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## A MULTIOBJECTIVE LINEAR PROGRAMMING ALGORITHM BASED ON SATISFACTORY GOALS AND INTERACTIVE UTILITY ASSESSMENT

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# UN NOUVEL ALGORITHME DE PROGRAMMATION LINEAIRE MULTIOBJECTIFS

## RESUME

Ce cahier présente une méthode interactive en programmation linéaire multi-objectifs. La méthode combine certains aspects de la Méthode des Buts Satisfaisants et de la théorie de l'utilité multiattribut. La fonction d'utilité du décideur est supposée localement additive. Elle est estimée par une forme linéaire par morceaux en utilisant le modèle de régression ordinaire UTA. Des techniques d'optimisation linéaire par morceaux sont utilisées pour proposer, à chaque itération, un nouveau compromis appartenant à l'ensemble des solutions efficaces. La méthode est illustrée sur un problème de production bicritère.

Mots-clés : Aide à la décision multicritère, Programmation linéaire multi-objectifs, Méthode interactive.

# A MULTIOBJECTIVE LINEAR PROGRAMMING ALGORITHM BASED ON SATISFACTORY GOALS AND INTERACTIVE UTILITY ASSESSMENT

## ABSTRACT

This paper presents an interactive method for Multiple Objective Linear Programming problems. The method combines the features of both the Method of Satisfactory Goals and of Multiattribute Utility theory. The decision maker's utility function is assumed to be locally additive and is assessed in a piecewise linear form, using the ordinal regression model UTA. Piecewise linear optimization techniques are used to estimate, at each iteration, a new compromise solution over the set of efficient solutions. The proposed method is illustrated by an application to a bicriterion production planning problem.

Keywords : Multiple Criteria Decision Making, Multiple Objective Linear Programming, Interactive Method.

## 1. INTRODUCTION

Whenever a decision is to be made, the objectives involved in the decision problem are multiple and in most cases competitive. However, simultaneous optimization of these objectives within the frame of Multiple Objective Mathematical Programming is usually unattainable due to their conflicting nature. Thus, problems of this kind call for implicit or explicit trade-off decisions in order to attain the best compromise solution. Texts and surveys on MOMP and its applications can be found in Zeleny [24], Hwang and Masud[11], Goicoechea et al.[9] Chankong and Haimes [4], Evans [7], Roy[18] and Cohon and Marks[6].

The problem dealt with in this paper is defined as follows

- (a) There are  $m$  decision variables  $\underline{x}=(x_1, x_2, \dots, x_m)$ .
- (b) There is a polyhedral set of alternatives  $A$ , which is implicitly dictated by a set of well defined linear constraints.
- (c) There are  $n$  explicitly defined linear objectives  $g_1, g_2, \dots, g_n$ , all real valued functions of  $\underline{x}$ .
- (d) There is a decision maker who has an implicit, unknown utility function  $U$ , such that if  $\underline{x}, \underline{y}$  are two alternatives of the set  $A$ ,  $\underline{x}$  is preferred to  $\underline{y}$  iff  $U(\underline{g}(\underline{x})) > U(\underline{g}(\underline{y}))$  and  $\underline{x}$  is indifferent to  $\underline{y}$  iff  $U(\underline{g}(\underline{x})) = U(\underline{g}(\underline{y}))$ , where  $\underline{g}(\underline{x}), \underline{g}(\underline{y})$  are the multicriteria consequences of the alternatives  $\underline{x}$  and  $\underline{y}$  respectively.

This is a typical Multiple Objective Linear Programming problem which can be expressed mathematically by the model

$$\max g_1(\underline{x}), \dots, \max g_n(\underline{x})$$

$$\underline{x} \in A$$

Hitherto, several interactive methods for solving such problems have been proposed. All these methods, based on a progressive articulation of preferences, aim to attain the best compromise solution, usually by means of single objective optimization related to the original MOLP. The various methods are distinguished by the kind of information required by the decision maker (i.e implicit or explicit

trade-off information, ranking order alternatives etc.) as well as the type of single objective program used to estimate a new compromise solution at each iteration.

Most, if not all, of the methods seem to suffer from various kinds of drawbacks. Naslund [17], Wallenius [21] and Hemming [10] discuss some properties such as convergence, simplicity of information required, insensitivity to wrong estimations, efficiency of the compromise solutions which provide relevant criteria to compare the various methods. Some of these methods like Benson's Method of Satisfactory Goals [3] applied on MOLP problems, require the decision maker to set and probably reset his aspiration levels for each objective in an interactive way throughout the process. Although these approaches seem to be attractive, since the final compromise lies within predetermined bounds for the objective values, the determination of initial feasible goals is rather difficult and usually time consuming. Furthermore, there is lack of rationalism for the compromise solutions obtained by this kind of methods.

Other methods exploit in a direct way the decision maker's utility function and seek the best compromise solution through successive maximizations of the utility functions assessed locally at each iteration. Representative methods of this kind are the method of Zionts and Wallenius [25] and some modifications of it [16,22], which are characterized by the assumption of linearity for the utility function. However, the compromise solutions obtained are efficient extreme points in the decision space, a fact that lies against the ideas of compromise programming.

Recently, Jacquet-Lagrèze, Méziani and Slowinski [13] presented a method which seeks the best compromise solution using an overall additive utility function and enables the decision maker to reach compromise solutions not necessarily extreme. The assessment of the decision maker's utility function is realized with the use of an interactive method named PREFCALC [12] which is based on an ordinal regression model. The overall utility function is elaborated on a preference order, externalized by the decision maker over a finite subset of feasible and efficient decision profiles. Indifference is excluded, thus the decision maker is forced to express only strict preference.

Jacquet-Lagrèze et.al suggest the first step of the algorithm of Choo and Atkins [5] for the construction of the alternative scenarios. This technique could be costly, so they use a heuristic in order to reduce the computing time.

The interaction of the method is limited only to the assessment of the decision maker's utility function. After such a function has been formalized the method becomes direct.

The main drawback of these latter methods is the fact that, although they satisfy the condition of rationalism, since the best compromise solution is reached by the maximization of a global criterion which reflects the preferences of the decision maker, the solutions obtained may lie outside inherent but undetermined satisfaction levels for each objective.

The method presented in this paper, extending the ideas of the method of Jacquet-Lagrèze et al, attempts to combine the advantageous features of both, the Method of Satisfactory Goals and Utility-Oriented approaches. The method provides a "two-level" interaction:

- Interactive assessment of the utility function.
- Interactive modification of the aspiration levels.

The utility function is assessed by analyzing the decision maker's preferences (strict preference and/or indifference) over a set of experimentally generated fictitious alternatives.

The paper is organized as follows. In section 2 the outline and the flow-chart of the method are presented. Section 3 provides an algorithm and some analysis. The method is illustrated by a numerical example in section 4. Finally, a discussion on the proposed method and concluding remarks are provided in section 5.

## 2. OUTLINE OF THE METHOD

The method presented here is for solving MOLP problems:

$$\begin{aligned} \max g_1(\underline{x}) = \underline{c}_1^T \underline{x}, \dots, \max g_n(\underline{x}) = \underline{c}_n^T \underline{x} \\ \underline{x} \in A \end{aligned} \quad (1)$$

where  $A = \{x \in R^m : \mathcal{R} x \leq b, x \geq 0 ; x, g_1, g_2, \dots, g_n \text{ as defined previously}; \mathcal{R}$  is the matrix of the coefficients of the constraints,  $b$  is the right-hand side of the constraints and  $c_j = (c_{j1}, \dots, c_{jm})$  are the coefficients of the objective  $g_j$ .

The flow-chart of the method is presented in figure 1.

The method consists of a preliminary and an iterative part.

In the preliminary part and in the first step, after the MOLP problem has been clearly formulated, each individual objective is optimized on the set of the feasible solutions. In this way, the ideal values (i.e the initial upper bounds for the objectives) are calculated and the pay-off table is constructed. The smallest (in the case of maximization) entries of the columns in the pay-off table define a vector, which is known in the literature as the nadir vector.

The anti-ideal values, which represent the initial lower bounds for the objectives, are obtained by minimizing each objective separately. However, in the bicriterion case, the lower bounds for the objectives can be defined alternatively by the components of the nadir vector, without excluding any part of the efficient set.

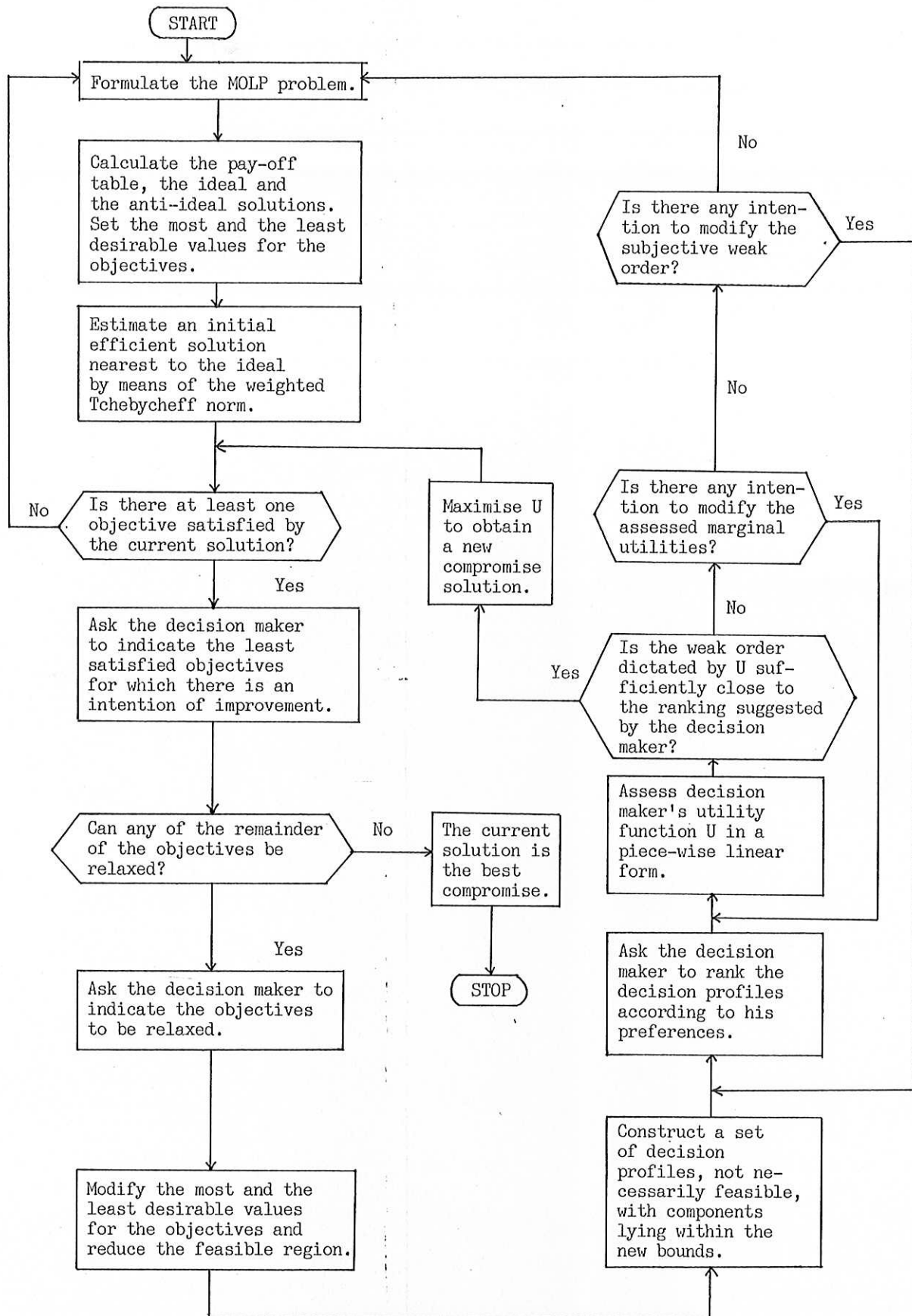
In a second step an initial weakly efficient point is estimated, closest to the ideal with respect to the weighted Tchebycheff norm. The weights are calculated mechanically in a way similar to that in Step Method (STEM) of Benayoun et.al [2].

Little attention is paid in this preliminary step to defining the initial upper and lower bounds for the objectives, as well as to estimating the initial solution, since the method itself enables the decision maker to modify these bounds progressively throughout the process and to establish consistent additive utilities in an interactive way. Thus, the decision maker is not involved in this stage.

The iterative part of the method can be resolved in four major successive stages.

STAGE I: At each iteration throughout the process, the decision maker is faced with a new compromise solution obtained with the

Figure 1: The flow-chart of the method.





maximization of his utility function, except for the initial solution which is reached in a different way described above. Whenever he thinks that there is at least one objective not satisfied by the current solution, he is asked to indicate the objectives he intends to relax in order to improve the value of the ones not satisfied. This piece of information given by the decision maker is used to establish new bounds for the objectives thus, reducing the feasible region. In fact, his intention to improve an objective value can be interpreted as his desire not to step down from the current value. Similarly, when he indicates that some objectives are to be relaxed it means that these objectives have already reached satisfactory levels far from which there is no need to go.

The iterative process terminates, during this stage, when a best compromise solution is reached, i.e. when the decision maker is not willing to relax any objective.

STAGE II: In this stage a simple generation technique is set up to construct a set of decision profiles. As long as none of these profiles is to be taken by the decision maker as an acceptable decision, they need not be efficient not even feasible. Thus, the set constructed consists of fictitious alternatives which will be offered later to the decision maker just to reveal his preferences toward them. The alternatives generated for this purpose should not dominate each other in order to protect the decision maker from facing trivial situations.

In stage I the decision maker indicates the objectives to be improved as well as the objectives which can be relaxed. The technique suggested here exploits this information, and, without violating the decision maker's indications generates the alternatives after having discretized each interval of varying the objectives. In this way the components of the decision profiles, being normally distributed along each interval of variation, allow the number of the generated alternatives to be completely controlled. This is a calculation stage, thus, the decision maker is not involved.

STAGE III: An additive utility function, which reflects the decision maker's preferences, is assessed by the ordinal regression model UTA by Jacquet-Lagrèze and Siskos [15] and Siskos and Yannacopoulos [20].

The information required of the decision maker in this stage is a weak

order, over the set of the decision profiles generated in stage II and it is obtained through pairwise comparisons of these profiles. In order to construct the pairs, an alternative is selected at random out of the whole set as a basis and the decision maker is asked to compare each other alternative with the basis. In this way, the initial set is partitioned into three subsets: the set of the alternatives preferred to the basis, the set of the alternatives not preferred to the basis and the set of the alternatives that are judged by the decision maker to be indifferent to the basis. This latter set consists of alternatives equivalent to each other. The process is repeated for every subset of alternatives which have not been compared to each other until an order of equivalent classes is achieved. The resulting preference ranking is used by the model UTA, through a trial-and-error process, to assess and justify the decision maker's additive utility function as consistent as possible to his ranking.

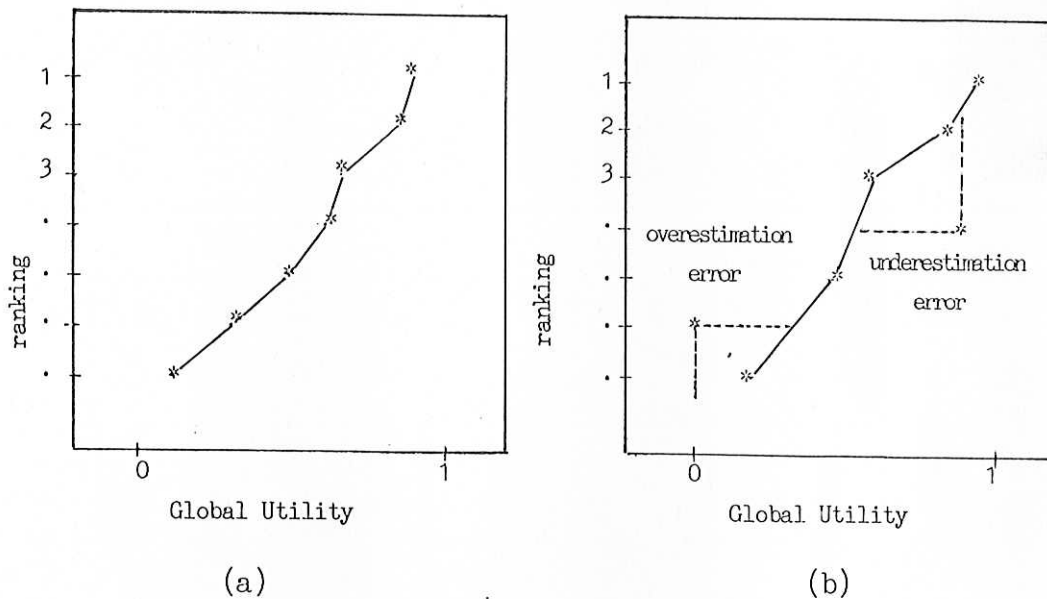


Figure 2 : Ordinal regression curves. (a) Full consistency achieved. (b) Case of inconsistencies.

The decision maker learns about any possible inconsistencies through pictorial information provided by the system. When full consistency is achieved, (fig. 2a) the assessment process is complete. In the case where inconsistencies appear (fig. 2b), i.e. when the decision maker is caught to have overestimated or underestimated some alternatives according to the utilities dictated by the model, the method sets up a dialogue focusing his attention on these inconsistencies.

The whole interactive process for building the decision maker's utility function is based on the principles of the decision support system MINORA [23] .

STAGE IV :The decision maker's utility function is optimized over the set of feasible solutions. For this purpose, piecewise linear programming techniques (see [8] for instance) are set up since the assessed utility function is piecewise linear in form. This is a calculation stage thus, the decision maker is not involved.

### 3. THE ALGORITHM

The complete algorithm and some analysis for solving problem (1) with the proposed method is given below.

Step 1 . Calculate  $h_i$ ,  $f_j$  and  $l_j$  , for  $i, j=1, \dots, n$  as follows:

$$(1.1) \quad h_i = \max_{\underline{x}} g_i(\underline{x}), \quad \underline{x} \in A.$$

Let  $\underline{x}_i^*$  be the optimal solution for this problem and

$$g_{ij} = g_j(\underline{x}_i^*) \text{ for every } i, j=1, \dots, n$$

$$(1.2) \quad f_j = \min_i \{g_{ij}\}; \quad i, j=1, \dots, n$$

$$(1.3) \quad \text{If } n=2 \text{ then set } l_j = f_j \quad \text{otherwise} \\ \text{set } l_j = \min_{\underline{x}} g_j(\underline{x}), \quad \underline{x} \in A$$

Step 2.

$$(2.1) \quad \text{Solve the linear program}$$

$$\begin{aligned}
 & \min z \\
 \text{s.t. } & \underline{x} \in A^0 = A \\
 & (h_i - g_i(\underline{x})) m_i \leq z, \quad i=1, \dots, n \\
 & z \geq 0
 \end{aligned} \tag{2}$$

where  $m_i = d_i / \sum_{k=1}^n d_k$  and  $d_i = (h_i - f_i) / h_i$ ,  $i=1, \dots, n$ .

Let  $\underline{x}^1$  and  $\underline{g}^1$  be respectively the optimal solution of problem (2) and its multiobjective consequences. The solution  $\underline{x}^1$  is weakly efficient and is the closest one to the ideal in the sense of the weighted Tchebycheff norm, the weights  $m_i$  reflecting the sensitivity of each objective in varying  $\underline{x}$ .

(2.2) Set  $q=1$ ,  $l_i^0 = l_i$ ,  $h_i^0 = h_i$  for every  $i=1, \dots, n$ .

Here,  $l_i^0$  and  $h_i^0$  are respectively the lower and the upper bounds for the objectives, dictated initially by the problem itself.

Step 3 . Modify the lower and the upper bounds for the objectives as follows.

(3.1) Ask the decision maker:

"Is there any objective satisfied in  $\underline{g}^q$ ?"

If NO, the multiple objective problem has no satisfactory solution. Ask the decision maker to review the formulation and restart from step 1.

If YES, go to (3.2) below.

(3.2) Ask the decision maker to indicate the least satisfied objectives he is willing to improve.

Let  $G$  be the whole set of objectives and  $GN$  the set of objectives indicated within this step.

(3.3) Ask the decision maker:

"Can any of the remainder of the objectives be relaxed?"

If NO,  $\underline{x}^q$  is the best compromise solution and  $\underline{g}^q$  its

consequences. STOP.

If YES, go to (3.4) below.

- (3.4) Ask the decision maker to indicate the objectives to be relaxed and form the set GY of the objectives indicated.

Consequently, the complementary of the set  $GN \cup GY$  in G (say  $\tilde{G}$ ) is the set of the objectives not mentioned by the decision maker.

- (3.5) Set  $l_i^q = g_i^q, h_i^q = h_i^{q-1} \quad \forall g_i \in GN$

$$l_i^q = l_i^{q-1}, h_i^q = g_i^q \quad \forall g_i \in GY$$

$$l_i^q = l_i^{q-1}, h_i^q = h_i^{q-1} \quad \forall g_i \in \tilde{G}$$

$$\text{Set } A^q = A^{q-1} \cap \{ \underline{x} \in R^m / l_i^q \leq g_i(\underline{x}) \leq h_i^q ; i=1, \dots, n \}$$

- Step 4. For a given integer s and  $k=0,1,\dots,s$  generate the alternative profiles  $\underline{g}^k = (g_{ik}^q), i=1, \dots, n$  with respective coordinates defined as follows:

$$g_{ik}^q = \begin{cases} l_i^q + (k/s) * (h_i^q - l_i^q) & \text{for } g_i \in GN \cup \tilde{G} \\ h_i^q - (k/s) * (h_i^q - l_i^q) & \text{for } g_i \in GY \end{cases} \quad (3)$$

The number of the alternatives generated by this procedure is  $s+1$ , but it is easy to consider any other criteria profile for comparison purposes.

These alternatives are fictitious, in the sense that they are not efficient, but suitable to be offered to the decision maker just to reveal his preferences toward them. They do not dominate each other since there is at least one objective increasing from its lower to its upper bound and at least one objective decreasing.

- Step 5. Present to the decision maker the whole set of the alternatives generated within step 4 and ask him to rank

order them as follows:

- (5.1) Select an alternative  $\underline{g}^p$  as a basis.
- (5.2) For every other alternative of the set, different to the basis, calculate the differences:

$$\underline{t}^k = \underline{g}^k - \underline{g}^p \quad k=0, \dots, s \text{ and } k \neq p$$

Here,  $\underline{t}^k$  are trade-offs expressing the pros and cons of preferring  $\underline{g}^k$  to  $\underline{g}^p$ .

- (5.3) Present the basis to the decision maker and for each  $\underline{t}^k$  ask him:

"Having  $\underline{g}^p$  in hand, do you accept the trade-offs in the objectives shown in  $\underline{t}^k$ ?"

If his answer is YES, it means that he prefers  $\underline{g}^k$  to  $\underline{g}^p$  (noted  $\underline{g}^k \succ \underline{g}^p$ ).

If his answer is NO, it means that he prefers  $\underline{g}^p$  to  $\underline{g}^k$  (noted  $\underline{g}^p \succ \underline{g}^k$ ).

If his answer is INDIFFERENT, it means that  $\underline{g}^p$  and  $\underline{g}^k$  are judged to be equivalent (noted  $\underline{g}^p \sim \underline{g}^k$ ).

- (5.4) Get the set of the alternatives preferred to the basis and the set of the alternatives not preferred to the basis. For each of these sets with cardinal number greater than one, repeat step 5 from (5.1) to (5.4) until the initial set be partitioned into equivalent classes.

Step 6. Assess additive utilities

$$u(\underline{g}) = \sum_{i=1}^n u_i(g_i), \text{ satisfying the normalization relations}$$

$$u_i(1_i^q) = 0 ; \quad i=1, \dots, n$$

$$\sum_{i=1}^n u_i(h_i^q) = 1$$

according to the steps of the UTA algorithm (c.f [15,19,20]).

- (6.1) Express the global utilities of the alternatives in the

ranking in terms of marginal utilities

$$u(\underline{g}^k) = \sum_{i=1}^n u_i(g_{ik}) \quad k=0,1,\dots,s \quad (4)$$

(6.2) Discretize each interval  $[l_i^q, h_i^q]$  as follows:

$$[l_i^q, h_i^q] = [l_i^q = g_i^1, \dots, g_i^j, g_i^{j+1}, \dots, g_i^{a_i} = h_i^q], \quad a_i \leq s+1$$

where

$$g_i^{j+1} = l_i^q + \frac{j(h_i^q - l_i^q)}{a_i - 1}, \quad i=1, \dots, n \quad \text{and} \quad j=0, \dots, a_i - 1$$

In order to assure that at least one  $g_{ik}$  lies within each interval  $[g_i^j, g_i^{j+1}]$  it must be  $a_i \leq s+1$  for every  $i$ . If  $a_i = s+1$ , for some,  $i$  then  $g_i^{j+1}$  and  $g_{ik}$  are identical for  $j=0, \dots, s$  and  $k=0, \dots, s$ .

(6.3) Using linear interpolation for  $g_{ik} \in [g_i^j, g_i^{j+1}]$  express the marginal utilities for  $g_{ik}$  in terms of the marginal utilities of  $g_i^j, g_i^{j+1}$  through the relations

$$u_i(g_{ik}) = \frac{g_i^{j+1} - g_{ik}}{g_i^{j+1} - g_i^j} u_i(g_i^j) + \frac{g_{ik} - g_i^j}{g_i^{j+1} - g_i^j} u_i(g_i^{j+1}) \quad (5)$$

(6.4) Introduce in (5) and consequently in (4) the variables

$$w_{ij} = u_i(g_i^{j+1}) - u_i(g_i^j) \quad \text{for } j=1, \dots, a_i - 1 \quad \text{through the relations}$$

$$u_i(g_i^1) = 0, \quad u_i(g_i^j) = \sum_{d=1}^{j-1} w_{id} \quad \text{for } j=2, \dots, a_i$$

(6.5) Assign an overestimation error function  $\sigma^+(\cdot)$  and an underestimation error function  $\sigma^-(\cdot)$  to each alternative in the preference ranking.

For each pair of consecutive alternatives in the ranking get the expressions

$$\Delta(\underline{g}^k, \underline{g}^{k+1}) = u(\underline{g}^k) - u(\underline{g}^{k+1}) + \sigma^+(\underline{g}^k) - \sigma^-(\underline{g}^k) - \sigma^+(\underline{g}^{k+1}) + \sigma^-(\underline{g}^{k+1}), \quad k=0, \dots, s-1$$

Here, for simplicity but without loss of generality it is assumed that the alternatives  $\underline{g}^k$  are ranked in increasing order of  $k$ , i.e.  $\underline{g}^k \succ \underline{g}^{k+1}$  or  $\underline{g}^k \sim \underline{g}^{k+1}$  for every  $k=0, \dots, s-1$ .

(6.6) Solve the following linear program to estimate a piecewise additive utility function  $U^q(\underline{g})$  as consistent as possible with the decision maker's ranking:

$$\begin{aligned} \min F &= \sum_{k=0}^s \{ \sigma^+(\underline{g}^k) + \sigma^-(\underline{g}^k) \} \\ \text{s.t. } \Delta(\underline{g}^k, \underline{g}^{k+1}) &\geq \delta, \text{ if } \underline{g}^k \succ \underline{g}^{k+1}, \text{ for } k=0, \dots, s-1 \\ &= 0, \text{ if } \underline{g}^k \sim \underline{g}^{k+1} \\ \sum_{i=1}^n \sum_{j=1}^{a_i-1} w_{ij} &= 1 \\ w_{ij} &\geq 0 \quad ; \quad i=1, \dots, n, \quad j=1, \dots, a_i-1 \\ \sigma^+(\underline{g}^k), \sigma^-(\underline{g}^k) &\geq 0 \quad k=0, \dots, s \end{aligned} \tag{6}$$

$\delta$  being a small positive number (c.f [15]).

(6.7) Set up a stability analysis and in case of multiple or near optimal solutions in (6) find those which maximize the weights

$$u_i(h_i^q) = \sum_{j=1}^{a_i-1} w_{ij} \quad ; \quad i=1, \dots, n.$$

In the last case, take as unique utility function  $U^q$  the mean of the  $n$  (near)optimal utilities (see [15] for more details).

(6.8) Calculate the global utility for each alternative



and provide the decision maker with the consequences of his judgment policy through the utility - ranking diagram shown in fig. 2.

(a) If full consistency is achieved, i.e. when the preference ranking is restituted by the model, go to step 7, otherwise, go to (b) below.

(b) Ask the decision maker if he intends to modify the marginal utilities dictated by the model in order to preserve his ranking.

If YES, let him modify the marginal utilities and go to (6.8)

If NO, go to (c) below.

(c) According to the pictorial information ask the decision maker if he intends to modify the previously established preference ranking.

If YES, ask him to reorder the alternatives involved in the inconsistencies, get the new weak-order and go to (6.1).

If NO, ask the decision maker to review the multiobjective model and restart from step 1.

Step 7.

(7.1) Solve the piecewise linear program

$$\max U^q = \sum_{i=1}^n \sum_{j=1}^{a_i} r_{ij} u_i(g_i^j) \quad (7)$$

$$\text{s.t. } \underline{x} \in A^q \quad (8)$$

$$g_i(\underline{x}) - \sum_{j=1}^{a_i} r_{ij} g_i^j = 0 ; i=1, \dots, n \quad (9)$$

$$\sum_{j=1}^{a_i} r_{ij} = 1 ; i=1, \dots, n \quad (10)$$

$$r_{ij} \geq 0 ; i=1, \dots, n, j=1, \dots, a_i \quad (11)$$

and at most, two adjacent  $r_{ij}$  be positive, for each  $i$ .

The information provided by the decision maker in step 3

is used to reduce the decision space. This is realized through the constraints (8). In (9)-(11), each objective is expressed as a linear combination of the breakpoints  $g_i^j$ ,  $j=1, \dots, a_i$  of the interval  $[l_i^q, h_i^q]$  through the variables  $r_{ij}$ . Throughout step 6, marginal utilities have been assigned to the breakpoints  $g_i^j$ , for each objective. The marginal utilities are piecewise linear in form and can be expressed by the analytic expressions

$$u_i(g_i(x)) = \sum_{j=1}^{a_i} r_{ij} u_i(g_i^j), \quad i=1, \dots, n$$

Since the global utility model is assumed to be additive, the overall utility function which will be maximized takes the form (7).

(7.2) Set  $q=q+1$ , let  $\underline{x}^q, \underline{g}^q$  be the compromise solution and its consequences obtained within (7.1) and go to step 3.

A simple numerical example, taken from [24], is provided in the next section as an illustration of the proposed method.

#### 4. ILLUSTRATION

Assume that the MOLP problem to be solved is the following:

$$\begin{aligned} \max \quad & g_1 = -4x_1 + 3x_2 \\ \max \quad & g_2 = 7x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 3 \\ & -2x_1 + 3x_2 \leq 12 \\ & 6x_1 + x_2 \leq 42 \\ & x_2 \leq 6 \\ & x_1, x_2 \geq 0 \end{aligned} \tag{12}$$

Let  $A^0$  be the feasible region of problem (12).

According to the presented algorithm the procedure lies in the following steps:

#### INITIALIZATION

In table 1, the solutions and the consequences, derived from the optimization of each individual objective of (12) are presented while the objective space is presented in figure 3.

Table 1 : The pay-off table of problem (12)

|       | $g_1$ | $g_2$ | $x_1$ | $x_2$ |
|-------|-------|-------|-------|-------|
| $g_1$ | 12    | 20    | 0     | 4     |
| $g_2$ | -6    | 72    | 6     | 6     |

The initial lower and upper bounds for the objectives  $g_1$  and  $g_2$  are respectively

$$l_1^0 = -6, h_1^0 = 12, l_2^0 = 20, h_2^0 = 72.$$

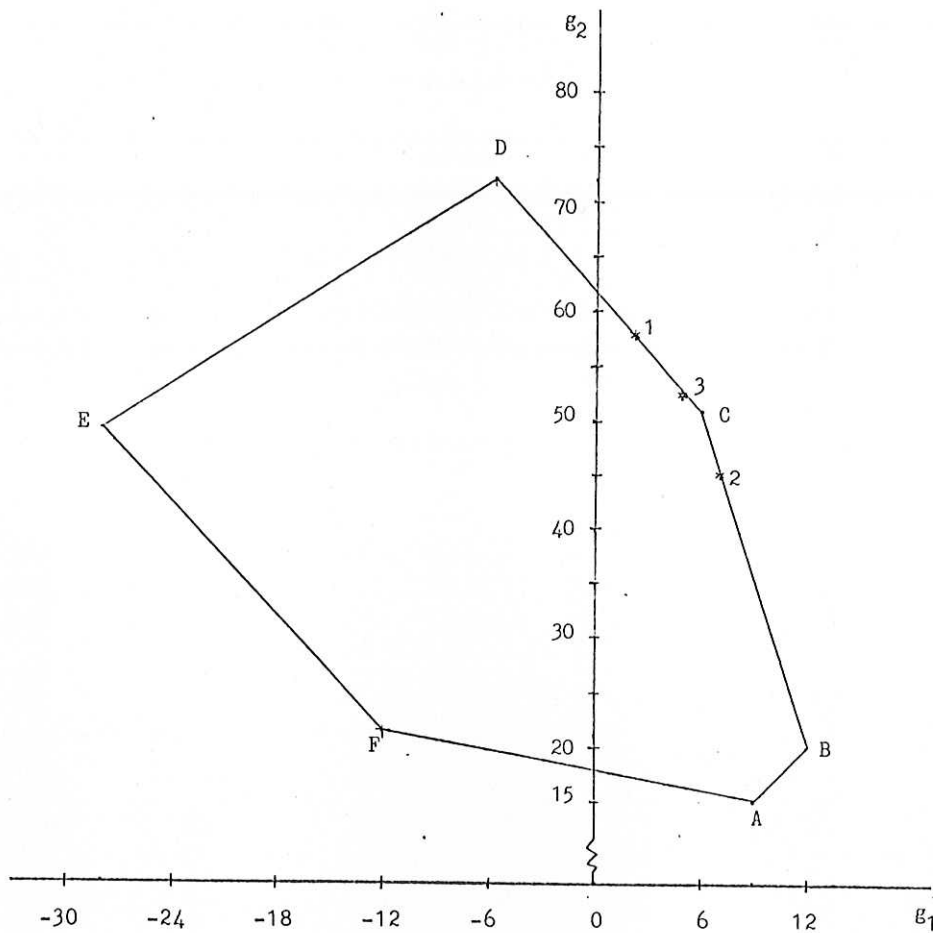


Figure 3 : The objective space of problem (12) with extreme points A(9,15), B(12,20), C(6,51), D(-6,72), E(-28,49), F(-12,21) and efficient frontier the line BCD.

The weights, reflecting the sensitivity of the objectives are  $m_1 = 0.68$  for the objective  $g_1$  and  $m_2 = 0.32$  for  $g_2$ . The linear program which leads to the nearest, weakly efficient solution to the ideal, by means of the weighted Tchebycheff norm takes the form

$$\begin{aligned}
 & \min \quad z \\
 & \text{s.t.} \quad \underline{x} \in A^0 \\
 & \quad -2.72x_1 + 2.04x_2 + z \geq 8.16 \\
 & \quad 2.24x_1 + 1.60x_2 + z \geq 23.04 \\
 & \quad z \geq 0
 \end{aligned}$$

An optimal solution to the above program is  $x_1^1 = 3.53$ ,  $x_2^1 = 6$  with consequences  $g_1^1 = 3.88$ ,  $g_2^1 = 57.71$  (point 1 in fig. 3).

ITERATION 1 ( $q=1$ ).

The solution  $\underline{g}^1 = (3.88, 57.71)$  is presented to the decision maker and he is asked:

"Is there any objective satisfied by the solution  $\underline{g}^1 = (3.88, 57.71)$ ?"

His answer is: YES.

He is then asked:

"Which is the least satisfied objective?"

His answer is:  $g_1$ .

He is then asked:

"Do you intend to relax a little bit the objective  $g_2$  in order to improve  $g_1$ ?"

His answer is: YES.

Throughout this dialogue the lower and upper bounds of the objectives are modified as follows:

$$l_1^1 = g_1^1 = 3.88, h_1^1 = h_1^0 = 12, l_2^1 = l_2^0 = 20, h_2^1 = g_2^1 = 57.71.$$

Through step 4 of the algorithm for  $s=5$  and  $k=0, \dots, 5$  the following alternatives are generated:

$$\underline{g}^0 = (3.88, 57.71), \underline{g}^1 = (5.50, 50.17), \underline{g}^2 = (7.12, 42.63),$$

$$\underline{g}^3 = (8.74, 35.09), \underline{g}^4 = (10.36, 27.55), \underline{g}^5 = (12.00, 20.00)$$

Let  $\underline{g}^0$  be the basis for the first set of pairwise comparisons and

$$\underline{t}^1 = \underline{g}^1 - \underline{g}^0 = (1.62, -7.54), \underline{t}^2 = \underline{g}^2 - \underline{g}^0 = (3.24, -15.08),$$

$$\underline{t}^3 = \underline{g}^3 - \underline{g}^0 = (4.86, -22.62), \underline{t}^4 = \underline{g}^4 - \underline{g}^0 = (6.48, -30.16),$$

$$\underline{t}^5 = \underline{g}^5 - \underline{g}^0 = (8.12, -37.71).$$

The differences  $\underline{t}^1, \underline{t}^2, \underline{t}^3, \underline{t}^4, \underline{t}^5$  are presented successively to the decision maker and he is asked to answer with a YES, NO or INDIFFERENT to a question such as:

"You have already got 3.88 for  $g_1$  and 57.71 for  $g_2$ . Do you think that an improvement of 1.62 units for  $g_1$  compensates for a loss of 7.54 units for  $g_2$ ? In that case would you prefer the resulting solution ? (Y/N/I)".

Suppose his answer is YES. That means that he prefers  $\underline{g}^1$  to  $\underline{g}^0$ .

After the dialogue has been completed and all the alternatives have been compared to the basis  $\underline{g}^0$  suppose that the decision maker suggests the following partition:

$$\{\underline{g}^1, \underline{g}^2\} \succ \{\underline{g}^0\} \succ \{\underline{g}^3, \underline{g}^4, \underline{g}^5\}$$

Repeating the same procedure for the sets  $\{\underline{g}^1, \underline{g}^2\}, \{\underline{g}^3, \underline{g}^4, \underline{g}^5\}$  suppose that the decision maker answers the questions in a way leading to the following strict order:

- (1)  $\underline{g}^2 = (7.12, 42.63)$ , (2)  $\underline{g}^1 = (5.50, 50.17)$ , (3)  $\underline{g}^0 = (3.88, 57.71)$ ,  
 (4)  $\underline{g}^3 = (8.74, 35.09)$ , (5)  $\underline{g}^4 = (10.36, 27.55)$ , (6)  $\underline{g}^5 = (12.00, 20.00)$

In order to assess the decision maker's utility function in a piecewise linear form, the intervals of varying the objectives are discretized with  $a_1=3$  and  $a_2=4$  :

$$[3.88, 7.94, 12], [20, 32.57, 45.14, 57.71]$$

Step 6 of the algorithm, with  $\delta = 0.08$  for the discrimination of two ranks, leads to the following optimal utility variations (only positive values are given):

$$w_{11}=0.34, w_{21}=0.13, w_{22}=0.53$$

with a total deviation  $F=0.05$ .

The marginal utilities for  $g_1$  and  $g_2$  are shown in figure 4, while the utilities assigned to the alternatives are:

$$u(\underline{g}^2)=0.826, u(\underline{g}^1)=0.796, u(\underline{g}^0)=0.660, u(\underline{g}^3)=0.576, u(\underline{g}^4)=0.418, u(\underline{g}^5)=0.340.$$

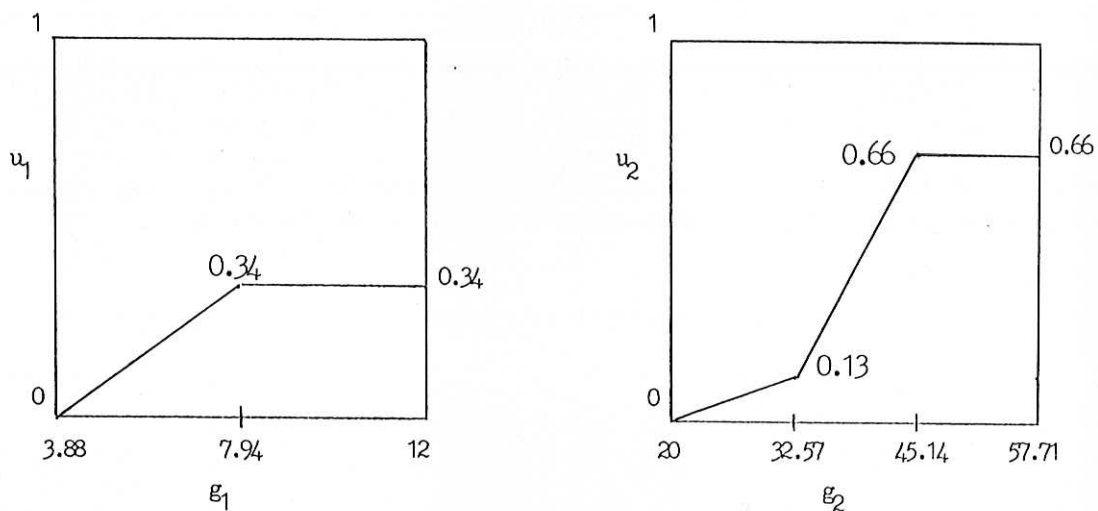


Figure 4 : Marginal utilities assigning a "weight" of 0.34 to  $g_1$  and 0.66 to  $g_2$ .

The above utilities lead to a perfect restitution of the decision maker's ranking.

According to step (6.7) of the algorithm (stability analysis) two new linear programs are solved. The weights  $w_{11} + w_{12}$  and  $w_{21} + w_{22} + w_{23}$  are maximized on the polyhedron, defined by the constraints of the previous linear program, bounded by the following constraint:

$$\sum_{k=0}^5 \sigma^+(\underline{g}^k) + \sigma^-(\underline{g}^k) \leq 0.05$$

The maximization of  $w_{11} + w_{12}$  leads to the same marginal utilities (fig. 4) while the maximization of  $w_{21} + w_{22} + w_{23}$

leads to the following, slightly different, marginal utilities:  
 $u_1(3.88)=0$ ,  $u_1(7.94)=0.33$ ,  $u_1(12)=0.33$ ,  $u_2(20)=0$ ,  $u_2(32.57)=0.15$   
 $u_2(45.14)=0.67$ ,  $u_2(57.71)=0.67$ .

The mean utility is then approximately identical to the utility estimated initially.

The piecewise linear program, which seeks the new efficient compromise solution rationalized by the maximization of the assessed utility function, takes the form

$$\max U^1 = 0r_{11} + 0.34r_{12} + 0.34r_{13} + 0r_{21} + 0.13r_{22} + 0.66r_{23} + 0.66r_{24}$$

$$\text{s.t.} \quad x_1 + x_2 \geq 3$$

$$-2x_1 + 3x_2 \leq 12$$

$$6x_1 + 3x_2 \leq 42$$

$$x_2 \leq 6$$

$$-4x_1 + 3x_2 \geq 3.88$$

$$7x_1 + 5x_2 \geq 20$$

$$-4x_1 + 3x_2 - 3.88r_{11} - 7.94r_{12} - 12r_{13} = 0$$

$$7x_1 + 5x_2 - 20r_{21} - 32.57r_{22} - 45.14r_{23} - 57.71r_{24} = 0$$

$$r_{11} + r_{12} + r_{13} = 1$$

$$r_{21} + r_{22} + r_{23} + r_{24} = 1$$

$$x_1, x_2, r_{11}, r_{12}, r_{13}, r_{21}, r_{22}, r_{23}, r_{24} \geq 0$$

and at most two adjacent  $r_{ij}$  be positive for  $i=1$  and  $i=2$ .

The optimal solution of the above program is  $x_1^2 = 2.43$ ,  $x_2^2 = 5.62$  with



consequences  $g_1^2 = 7.13$ ,  $g_2^2 = 45.14$  (point 2 in fig. 3). This is the new compromise for the second iteration.

ITERATION 2 ( $q=2$ ).

The new compromise solution is presented to the decision maker and he is asked:

"Is there any objective satisfied by the solution  $\underline{g}^2 = (7.13, 45.14)$ ?"

His answer is: YES.

He is then asked:

"Which is the least satisfied objective?"

His answer is:  $g_2$ .

He is then asked:

"Do you intend to relax a little bit the objective  $g_1$  in order to improve  $g_2$ ?"

His answer is: YES.

The new bounds for the objectives are now as follows:

$$l_1^2=3.88, h_1^2=7.13, l_2^2=45.14, h_2^2=57.71$$

For  $s=5$  and  $k=0, \dots, 5$  the following six decision profiles are generated.

$$\underline{g}^0 = (7.13, 45.14), \underline{g}^1 = (6.48, 47.65), \underline{g}^2 = (5.83, 50.17)$$

$$\underline{g}^3 = (5.18, 52.68), \underline{g}^4 = (4.53, 55.20), \underline{g}^5 = (3.88, 57.71)$$

Through step 5 of the algorithm the decision maker provides the following strict preference order:

$$(1) \underline{g}^1 = (6.48, 47.65), (2) \underline{g}^3 = (5.18, 52.68), (3) \underline{g}^0 = (7.13, 45.14)$$

$$(4) \underline{g}^2 = (5.83, 50.17), (5) \underline{g}^4 = (4.53, 55.20), (6) \underline{g}^5 = (3.88, 57.71)$$

The intervals of varying the objectives are discretized with  $a_1=3$  and  $a_2=4$  :  $[3.88, 5.51, 7.13]$ ,  $[45.14, 49.33, 53.52, 57.71]$ .

According to the UTA algorithm the marginal utilities assessed for  $g_1$  and  $g_2$  are presented in figure 5 and the utilities for each

decision profile in the ranking are (stability analysis has been omitted due to abbreviation):

$$u(\underline{g}^1)=0.740, u(\underline{g}^3)=0.805, u(\underline{g}^0)=0.725, u(\underline{g}^2)=0.800, u(\underline{g}^4)=0.565, \\ u(\underline{g}^5)=0.275.$$

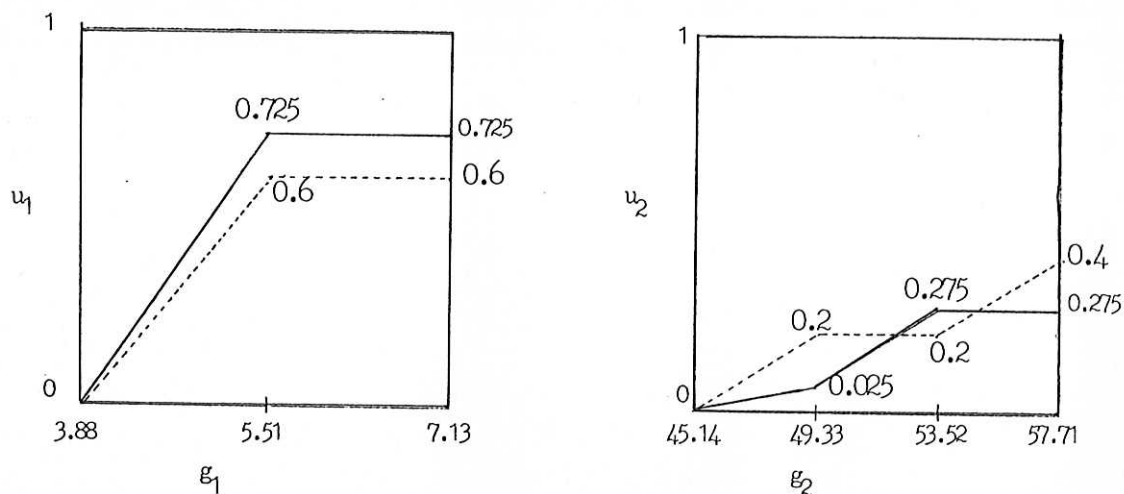


Figure 5 : Marginal utilities assigning a "weight" of 0.725 to the objective  $g_1$  and 0.275 to the objective  $g_2$ . The dotted curves represent the modified marginal utilities.

As it is shown in figure 6 some inconsistencies appear between the decision maker's ranking and the assessed utilities. Actually the model suggests that the decision maker has underestimated the alternatives  $\underline{g}^3$  and  $\underline{g}^2$  since it assigns the highest utilities to these alternatives.

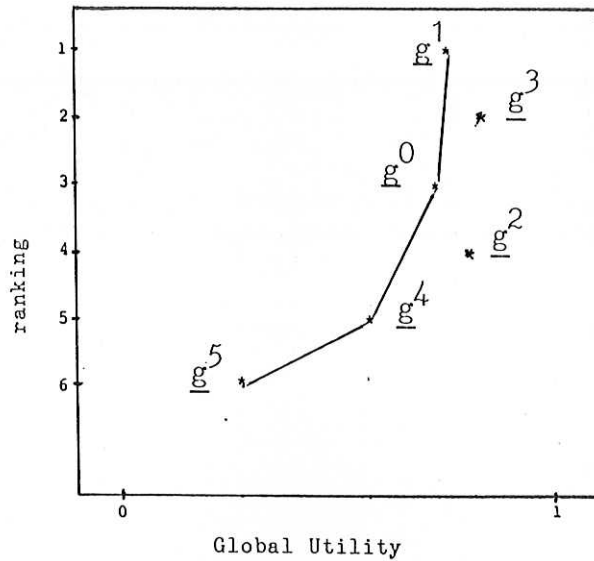


Figure 6 : Global utility versus ranking:  
Two inconsistencies appeared.

The decision maker believes that the alternative  $g^3$  must be kept after  $g^1$ , in the initial order in contradiction with the suggestions of the model. So he modifies the marginal utilities as it shown in figure 5 (dotted curves).

These modifications lead to the utilities  $u(g^1)=0.720$ ,  $u(g^3)=0.680$ ,  $u(g^0)=0.600$ ,  $u(g^2)=0.800$ ,  $u(g^4)=0.520$ ,  $u(g^5)=0.400$  and to a new utility - ranking diagram. At this stage the decision maker decides to alter his initial ranking according to the suggestions of the model, i.e. to raise the alternative  $g^2$  on the top of the ranking.

Based on the new ranking, the UTA algorithm estimates new marginal utilities, shown in figure 7.

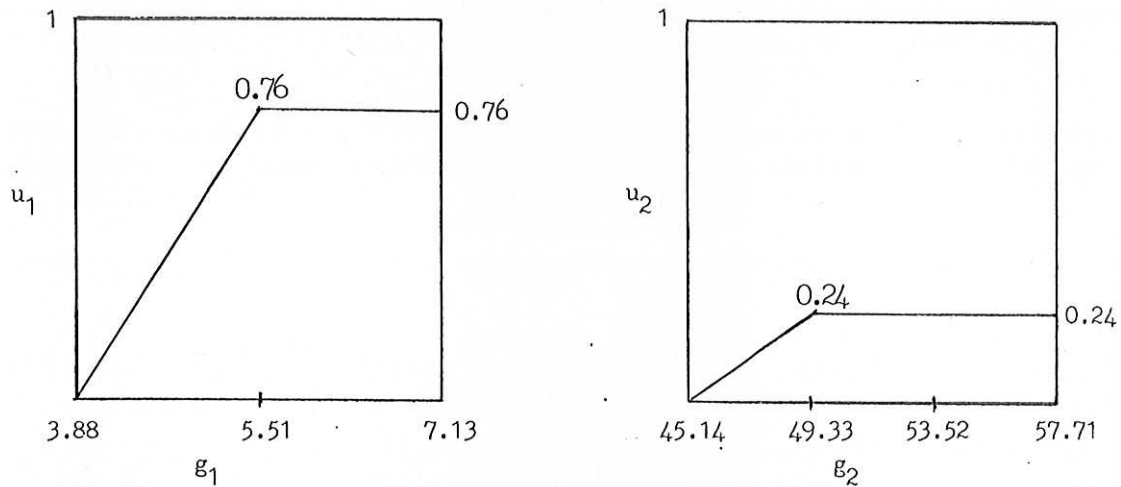


Figure 7 : Marginal utilities for the objectives  $g_1$  and  $g_2$ .

The utilities assigned to the alternatives are:

$u(\underline{g}^2)=1$ ,  $u(\underline{g}^1)=0.904$ ,  $u(\underline{g}^3)=0.848$ ,  $u(\underline{g}^0)=0.760$ ,  $u(\underline{g}^4)=0.544$ ,  
 $u(\underline{g}^5)=0.240$ . These utilities lead to a perfect restitution of the preference ranking.

An optimal solution, corresponding to a maximum utility, is  $x_1^3=3.21$ ,  $x_2^3=6$  and its consequences are  $g_1^3=5.16$ ,  $g_2^3=52.47$  (point 3 in fig. 3).

ITERATION 3 ( $q=3$ )

The solution  $\underline{g}^3=(5.16, 52.47)$  is provided to the decision maker and he is asked:

"Is there any objective satisfied by the solution  $\underline{g}^3=(5.16, 52.47)$ ?"

His answer is: YES.

He is then asked:

"Which is the least satisfied objective?"

His answer is :  $g_1$

He is then asked:

"Do you intend to relax a little bit the objective  $g_2$  in order to improve  $g_1$ ?"

His answer is NO.

As long as the decision maker is not willing to relax further any objective the process terminates here with the best compromise solution achieved being  $x_1=3.21$ ,  $x_2=6$  with corresponding objective values  $g_1=5.16$  and  $g_2=52.47$ .

## 5. CONCLUDING REMARKS

A new interactive method for solving MOLP problems is outlined in this paper. The method asserts that the final compromise solution lies within satisfactory levels and is efficient but not necessarily extreme. Furthermore, the final decision, having been achieved by the maximization of the decision maker's utility function, is rationalized by his needs. The decision maker is free to alter his judgment policy, during each iteration, in order to erase undesirable consequences which emerged from wrong estimations. However, the method is still based on the assumption that the decision maker is rational and consistent in providing the information required, especially in stage I. The information required in this stage, which establishes new bounds for the objectives, is needed to reduce the feasible region of the decision space. Although this restriction helps the convergence of the method, we believe, subject to further research, that the method could be modified in order to remove this restriction.

The information required by the decision maker within stage III is simple since, it is provided by him through pairwise comparisons of some decision profiles. However, it must be mentioned that the comparison of two decision profiles is rather problematic when it is realized by means of trade-offs involving all the objectives simultaneously. This happens if the objectives involved in the decision problem are more than two ( $n > 2$ ). Thus, it would be helpful if the decision maker was faced with trade-offs involving only two objectives at a time.

The above should be seriously considered when developing a Decision Support System.

The number of the pairwise comparisons made by the decision maker is strictly depended on  $s$ . It can easily be shown that the number of comparisons leading to a preference ranking is bounded by  $s(s-1)/2$  and takes its maximum value when, during each cycle of comparisons, no indifferences are involved and either the set of the alternatives preferred to the basis or the set of the alternatives not preferred to the basis is null.

The method helps the decision maker in understanding the situations he faces, through pictorial information analyzing the consequences of his judgment policy. This information could be enriched further with histograms representing the percentage achievement of the objectives with respect to the compromise solutions obtained at each iteration.

The information provided by the decision maker during previous iterations should be considered when starting a new iteration. Thus, previously generated decision profiles whose components lie within the new bounds can easily be embodied to the new ranking without, of course, violating the ranking already established on them.

The method requires standard and piecewise linear programming techniques in order to assess and optimize the decision maker's utility function. The number of programs solved at each iteration, as well as their type and their dimensions are presented in table 2. Recall here that  $n$ ,  $m$ ,  $s+1$  and  $a_i$  are respectively the number of the objectives, the number of the decision variables in the original problem, the number of decision profiles generated at each iteration and the number of discrete points taken in the interval of varying the objective  $g_i$ . Additionally, let  $N$  be the number of the constraints in the original MOLP problem. In stage III, a new linear program is solved whenever the decision maker alters the subjective ranking of the decision profiles. Thus, the number of programs solved within this stage depends on the number (say  $K$ ) of reorderings.

In stage IV, the estimation of a new compromise solution is realized by the solution of a piecewise linear program. Such a program has the typical linear form except for the restriction that at most two

adjacent  $r_{ij}$  are positive for each  $i$  and can be solved using the standard simplex algorithm with a modified basis-entry rule. According to the modified rule, a nonbasic variable  $r_{ij}$  enters the basis if it improves the value of the objective function and if there are no more than two adjacent  $r_{ij}$  positive, for each  $i$ , in the new basis. Furthermore, it has been proved [1] that if all marginal utility functions are concave the modified basis-entry rule can be omitted and the related problem can be solved as a typical linear program.

Table 2 : Programs and their dimensions.

| STAGE          | PURPOSE                                      | TYPE OF PROGRAM  | NUMBER OF PROGRAMS | NUMBER OF VARIABLES                    | NUMBER OF CONSTRAINTS |
|----------------|----------------------------------------------|------------------|--------------------|----------------------------------------|-----------------------|
| Initialization | Construction of the pay-off table.           | Linear           | $n$                | $m$                                    | $N$                   |
|                | Estimation of an initial efficient solution. | Linear           | 1                  | $m+1$                                  | $N+n$                 |
| Stage III      | Utility assessment                           | Linear           | $K$                | $2(s+1) + (\sum_{i=1}^n \alpha_i) - n$ | $s+1$                 |
| Stage IV       | Estimation of a new compromise.              | Piecewise linear | 1                  | $m + \sum_{i=1}^n \alpha_i$            | $N+2n$                |

## REFERENCES

- [ 1 ] Bazarra, M.S., C.M. Shetty: Nonlinear Programming. Wiley, New York, 1979.
- [ 2 ] Benayoun, R., J.de Montgolfier, J. Tergny, O. Larichev: Linear Programming with Multiple Objective Functions: Step Method (STEM). Mathematical Programming, Vol.1, No 3, pp.366-375, 1971.
- [ 3 ] Benson, R.G.: Interactive Multiple Criteria Optimization Using Satisfactory Goals. Ph.D. Thesis, University of Iowa, 1975.
- [ 4 ] Chankong, V.,Y. Haimes: Multiobjective Decision Making. Theory and Methodology. North-Holland, New York, 1983.
- [ 5 ] Choo, E.U., D.R. Atkins: An Interactive Algorithm for Multicriteria Programming. Compt. and Ops Res. Vol. 7 ,pp. 81-87, 1980.
- [ 6 ] Cohon, J.L., D.H. Marks: A Review and Evaluation of Mutiobjective Programming Techniques. Water Resources Research, Vol.11, No 2, pp.208-220, 1975.
- [ 7 ] Evans, W.G.: An Overview of Techniques for Solving Multiobjective Mathematical Programs. Management Science, Vol. 30, No 11, pp.1268-1282, 1984.
- [ 8 ] Fourer, R.: Piecewise Linear Programming. Report, Department of Industrial Engineering and Management Science, Northwestern University, Evanston, 1983.
- [ 9 ] Goicoechea, A., D.R. Hansen, L. Duckstein: Multiobjective Decision Analysis with Engineering and Business Applications. Wiley, New York, 1982.
- [ 10 ] Hemming, T.: Multiobjective Decision Making Under Certainty. Dissertation, Stockholm, 1978.
- [ 11 ] Hwang C-L, A. Masud: Multiple Objective Decision Making-Methods and Applications. Springer-Verlag, Berlin, 1979.
- [ 12 ] Jacquet-Lagrèze, E.: PREFCALC - Evaluation et Décision Multicritères. Euro-Décision, 1983.
- [ 13 ] Jacquet-Lagrèze, E., R. Méziani, R. Slowinski: MOLP with an Interactive Assessment of a Piecewise Linear Utility Function.. European Journal of Operational Research, Vol. 31, No 3 pp.350-357, 1987
- [ 14 ] Jacquet-Lagrèze, E., M. Shkun: Decision Support Systems for Semi-structured Buying Decisions. European Journal of Operational



Research, Vol. 16, No 1, pp. 48-58, 1984.

- [ 15 ] Jacquet-Lagrèze, E., J. Siskos: Assessing a set of Additive Utility Functions for Multicriteria Decision Making: The UTA Method. European Journal of Operational Research, No 10, pp.151-164, 1982.
- [ 16 ] Marcotte, O., R.M. Soland: An Interactive Branch and Bound Algorithm for Multiple Criteria Optimization. Report No T-442, George Washington University, 1981.
- [ 17 ] Naslund, B.: Interactive Methods in Multiple Criteria Optimization. EFI working paper No 6004, Stockholm School of Economics, Stockholm, 1973.
- [ 18 ] Roy, B.: Problems and Methods with Multiple Objective Functions. Mathematical Programming, Vol.1, No 2, pp.239-266, 1971.
- [ 19 ] Siskos, J.: Evaluating a System of Furniture Retail Outlets Using an Interactive Ordinal Regression Method. European Journal of Operational Research, No 23, pp.179-193, 1986.
- [ 20 ] Siskos, J., D. Yannacopoulos: UTASTAR: An Ordinal Regression Method for Building Additive Value Functions. Investigacao Operational, Vol 5, No 1, pp.39-53, 1985.
- [ 21 ] Wallenius, J.: Interactive Multiple Criteria Decision Methods: An Investigation and an Approach. The Helsinki School of Economics, Helsinki, 1975.
- [ 22 ] White, D.J.: Multi-objective Interactive Programming. J. Oper. Res. Soc. Vol. 31, pp.517-523, 1980.
- [ 23 ] Yannacopoulos, D.: Mise en place et expèrimentation d'un Système Interactif d'Aide à la Décision Multicritère. Thèse 3<sup>e</sup> cycle, Université de Paris - Dauphine, Paris, 1985.
- [ 24 ] Zeleny, M.: Multiple Criteria Decision Making. McGraw-Hill, New York, 1982.
- [ 25 ] Zionts, S., J. Wallenius: An Interactive Programming Method for Solving the Multiple Criteria Problem. Management Science, Vol. 22, pp.652-663, 1976.