recognize here, in a precise context, the ideas of concordance and non discordance of Electre I and II methods - see Roy (1968) and Roy and Bertier (1973).

The fourth part of TACTIC aims at clearly presenting to the decision-maker the preference relation constructed in the third part. In order to obtain an interesting graphic representation, the simply connected components of the relation are isolated and, in each component, the alternatives are grouped in suitable levels after elimination of possible cycles.

For a more detailed presentation of TACTIC, we refer to Vansnick et al. (1984).

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