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MOLP WITH AN INTERACTIVE ASSESSMENT OF A
PIECEWISE LINEAR UTILITY FUNCTION (*)

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PROGRAMMATION LINÉAIRE MULTICRITÈRE
AVEC CONSTRUCTION INTERACTIVE D'UNE FONCTION D'UTILITÉ

RESUME

Ce cahier présente une nouvelle méthode de programmation linéaire multicritère.

Elle consiste en trois étapes :

- 1) Génération d'un ensemble de solutions efficaces (de 10 à 50) aussi représentatives que possible de l'ensemble des solutions efficaces.
- 2) Construction d'une fonction d'utilité additive à l'aide d'une méthode interactive (PREFCALC).
- 3) Optimisation de la fonction d'utilité sur l'ensemble des solutions réalisables de départ.

Cette méthode permet à l'utilisateur de trouver une solution efficace qui peut être différente d'un sommet du polyèdre. Elle est particulièrement adaptée à la résolution de problèmes de grande taille où des méthodes existantes sont trop coûteuses à utiliser. En effet, dans cette méthode, seule la seconde phase est interactive et a lieu sur micro-ordinateur, les phases 1 et 3 pouvant être exécutées sur un gros ou mini-ordinateur.

Une version micro-ordinateur de la méthode (PREFCHAT) est opérationnelle et actuellement en cours de test à AIR FRANCE.

MOLP WITH AN INTERACTIVE ASSESSMENT
OF A PIECEWISE LINEAR UTILITY FUNCTION

ABSTRACT

The paper presents a methodology for Multi-Objective Linear Programming (MOLP) problems.

It relies on three steps :

- 1) Generation of a subset of feasible efficient solutions (from 10 to 50) as representative as possible of the efficient set.
- 2) Assessment of an additive utility function using an interactive method (PREFCALC).
- 3) Optimization of the additive utility function on the original set of feasible alternatives.

Following this methodology enables the user to find compromise solutions which can be different from the vertices. It is particularly adapted for large scale linear programs where traditional multiobjective methods would be too costly to use, since the interactive phase is limited to step 2, using PREFCALC on a micro-computer.

A micro-computer version of the method (PREFCHAT) is available and is currently tested at AIR FRANCE.

1 - OUTLINE OF THE METHOD

This paper presents a new method to support Decision Makers (DM) in solving Multi-Objective Linear Programming (MOLP) problems:

$$\begin{aligned} \max \quad & \left[\begin{array}{l} g_1(\underline{x}) = \sum_{j=1}^n c_{1j} x_j \\ \dots \dots \dots \\ g_K(\underline{x}) = \sum_{j=1}^n c_{Kj} x_j \end{array} \right] \\ \text{s.t.} \quad & \sum_{j=1}^n a_{ij} x_j \leq b_i \quad (i=1, \dots, m) \\ & x_j \geq 0 \quad (j=1, \dots, n) \end{aligned}$$

Why a new method ? Many have been proposed, most of them are interactive, many are operational under the form of computer programs (see for instance SLOWINSKI (13), VINCKE (15) ,...).

Most of the previous methods, when they are interactive, use the concept of "local preferences" . In consequence, one has to solve one or more optimizations of the initial linear program at each iteration. This can be costly and time consuming, preventing quick answers in the process, and making the method less interactive .

The method proposed here relies on a complete different principle. It consists in three steps .

(1) Generation of a subset of efficient solutions.

Preferences do not always pre-exist. It is therefore important to show some feasible alternatives to the decision-maker and their values on the criteria in order to enable him to react entering so in a learning process of his preferences. In order to show him interesting solutions, we suggest to generate a few efficient solutions. This technical step is not interactive and does not require the presence of the DM.

(2) Interactive assessment of an overall utility function.

We propose to use PREFCALC Method (see JACQUET-LAGREZE and SHAKUN (7), JACQUET-LAGREZE (6)). The DM works with a rather small subset of efficient solutions generated in step 1, using a micro-computer, and the output of this process is an additive piecewise-linear utility function. The method is highly interactive and the computer program user friendly supporting in an efficient way the learning process of the DM.

(3) Optimization of the utility function on the original set of feasible solutions.

This is again a non interactive step. We use procedures developed in the context of separable convex programming to solve the problem. Briefly speaking, it is possible to transform the problem into a new linear program.

Although this method is fully operational on a micro-computer, whenever the utility assessed in step 2 is concave for each marginal utility function, it seems particularly adapted as a part of a DSS (Decision Support System) for large problems which would require the power of a mainframe as suggested in fig.1

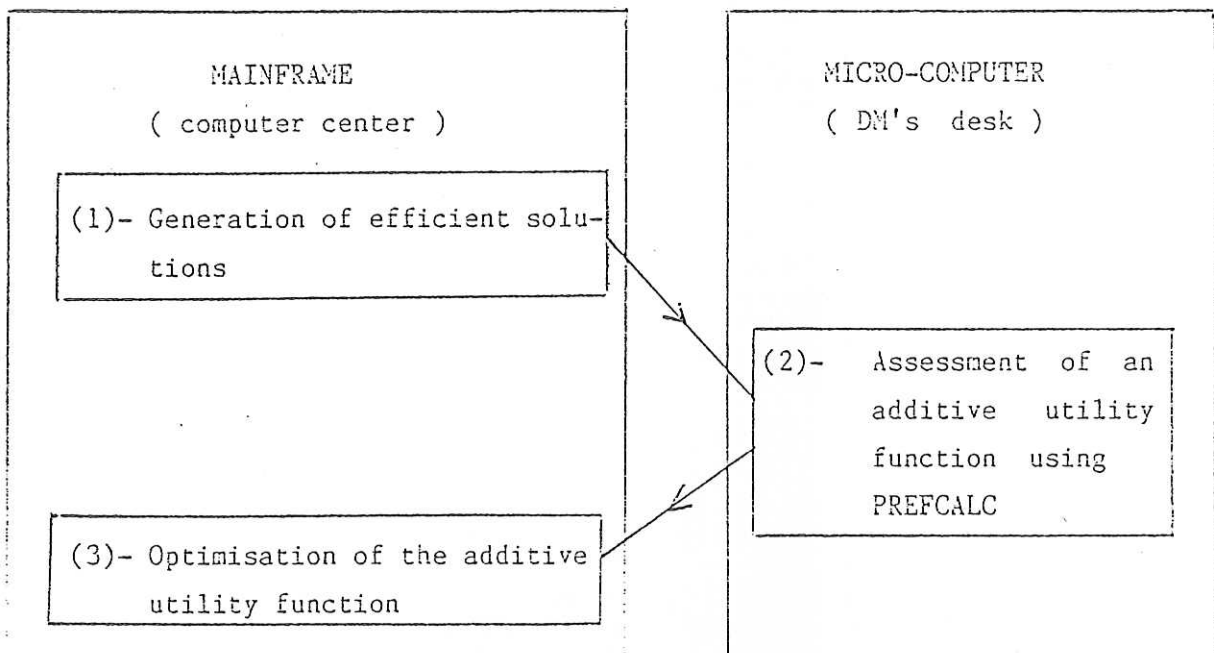


Fig.1 The method, in 3 steps

Steps 1 and 3 which are not interactive could be solved on a mainframe (or mini) in a computer center, or even on a system which is at a long distance from the DM using telecommunication to transmit intermediary data files .

Step 2 is to take place on the DM's desk , in his office, using his personal computer. It is the subjective part of the method which requires his presence, far from the technical problems he thinks in a quiet and familiar environment to the pros. and cons. of the problem he has to solve, having a clear idea of what is feasible and interesting (the efficient solutions).

In the next 3 sections (2 to 4) we present in detail each step, using a simple illustrative example proposed by GOICOECHEA, HANSEN and DUCKSTEIN (4).

2- STEP 1 - GENERATION OF EFFICIENT POINTS

The objective of this step is to propose to the DM a sample of feasible and interesting solutions. We suggest to compute efficient solutions although non efficient solutions could also be generated at the condition they also are interesting to consider for assessing the DM's preferences.

2-1. The method

There are several possible ways of generating efficient points of a multiobjective linear program. Let us only mention the possibility of combining the multiparametric linear programming methods (5),(16),(17),(18), with filtering techniques presented in (10),(14).

In this paper we shall use, however, another method which is particularly tractable and permits an easy control of dispersion of the generated efficient points, not confined to vertices of the efficient region only; moreover it uses linear programming as a main computational procedure. This is the first step of the interactive algorithm for multicriteria programming by CHOO and ATKINS (2). Let us start with an informal description of the method .

Let us denote by S the feasible set, $G = (G_1, G_2, \dots, G_K)$ the ideal point where $G_k = \max_{x \in S} g_k(x)$ ($k=1, \dots, K$), and the efficient

solution $\underline{y}^0 = (y_1^0, y_2^0, \dots, y_K^0)$ closest to G in the sense of the weighted Chebyshev norm in the criteria space :

$$\max_k \beta_k (G_k - y_k^0) = \min_{x \in S} \max_k \beta_k (G_k - g_k(x)).$$

G is defined by the diagonal of the pay-off table calculated for the MOLP problem, and $\underline{y}^0$ results from an optimal solution to the following LP problem :

$$(P1) \quad \min z$$

$$\text{s.t.} \quad z \geq \beta_k (G_k - g_k(x)) \quad (k=1, \dots, K)$$

$$x \in S$$

The weights $\beta_k > 0$ are chosen a priori so as keep $\underline{y}^0$ in the "central" area of the feasible region. β_k can be calculated as follows :

$$\beta_k = 1 / (G_k - F_k) \quad (k=1, \dots, K) \quad (1)$$

where F_k is a smallest element in the k -th column of the pay-off table.

F is usually called the nadir point and let us observe that $1/\beta_k$ ($k=1, \dots, K$) are direction coefficients of the straight line passing through the ideal and the nadir points, and that $\underline{y}^0$ lies on this direction too. Point $\underline{y}^0$ becomes a starting point for generating other efficient points using the search direction from G to F .

Let us recall after (9), that a point $\underline{x} \in S$ is said to be weakly efficient iff there does not exist another $\underline{x}' \in S$ such that $g_k(\underline{x}') > g_k(\underline{x})$ for all k . In other words, weak efficiency means that the obtained point can be dominated only by another feasible point lying on the boundary of the positive orthant with the origin at the obtained point. This feature does not decrease, however, the usefulness of the generation method since the efficiency of solutions belonging to the reference set is not necessary requirement of our approach.

A detailed description of the generation method is presented in the following steps :

- Step 0. Calculate the pay-off table of the MOLP problem and define the ideal and the nadir points, G and F , respectively.
- Step 1. Calculate β_k ($k=1, \dots, K$) according to (1).
- Step 2. Solve problem (P1). We obtain $\underline{x}^*$ and the first (weakly) efficient point $\underline{y}^0 = (y_1^0, y_2^0, \dots, y_K^0)$, where $y_k^0 = g_k(\underline{x}^*)$ ($k=1, \dots, K$).
- Step 3. For a given s , $r=1, \dots, s$ and $k=1, \dots, K$, solve the following LP problem :

$$\begin{aligned}
 (P2) \quad & \max \quad g_k(\underline{x}) \\
 \text{s.t.} \quad & g_j(\underline{x}) \geq y_j^0 - r/s (y_j^0 - F_j) \quad (j=1, \dots, K; j \neq k) \\
 & \underline{x} \in S
 \end{aligned}$$

We obtain (weakly) efficient points $\underline{y}^{kr}$ ($k=1, \dots, K$; $r=1, \dots, s$).

If the values taken by different objective functions differ a lot for the same $\underline{x}$, it is worth normalizing them by multiplying each objective k by $\alpha_k = 1 / (\sum_j c_{kj}^2)^{1/2}$, where c_{kj} are cost coefficients of g_k ($k=1, \dots, K$; $j=1, \dots, n$).

2-2. Example

To illustrate the above generation method, consider the following two-criteria LP problem, taken from ((4), example 2.3-1):

(EX)

$$\max \begin{bmatrix} g_1(\underline{x}) = x_1 - 3x_2 \\ g_2(\underline{x}) = -4x_1 + x_2 \end{bmatrix}$$

$$\begin{aligned} \text{s.t. } & -x_1 + x_2 \leq 3.5 \\ & x_1 + x_2 \leq 5.5 \\ & 2x_1 + x_2 \leq 9 \\ & x_1 \leq 4 \\ & x_1, x_2 \geq 0 \end{aligned}$$

The feasible region in the criteria space is shown in fig.3. The thicken border of the feasible region can be identified as the set of all efficient points. The results of particular steps of the algorithm are as follows :

Step 0. The pay-off table :

	g_1	g_2
g_1	4.0	-16
g_2	-10.5	3.5

Hence, the ideal point $G = (4.0, 3.5)$, and the nadir point $F = (-10.5, -16)$

Step 1. $\beta_1 = 0.069$, $\beta_2 = 0.051$

Step 2. $\underline{x}^* = (0.35, 0.0)$
 $\underline{y}^0 = (0.35, -1.4)$

Step 3. For $s=5$ we obtain the efficient points given in table 1 and shown in fig.3 .

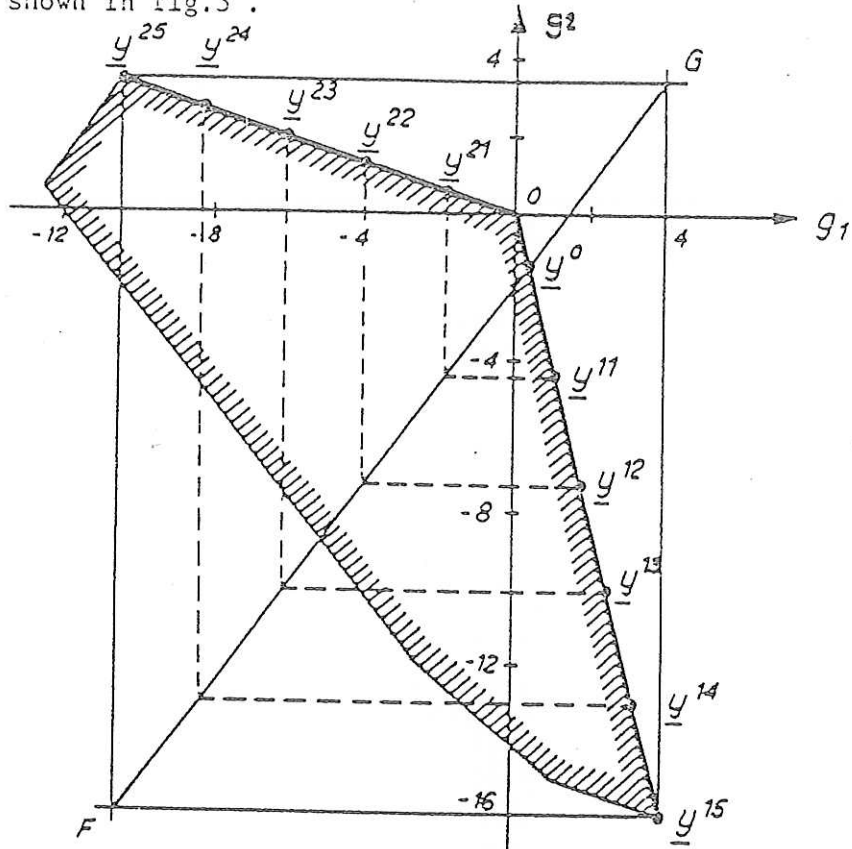


Fig.3 Illustration on the example

Trial eff. point	g_1	g_2
y^0	0.35	-1.4
y^{11}	1.08	-4.33
y^{21}	-1.83	0.61
y^{12}	1.81	-7.25
y^{22}	-4.0	1.33
y^{13}	2.54	-10.17
y^{23}	-6.17	2.06
y^{14}	3.27	-13.09
y^{24}	-8.34	2.78
y^{15}	4.0	-16.0
y^{25}	-10.5	3.5

Table 1

3 - STEP 2 - ASSESSING AN ADDITIVE PIECEWISE-LINEAR UTILITY FUNCTION

This step is interactive and requires the participation of the DM.

3-1. Brief presentation of PREFCALC

As said before, we use the PREFCALC method and micro-computer package to support the DM in the task .

The data we need to use PREFCALC are on a file containing a list of discrete alternatives (i.e. the efficient solutions) and their values on each criterion. Formally speaking, we know set A , $g_k(a)$ for all $a \in A$ and $k=1, \dots, K$.

We assume that the preference is a non-decreasing or non-increasing function of each criterion (i.e. $u_k(g_k)$ is monotone), and the overall preference can be represented by an additive piecewise-linear function :

$$u(\underline{g}) = \sum_{k=1}^K u_k(g_k) \quad , \quad u_k(g_k) \geq 0$$

$$u_k(g_{k*}) = 0 \quad , \quad \sum_{k=1}^K u_k(g_k^*) = 1$$

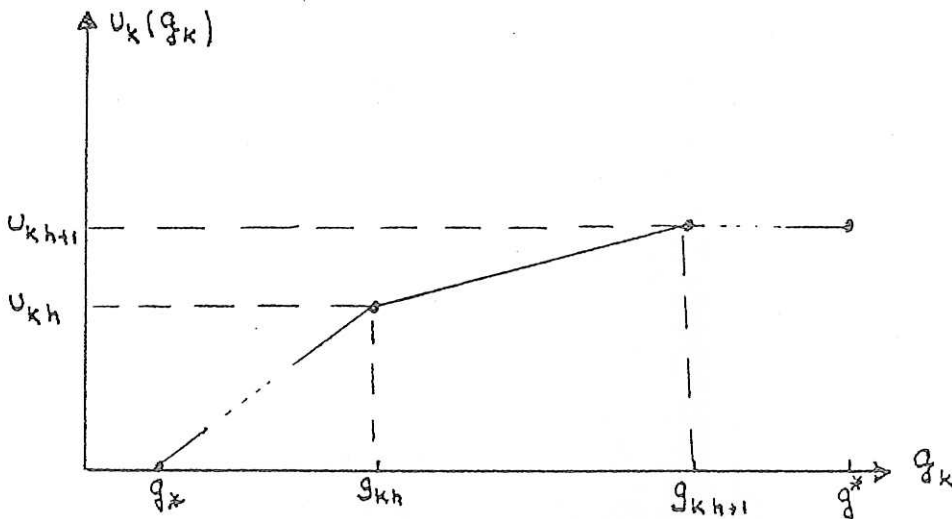


Fig.4 Piecewise marginal utility function

where g_{k*} and g_k^* represent the extremal values encountered in the data file used as an input for PREFCALC. Generally these values correspond to the least and the most preferred values on criterion g_k , respectively.

Using PREFCALC, the DM can perform the following tasks :

- Modify g_{k*} or g_k^* . In particular increase the minimum value means add a subjective constraint such that no solution giving a value under this level should be considered.
- Weight the criteria (give value for $u_k(g_k^*)$).
- Draw marginal utility function (give a certain number of linear pieces g_{kh} , $h=1, \dots, \alpha_k$) and give the values of $u_k(g_{kh})$.
- Estimate indirectly the values $u_k(g_{kh})$ so that the utility function is as consistent as possible with a wholistic preference on some alternatives (ordinal regression).

The last option has been described in detail by JACQUET-LAGREZE and SISKOS (8) and SISKOS (12). Small linear programs are solved to perform the task.

3-2. Application to the example

- The DM examines 11 efficient solutions. He does not want to modify the values of g_{k*} and g_k^* .
- Instead of giving weights to the criteria, he prefers to start the interactive procedure by ordering the following five alternatives.

	G_1	G_2	PREFERENCE
$\underline{y}^{12} = S4$	1.8	-7.3	1
$\underline{y}^{15} = S10$	4.0	-16	2
$\underline{y}^0 = S0$	0.4	-1.5	3
$\underline{y}^{23} = S7$	-6.2	2.1	4
$\underline{y}^{25} = S11$	-10.5	3.5	5

Table 2

With one linear piece for each marginal utility function, it is not possible to assess a function. As suggested by the program, the DM increases the number of linear pieces. Actually he uses 2 linear pieces for the 2nd criterion feeling that his preference does not increase in a linear way with this criterion.

- As a result, the preference order becomes consistent, giving the solution of fig. 5.

Note the 3 curves for each marginal function; these curves represent the stability (imprecision) in the estimation procedure (post-optimal analysis in the ordinal regression phase of PREFCALC). The utility function proposed by PREFCALC is an average one (giving weights .80 and .20).

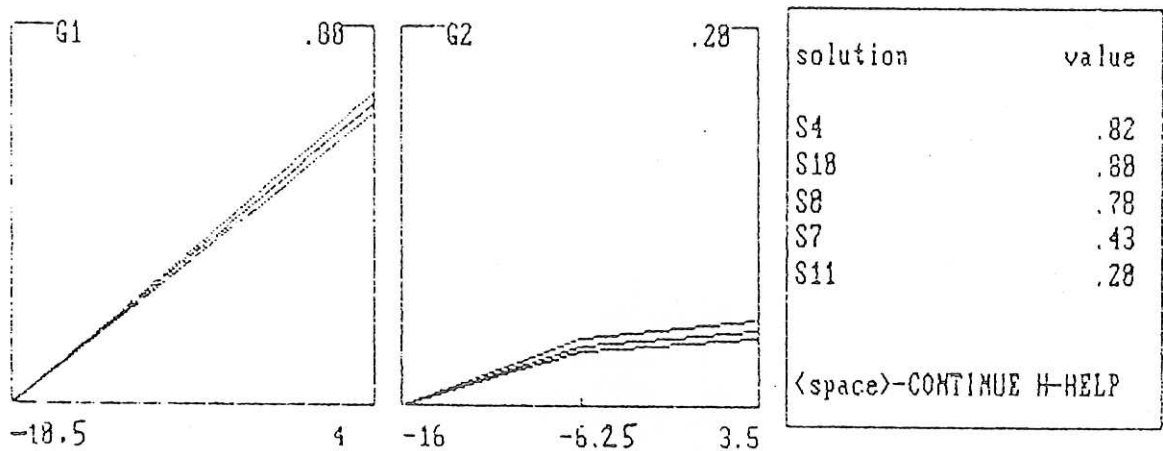


Fig. 5 Estimated utility function with PREFCALC

- DM agrees with the marginal utility function on criterion g_2 which becomes almost flat above value -6.25 and actually he decides to use the utility function shown in fig. 6 with weights .8 and .2 . He saves that function and sends it back for optimization in step 3 .

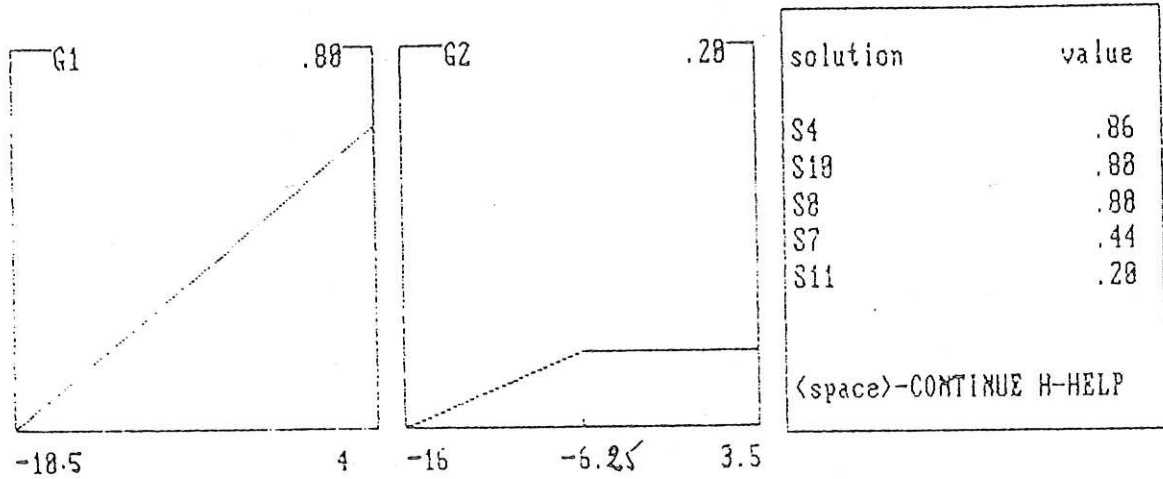


Fig. 6 Utility function chosen by the DM

4 - STEP 3 - SOLUTION OF THE PIECEWISE-LINEAR PROGRAM

4-1. Problem formulation and the algorithm

As the result of using PREFCALC, we obtain an additive utility function and each marginal utility function $u_1(g_1)$, $u_2(g_2), \dots, u_K(g_K)$ having a piecewise-linear form. Let (g_{kh}, u_{kh}) be the breakpoints of $u_k(g_k)$ ($h=1, \dots, \alpha_k$; $k=1, \dots, K$) which are known (see fig. 4). Then each objective function g_k and each partial utility $u_k(g_k)$ can be expressed as a combination of the breakpoints :

$$g_k = \sum_{h=1}^{\alpha_k} \lambda_{kh} g_{kh}$$

$$u_k(g_k) = \sum_{h=1}^{\alpha_k} \lambda_{kh} u_{kh}$$

$$\sum_{h=1}^{\alpha_k} \lambda_{kh} = 1$$

$$\lambda_{kh} \geq 0 \text{ for } h=1, \dots, \alpha_k ; k=1, \dots, K$$

where, at most, two adjacent λ_{kh} 's are positive for each k .

Since the overall utility function U is the sum of partial utility functions, and $g_k = \sum_{j=1}^n c_{kj} x_j$ ($k=1, \dots, K$), the problem of finding $\underline{x}$ maximizing U can be formulated as follows :

$$(P3) \quad \max \quad U = \sum_{k=1}^K \sum_{h=1}^{\alpha_k} \lambda_{kh} u_{kh}$$

$$\text{s.t.} \quad \sum_{j=1}^n a_{ij} x_j \leq b_i \quad (i=1, \dots, m)$$

$$\sum_{j=1}^n c_{kj} x_j - \sum_{h=1}^{\alpha_k} \lambda_{kh} g_{kh} = 0 \quad (k=1, \dots, K)$$

$$\sum_{h=1}^{\alpha_k} \lambda_{kh} = 1 \quad (k=1, \dots, K)$$

$$\lambda_{kh} \geq 0 \quad (k=1, \dots, K ; h=1, \dots, \alpha_k)$$

$$x_j \geq 0 \quad (j=1, \dots, n)$$

And, at most, two adjacent λ_{kh} 's are positive for each k .

With the exception of the constraint that, at most two adjacent λ_{kh} are positive for each k , problem (P3) is a linear program. For solving problem (P3) one can use the simplex method with the following restricted-basis-entry rule. A non-basic variable λ_{kh} is introduced into the basis only if it improves the objective function and if the new basis has no more than two adjacent λ_{kh} 's that are positive for each k .

It can easily be proved ((1), theorem 11.3.1) that if all partial utility functions are concave, we can discard the restricted basis entry rule and adopt the ordinary simplex algorithm. Recently, a particularly efficient version of the simplex algorithm that can maximize any concave separable piecewise-linear objective, subject to linear constraints, has been suggested by FOURER (3), along with a comprehensive unified treatment of both theory and algorithms for piecewise linear programming. FOURER extends the linear bounded-variable simplex algorithm to handle arbitrary concave separable piecewise-linear objectives. The steps of the algorithm are found to require about the same work as their counterparts in common linear simplex algorithms.

To end with the concave case, let us remark that an optimal solution of (P3) is either an efficient vertex of the feasible set or another point of the efficient border of the MOLP problem; the first case occurs if in the optimal solution exactly two λ_{kh} are positive for each k , and the second, if for at least one k , there exists h such that $\lambda_{kh} = 1$. In the latter case, the optimal solution is a tangential point of the efficient border and an isopreference curve in its breakpoint (case of the example, see fig. 7).

For non-concave partial utility functions, the solution obtained using the simplex algorithm with the restricted-basis-entry rule is an optimal solution of problem (P3) in a local sense only; if the problem possesses more than one local optimum, there is no guarantee that best from among them will be found (10).

4-2. Example

Let us continue with example (EX). As result of using PREFCALC, we have :

$$\begin{aligned}g_1 &= -10.5 \lambda_{11} + 4 \lambda_{12} \\g_2 &= -16 \lambda_{21} - 6.25 \lambda_{22} + 3.5 \lambda_{23}\end{aligned}$$

$$\begin{aligned}u_1(g_1) &= 0 \lambda_{11} + 0.8 \lambda_{12} \\u_2(g_2) &= 0 \lambda_{21} + 0.2 \lambda_{22} + 0.2 \lambda_{23}\end{aligned}$$

$$\begin{aligned}\lambda_{11} + \lambda_{12} &= 1 \\ \lambda_{21} + \lambda_{22} + \lambda_{23} &= 1\end{aligned}$$

$$\lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{23} \geq 0$$

and, at most, two adjacent λ_{kh} 's are positive.

Thus, problem (P3) takes the form :

$$\text{max} \quad U = 0.3 \lambda_{12} + 0.2 \lambda_{22} + 0.2 \lambda_{23}$$

$$\begin{aligned} \text{s.t.} \quad & -x_1 + x_2 \leq 3.5 \\ & x_1 + x_2 \leq 5.5 \\ & 2x_1 + x_2 \leq 9 \\ & x_1 \leq 4 \end{aligned}$$

$$\begin{aligned} & x_1 - 3x_2 + 10.5 \lambda_{11} - 4 \lambda_{12} = 0 \\ & -4x_1 + x_2 + 16 \lambda_{21} + 6.25 \lambda_{22} - 3.5 \lambda_{23} = 0 \end{aligned}$$

$$\begin{aligned} & \lambda_{11} + \lambda_{12} = 1 \\ & \lambda_{21} + \lambda_{22} + \lambda_{23} = 1 \end{aligned}$$

$$x_1, x_2, \lambda_{11}, \lambda_{12}, \lambda_{21}, \lambda_{22}, \lambda_{23} \geq 0$$

Since partial utility functions are concave, we can discard the constraint that, at most, two adjacent λ_{kh} 's are positive.

The optimal solution of the above LP problem is the following :

$$U^* = 0.8655$$

$$g_1 = 1.5625, \quad g_2 = -6.25$$

$$x_1^* = 1.5625, \quad x_2^* = 0$$

$$\lambda_{11}^* = 0.1681, \quad \lambda_{12}^* = 0.8319$$

$$\lambda_{21}^* = 0, \quad \lambda_{22}^* = 1, \quad \lambda_{23}^* = 0$$

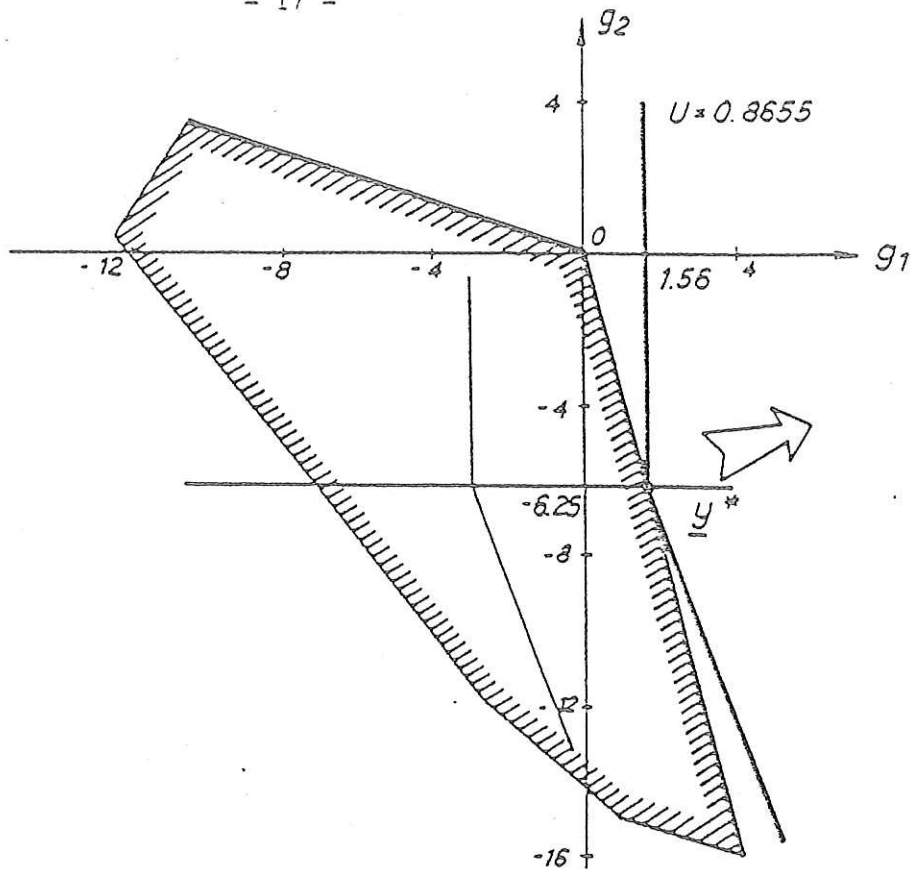


Fig. 7 Optimal solution of the piecewise utility function

This solution is represented in fig.7 . Let us remark that in the criteria space of problem (EX), the utility function U defines a family of isopreference curves in the form of parallel piecewise-linear curves; two of them are shown in fig.7 and the direction of the growth of utility is marked with an arrow. The optimal solution is a tangential point of the efficient border and an isopreference curve in its breakpoint. It is worth stressing that it is an efficient point which is not a vertex of the feasible set.

5 - DISCUSSION

A micro-computer version of the method (PREFCHAT) is now available on an IBM-PC and it is possible to test it on medium size problems (50 variables, 50 constraints and 10 criteria) on an IBM-PC. The computing time in step 1 has been greatly reduced by avoiding to optimize separately the objective functions. We start from an optimal feasible solution for one criterion in order to find an optimal solution for another criterion. The order chosen is not arbitrary and an heuristic enables to reduce the computing time. These results should be reported in a next paper.

As suggested before it is not necessary to compute efficient solutions in order to initialize the second step. Therefore PREFCHAT allows also to generate non efficient solutions computed as weighted averages of the K solutions computed to get the ideal point.

In practical problems the solutions proposed to the DM at the end of step 1 should be presented in the most attractive and meaningful way to the DM to facilitate his reactions and learning process of his preferences. In the problem currently studied for AIR FRANCE, it is possible to represent any x solution in a natural graphic form which is familiar to the DM. With such graphic representations of the solutions the DM is able to compare them more easily and rank them according to his preferences. Therefore the ordinal regression implemented in PREFCALC (step 2) should become a relevant mean to support the assesment of the utility function.

The utility function assessed using PREFCALC is additive. This means that preferential independence has to be satisfied. It could be a limitation for some problems although we never encountered such limitations in the practical discrete cases for which PREFCALC has been used. The advantage of the simplicity to use the method seems to compensate this theoretical limitation. Nevertheless other means to assess some non additive utility functions exist and could be used instead of PREFCALC for the second step.

The computer program for step 3 works only when the marginal utility functions are concave. The extension to the convex case as discussed earlier should be implemented. It could be also interesting to perform some sensitivity analysis in step 3 depending on some residual imprecision of the assessment of the utility function. For instance the ordinal regression routine implemented in PREFCALC yields often a set of utility functions all consistent with the preference of the DM. An interesting question is then what is the set of the optimal solutions in step 3 when we consider not a single utility function but a set of them, all equally representative of the preference of the DM.

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