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THEORETICAL FOUNDATIONS FOR DECISION SUPPORT SYSTEMS BASED ON REFERENCE POINTS

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RESUME

Les points de référence sont souvent utilisés comme une source d'information additionnelle dans les problèmes de choix d'une décision multicritère. Cependant, si nous ne pouvons pas faire l'hypothèse qu'un point de référence domine le point idéal, alors l'efficacité d'un point admissible le plus proche du point de référence n'est pas garantie. Dans cet article, nous étudions les conditions qui doivent être satisfaites par un point de référence x_0 pour que la solution du problème de scalarisation $\|x_0 - F(u)\| \to \min$, $u \in E$ soit efficace pour un problème d'optimisation multicritère $(F : U \to E) \to \min(\theta)$ où E est l'espace des critères partiellement ordonné par le cône convexe fermé θ. Nous introduisons la notion de points strictement dominants dans E puis nous démontrons que, moyennant quelques conditions additionnelles, si x_0 est un point de référence strictement dominant, alors la méthode de scalarisation par distance de x_0 donne une solution efficace.

Ce théorème peut être appliqué dans les systèmes d'aide à la décision basés sur l'utilisation de points de référence. Nous présentons également une caractérisation constructive de l'ensemble de tous les points strictement dominants et étudions leurs propriétés géométriques.

THEORETICAL FOUNDATIONS FOR DECISION SUPPORT SYSTEMS BASED ON REFERENCE POINTS

ABSTRACT

Reference points often serve as a source of additional information in MCDM problems ; however, if we may not assume that they dominate the utopia point, then Pareto optimality of compromise solutions resulting from a distance scalarization procedure is an open question. In this paper, we investigate the conditions which should be satisfied by a reference point x_0 to ensure that the solution of the scalarization problem $\|x_i - F(u)\| \to \min,\ u \in E$, be Pareto optimal for a multicriteria optimization problem $(F : U \to E) \to \min(\theta)$. E is a criteria space partially ordered by a closed and convex cone θ. We introduce the idea of strictly dominating points in E and prove that, under some additional conditions, if x_0 is a strictly dominating reference point, then the distance scalarization with respect to x_0 results in a Pareto optimal point.

This theorem can be applied in decision support systems based on reference points. We will also give a constructive characterization of the set of strictly dominating points and study its geometric properties.

1. Introduction.

A distance minimization procedure is a well-known tool to generate a compromise solution to a vector optimization problem. A variety of methods has been proposed by numerous authors making this approach one of most classical in vector optimization. The fundamental question which arises is as follows: under which assumptions the minimum of the distance scalarizing function

$$g(u) := \|x_0 - F(u)\| , \tag{1}$$

where x_0 is an element of the criteria space, exists and is nondominated in a set of decisions U.

This problem has been studied by several authors - cf.e.g. Salukvadze (1971), Dinkelbach and Dürr (1972), Yu (1973), Zeleny (1973), Wierzbicki (1975), Rolewicz (1975), and others who considered the case where x_0 is so-called ideal point x^*, i.e. the vector in the criteria space with the coordinates equal to the optimal values of scalar criteria evaluated individually. So defined x_0^* dominates all attainable values of F. In this case, as well as in its simple generalization, where x_0 dominates the ideal point, one can prove that under relatively weak assumptions concerning the set of attainable values of criteria each point minimizing the function (1) is nondominated or even properly nondominated (cf. Gearhart (1979)). Similar results can be obtained for abstract problems in Hilbert and Banach spaces ordered by a closed, convex, and pointed cone satisfying certain additional assumptions (cf. Rolewicz (1975)). Jahn (1984) presented a collection of general results on properties of scalarization methods including norm scalarization as one of the subcases. Recently, Wierzbicki (1986) proposed a similar approach based on modified distance functions.

However, in distance scalarization there are still open questions. An attempt to give an answer to one of them is presented in this paper, namely we will pay our attention to the case where x_0 is a point which dominates some but not all nondominated points. Elements of criteria space of this property will be called partly dominating points. We will impose certain additional condition on partly dominating points which will ensure that the scalarizing function (1) will admit its minimum at a nondominated point. The points satisfying this condition will be called strictly dominating points. We will also study the geometric properties of the set of strictly dominating points.

2. Basic definitions and properties.

We will refer to vector optimization problems of the form

$$(F : U \rightarrow E) \rightarrow \min (\Theta), \qquad (2)$$

where U is the set of admissible decisions, E is the space of criteria - a Banach space partially ordered by a closed, convex and pointed cone Θ, and F is a vector objective to be minimized with respect to the partial order introduced by Θ.

Let us recall that a cone Θ is <u>pointed</u> iff $\Theta \cap (-\Theta) = \{0\}$. The partial order $\leq_\Theta$ introduced by Θ is defined by the relation

$$x \leq_\Theta y \Leftrightarrow y - x \in \Theta$$

In further considerations F and U will play no separate role since we will concentrate our attention on the set

$$X := F(U) \subset E$$

and its relation to a point x_0 occuring in the scalarizing function (1).

The key definition in vector optimization can be expressed as follows.

Definition 2.1. An element y of X fulfilling the condition

$$(y - \Theta) \cap X = \{y\}$$

will be called Θ - minimal or nondominated in X.
A $(-\Theta)$- minimal point will also be called Θ - maximal.
The set of all Θ - minimal points in X will be denoted by $P(X,\Theta)$. There exists a great deal of conditions ensuring that $P(X,\Theta)$ is nonempty which will not be discussed here (cf.e.g. Sawaragi et.al. (1985), Chapter 3).

Throughout the paper we will assume that X is Θ- closed and Θ-complete, i.e. $X + \Theta$ is closed and

$$\forall x \in X \ \exists y \in P(X,\Theta) : \quad y \leq_{\Theta} x, \tag{3}$$

which is a sufficient condition for the existence of solutions to the scalarization problem (1).

Remark 2.1. A set X satisfying condition (3) is sometimes called externally stable or having the domination property.

In vector optimization an important role is played by so-called ideal or utopia points which express the best values of coordinates of criteria considered separately. Here we will give a more abstract definition related to the general formulation of the problem (2) and to the notion of totally dominating points.

Definition 2.2. A point $x \in E$ such that

$$X \subset x + \Theta$$

will be called a totally dominating point for X.

The set of totally dominating points will be denoted by $TD(X,\Theta)$.

Definition 2.3. By an ideal point for X we will call any $(-\Theta)$ - minimal element of $TD(X,\Theta)$. If the ideal points is unique, it will be denoted by $x^{*}(X,\Theta)$.

The uniqueness of ideal points is implied by the properties of Θ, namely we have the following.

Proposition 2.1. Suppose that the set of ideal points for a subset X of E is non-empty. Then the following conditions are equivalent:

a) There exists the unique ideal point $x^{*}(X,\Theta)$ for X,

b) For every two points x_1 and x_2 there exist $y \in E$ such that $x_i \leq_{\Theta} y$ for $i=1, 2$(i.e. E is a Banach lattice)

c) Θ is pointed and $\Theta - \Theta = E$

d) Θ is pointed and contains a base of E.

Let us note that $TD(X,\Theta)$ can be expressed in the form

$$TD(X,\Theta) = x^{*}(X,\Theta) - \Theta \, .$$

Besides of totally dominating and ideal points an important role in distance scalarization is played by partly dominating points.

Definition 2.4. A point $y \in E$ such that $(y+\Theta) \cap P(X,\Theta) \neq \emptyset$ will be called a partly dominating points for X. The set of partly dominating points will be denoted by $PD(X,\Theta)$.

A dual nature of partly dominating points is expressed by the following property.

Proposition 2.2. The set of Θ - maximal points of $PD(X,\Theta)$ is the same as the set of Θ - minimal points of X. i.e.

$$P(\ PD(X,\Theta),\ \ (-\Theta)\ ) = P(X,\Theta) \tag{4}$$

Moreover,

$$PD(X,\Theta) = \bigcup \{x-\Theta:\ x \in P(X,\Theta)\} \tag{5}$$

Proof: By Def. 2.4. each Θ-minimal point of X belongs to $PD(X,\Theta)$, i.e.

$$P(X,\Theta) \subset PD(X,\Theta) \tag{6}$$

If y is dominated by an element of $P(X,\Theta)$ then $(y + \Theta) \cap P(X,\Theta) \neq \emptyset$, consequently $P(X,\Theta) \subset P(PD(X,\Theta),\ (-\Theta))$. Conversely, if $z \in P(PD(X,\Theta),\ (-\Theta))$ then $z \leq_{\Theta} x$ for certain $x \in P(X,\Theta)$, hence and from (4) it follows that $z=x$.

To prove the relation (5) suppose that $y \in PD(X,\Theta)$. By Def. 2.3. there exists $x \in P(X,\Theta)$ such that $x \in y + \Theta$, i.e. $y \in x - \Theta$.

If $x \in P(X,\Theta)$ and $y \in x - \Theta$ then $x - y \in \Theta$, i.e. y dominates x, consequently, y is a partly dominating point for X.

Corollary 2.1.

If $P(X,\Theta)$ and Θ are closed then so is $PD(X,\Theta)$.

Proof: By (5) $PD(X,\Theta)$ is expressed as the range of the closed - valued multifunction

$$T:\ \ P(X,\Theta) \ni a \longrightarrow a - \Theta \subset E$$

defined on the closed set $P(X,\Theta)$.

For the closedness of $PD(X,\Theta)$ it is sufficient to prove that T is continuous.

However, if a and b are such that $\|a - b\| < \delta$ then the Hausdorff distance, d_H, of T(a) and T(b) can be estimated as follows:

$$d_H\ (\ T(a),\ T(b)\ ) = d_H\ (a - \theta,\ b - \theta) =$$
$$= d_H\ (a - \theta,\ (b - a)+(a - \theta)) \leqq \|a-b\| \leqq \delta.$$

Hence T is Hausdorff - continuous and the range of T, $PD(X,\theta)$, is closed, which ends the proof

q.e.d.

Remark 2.2. Let us note that in general the converse statement is not true, i.e. the closedness of $PD(X,\theta)$ does not imply that $P(X,\theta)$ is closed.

The necessary conditions for θ - minimality in distance-minimization derived previously by Dinkelbach and Dürr (1972), Rolewicz (1975), Jahn (1984), Wierzbicki (1986), and others, touched upon the reference points being ideal, or totally dominating points for X, with some corollaries regarding partly dominating reference points with additional constraints in the criteria space.

In this paper we will introduce a new class of dominating points, called strictly dominating.

Definition 2.5. A point $x \in E$ is called a strictly dominating point for X iff

$$P((x+\theta)\cap X,\ \theta) = P(X,\theta)\cap(x+\theta). \qquad (7)$$

The set of strictly dominating points for X will be denoted by $SD(X,\theta)$.

Observe that the inclusion „ $\subset$ " is always satisfied and the condition (7) means that no point of $P(X,\theta)\cap(x+\theta)$ is dominated by another attainable point, i.e. the set $(x+\theta)\cap X$ does not contain new nondominated points created by the constraints $z \leqq_\theta x,\ z \in X$.

The properties of strictly dominating points will be discussed in a more detailed way in Sec.4. Now let us make the following simple observation.

Proposition 2.3. If X is θ - complete then

$$TD(X,\theta) \subset SD(X,\theta) \subset PD(X,\theta) . \qquad (8)$$

An example of sets of totally, strictly, and partly dominating points for a bicriteria optimization problem with the natural partial order is shown in Fig. 2.1.

3. Distance minimization with respect to dominating points

Now we will prove several theorems on θ-optimality in scalarization via distance functions. Let us note that there exist scalarization methods based on transformed norms(cf. Wierzbicki (1986))which let avoid much of difficulties with classical distance functions. However, one can show that in some cases the latters model the decision-maker preferences in a most appropriate way that justifies their use. This is also the reason why we will not be concerned here on other features of scalarization methods such as completness of characterization of properly efficient points or computational difficulties - a distance function will be treated here as a value function for VOP.

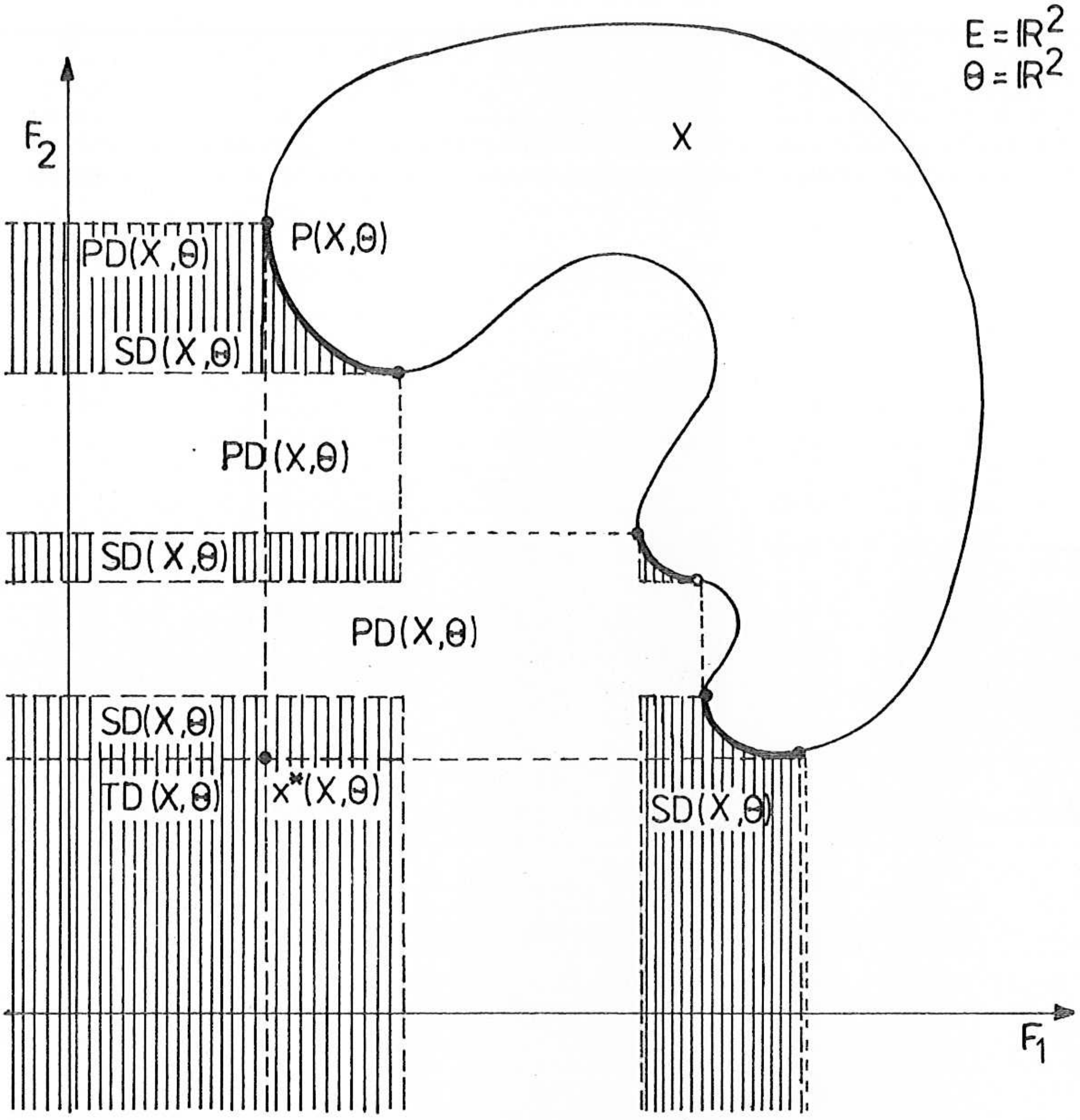


Fig. 2.1. An example of sets $TD(X,\Theta)$, $SD(X,\Theta)$, and $PD(X,\Theta)$.

Following earlier results concerning the finite-dimensional criteria space with the natural partial order, Rolewicz (1975) formulated the following geometric condition ensuring Θ-optimality of least-distance points:

Theorem 3.1. If the cone Θ is closed, convex, and pointed, X is a Θ-closed subset of E, and the norm in E is such that

$$\forall\, x \in E \qquad \Theta \cap (x - \Theta) \subset k_{\|x\|}(0) \cup \{x\}\,, \tag{9}$$

where $k_{\|x\|}(0)$ denotes the open ball with center 0 and radius $\|x\|$, then for each $x_o \in TD(X, \Theta)$ the scalarizing function $z \to \| x_o - z \|$ admits its infimum on X at a Θ- minimal point of X.

The geometric interpretation of condition (9) for a translated cone $y + \Theta$ is shown in Fig. 3.1.

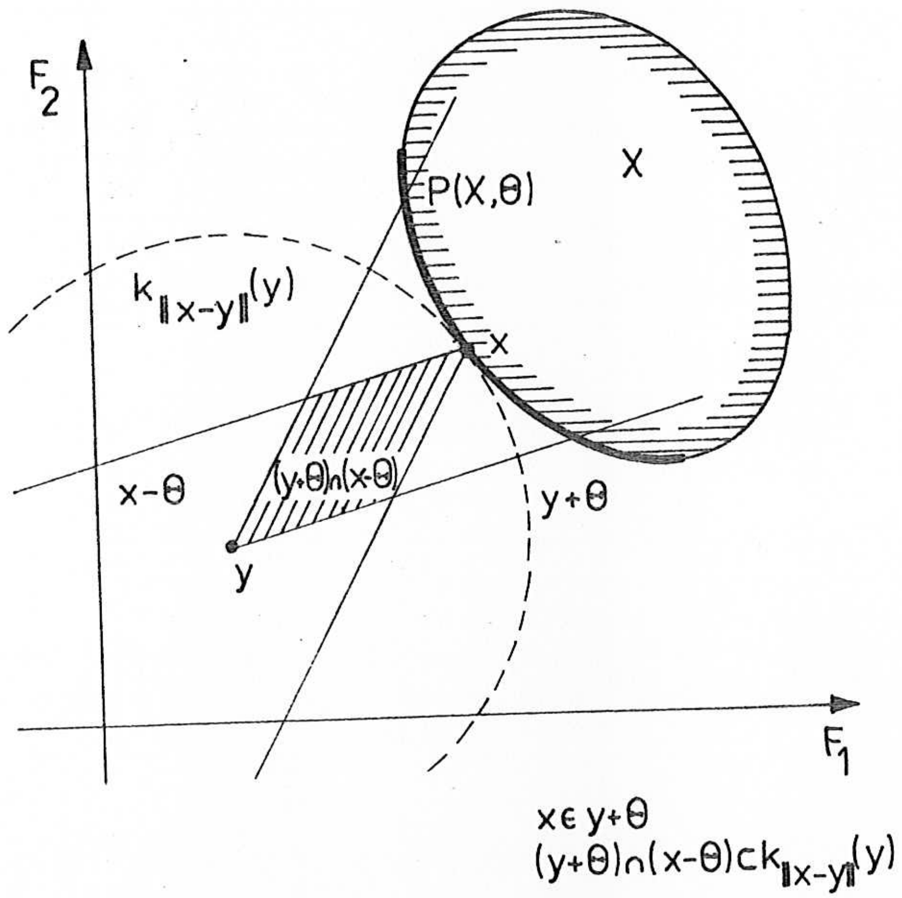


Fig. 3.1. A geometric interpretation of condition (9).

Notice that if E is a Hilbert space, then Thm. 3.1. generalizes earlier result of Wierzbicki (1975) who proved under same assumptions concerning Θ, X, and x_0 that the condition

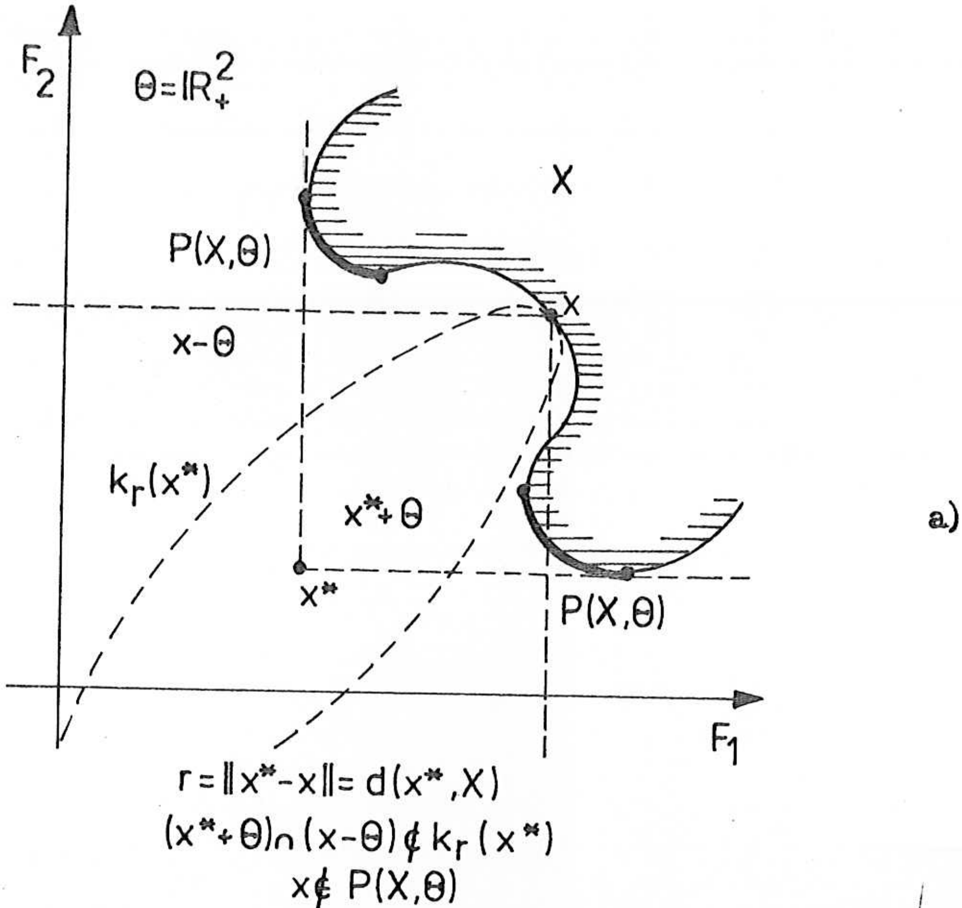
$$\Theta = \Theta^{*} , \qquad (10)$$

where $\Theta^{*} := \{y \in E: \ \forall z \in \Theta \ \langle y,z \rangle \geq 0\}$ is the dual cone, is sufficient for Θ-minimality of least distance elements of X. Namely, it turns out that (9) and (10) are equivalent in the case of a Hilbert criteria space (cf. Rolewicz (1975)).
On the other hand, Jahn (1984) proved some general theorems on scalarization methods, including distance scalarization, using the notion of strongly monotonically increasing functionals, by definition, $f : E \to \mathbb{R}$ is strongly monotonically increasing (s.m.i.) on E iff

$$x \leq_{\Theta} y, \ \ x \neq y \Rightarrow f(x) < f(y). \qquad (11)$$

Assuming that the norm in E is s.m.i. and $x_0 \in TD(X,\Theta)$, one can easily prove that the least-distance solution is Θ-minimal, however, it is not hard to see that (9) holds iff the norm is s.m.i., therefore those theorems coincide.

Condition (9) is evidently not necessary, but when not satisfied, Θ-minimality of points minimizing (1) depends on the shape of X, and situation of x_0 with respect to X, which are usually not a priori known. Several examples of situations, where the least-distance element fails to be Θ-minimal are presented in Fig. 3.2. and Fig. 3.3.



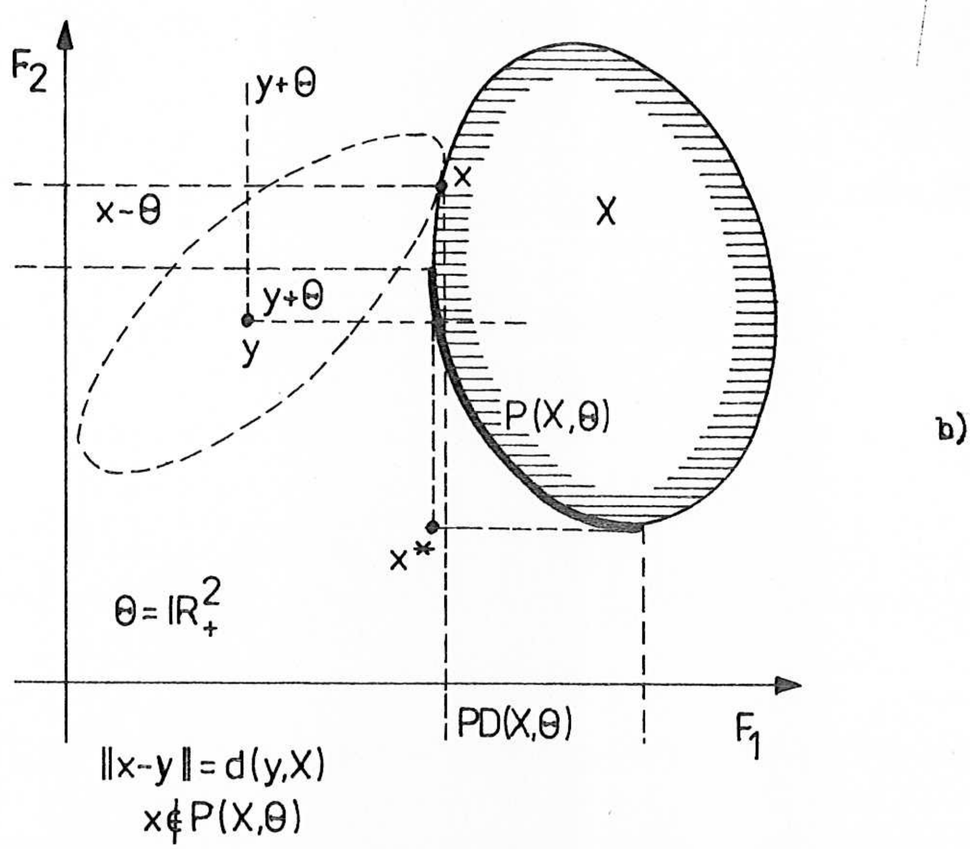


Fig. 3.2. Examples of situations, where the least-distance element in X to a reference point x_o is not Θ-minimal:
a) $x_o \in TD(X,\Theta)$ but the condition (9) is not satisfied
b) $x_o \in PD(X,\Theta)$, X is convex but (9) does not hold.

Remark 3.1. Let us note that the statement „ If $d(p,X) = \|p - x\|$, $x \in X$, $p \leq_{\Theta} x$, and (9) holds then $x \in P(x,\theta)$" (cf. Rolewicz (1975), Thm. 1') may not be true when X is not convex, which is exemplified in Fig. 3.3.

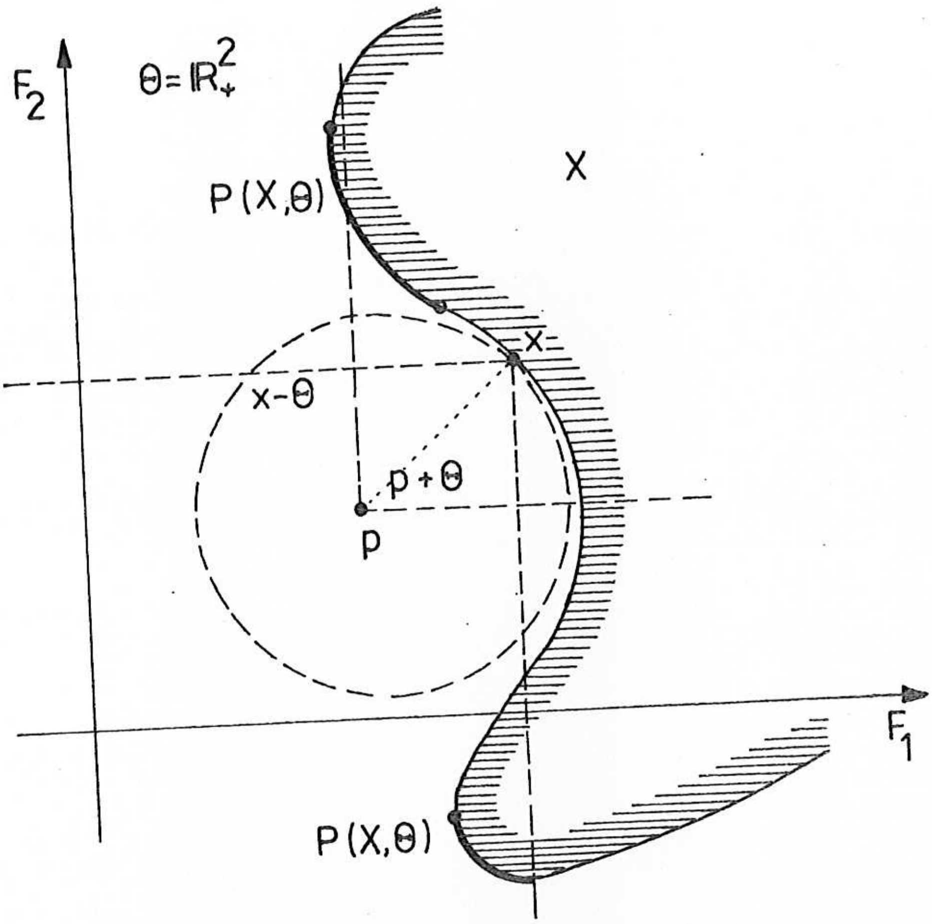


Fig. 3.3. An example of situation where (9) is fulfilled and a least-distance point x_o lies within p+Θ, but x_o is not Θ-minimal.

Let us note that in the case $E = \mathbb{R}^2$ the condition (9) is not necessary, namely one can prove

Proposition 3.1. Suppose that $E = \mathbb{R}^2$, and Θ is an arbitrary closed and convex cone such that $x^*(X,\Theta)$ and $P(X,\Theta)$ are non-empty, and X is Θ-convex.

If f is a non-constant convex function defined on E, having its global minimum attained at a point q belonging to

$$Z(X,\Theta) := (x^*(X,\Theta) + \Theta) \cap PD(X,\Theta) \tag{12}$$

then the minimum of f on X is attained at a Θ-minimal point.

Proof: Let x be a point of X such that the minimum of f on X is attained at x. The function f is convex then for each point y from the interval $[q,x]$

$$f(y) \leq f(x), \quad \text{and } \bar{f}(t) := f(tq + (1 - t)\, x)$$

in convex and non-decreasing on $[0,1]$.

Let $\bar{t}$ be the infimum of $t \in [0,1]$ such that $\bar{f}(t) = f(x)$ and let us denote $\bar{x} := \bar{f}(\bar{t})$.

From the introductory assumptions it follows that the boundary of Θ consists of two half-lines and $P(X,\Theta)$ is a curve which separates dominated and dominating points in $x^*(X,\Theta) + \Theta$.

If x is a dominated element of X then there exists $\bar{y} \in [q,\bar{x}) \cap P(X,\Theta) \subset X$, consequently, $f(\bar{y}) < f(x)$ which leads to a contradiction with the minimality of $f(x)$. Therefore $x \in P(X,\Theta)$.

q.e.d.

Corollary 3.1. Under the assumptions of Prop. 3.1. concerning X and Θ, in case $E = \mathbb{R}^2$ every solution to the scalarization problem $\min\{\|x-y\| : x \in X\}$ with the reference point $y \in Z(X,\Theta)$ is Θ-minimal.

Remark 3.2. In the above proof we used only the property that in convex bicriteria problems the set $P(X,\Theta)$ is a curve sufficiently smooth to topologically divide $x^*(X,\Theta) + \Theta$ in two disjoint subsets. Hence Prop. 3.2. and Corollary 3.1. remain true if $P(X,\Theta)$ is an arbitrary surface dividing $x^*(X,\Theta) + \Theta$, irrespectively of the dimension of E and the convexity of $X + \Theta$.

Corollary 3.2. Suppose that $X = F(U)$ is a closed subset of $\mathbb{R}^2$. If $F = (F_1, F_2)$ and $\inf\{F_1(u) : u \in U\} = \inf\{F_2(u) : u \in U\} = -\infty$, then for each convex function f having its global minimum attained at a partly dominating point of X the solution to the scalarization problem $(f : X \to \mathbb{R}) \to \min$ is Θ-optimal.

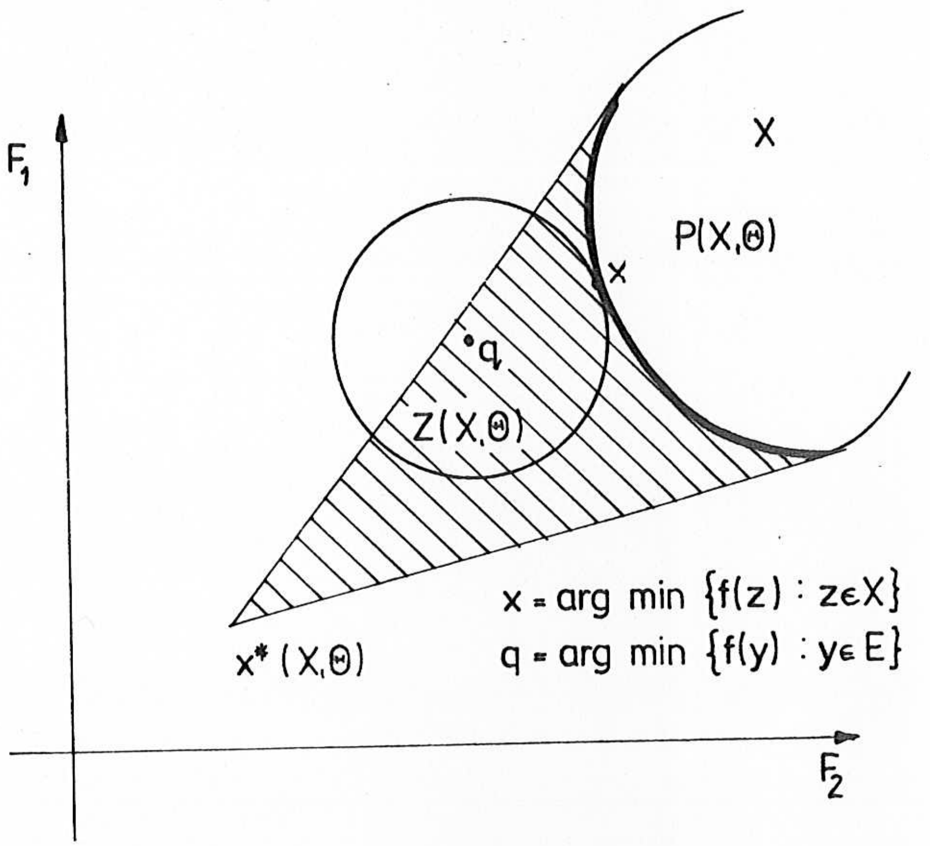


Fig.3.4. An illustration of the proof of Prop. 3.1.

If turns out, however, that in $\mathbb{R}^2$ the condition (9) warrants that a solution to a scalarization problem (1) is Θ-optimal for all $x_0 \in PD(X,\Theta)$.

<u>Theorem 3.2.</u> If $X \subset \mathbb{R}^2$ is convex and Θ-closed, Θ is closed, pointed, and satisfies condition (9), then a solution to the scalarization problem

$$\| y - x \| \longrightarrow \min (\Theta), \; x \in X$$

is Θ-minimal for each partly dominating reference point y.

Proof: Let y be an arbitrary partly dominating point, and let x be an element of X such that $\| x - y \| = d(y,X)$. Suppose first that the cone Θ does not degenerate to a half-line. The set $PD(X,\Theta)$ be decomposed into the disjoint union of sets $Z(X,\Theta) = x^*(X,\Theta) + \Theta) \cap PD(X,\Theta)$, $TD(X,\Theta) \setminus x^*(X,\Theta)$,

$A_1 := \{ (z_1,z_2) \in PD(X,\Theta) \; : \; z_1 > x_1^*, \; z_2 < x_2^* \}$,

$A_2 := \{ (z_1,z_2) \in PD(X,\Theta) \; : \; z_1 < x_1^*, \; z_2 > x_2^* \}$,

(some of them may be empty),

where x_1^* and x_2^* are coordinates of $x^*(X,\Theta)$ and all coordinates are related to a basis spanning Θ. Let us note that this theorem is already proved for y belonging to $Z(X,\Theta)$ or $TD(X,\Theta)$ (cf. Prop. 3.1. and Thm. 3.1. respectively).

If $y \in A_1$ or $y \in A_2$ and $y \leq_\Theta x$ then $x \in P(X,\Theta)$ by Corollary 3.1.

Suppose that $y \in A_1$ and x is non-comparable with y. The set of dominated points in X, non-comparable with ahy point of A_1 is separated from A_1 by sets $Z(X,\Theta)$ and A_2.

Therefore the interval $[y,x]$ must have a common point v either with A_0 or A_2. Let w be a least-distance point to v in X.

Of course, $\|w - v\| \leq \|x - v\|$, therefore by the triangle inequality $\| y - w \| \leq \| y - x \|$, consequently w is also least distant for y ($w = x$ when the balls in $\mathbb{R}^2$ are strictly convex). If $v \in Z(X,\Theta)$ then w is Θ-minimal, if $v \in A_2$ then $v \leq_\theta w$ - otherwize w would be dominated by y which was excluded, hence $w \in P(X,\Theta)$. Similarly we conclude that $w \in P(X,\Theta)$ for $y \in A_2$, and $x \in P(X,\Theta)$. If Θ degenerates to a half-line then the above reasoning is true for any non-degenerated cone Θ_1 containing Θ and fulfilling the assumptions of this Thm. Since $P(X,\Theta_1) \subset P(X,\Theta)$ then $w \in P(X,\Theta)$ in this case as well. Therefore $x \in P(X,\Theta)$.

q.e.d.

Now we will give a sufficient condition for Θ-optimality for non-convex attainable set X involving the use of strictly dominating points.

We will finish this section with a general theorem giving a sufficient condition for θ-minimality for strictly dominating reference points.

Theorem 3.3. Suppose that X is Θ-closed, the cone Θ is closed, convex, pointed, and satisfies condition (9). Then for each strictly dominating point x_0 the solution to the scalarization problem

$$\| x_0 - x \| \longrightarrow \min(\theta),\ x \in X,\ x_0 \leq_\theta x$$

is Θ-optimal in X.

Proof: Let us take an arbitrary $x_0 \in SD(X,\Theta)$. By the definition of strictly dominating points all Θ-minimal points in the set $P(X \cap (x_0 + \Theta),\Theta)$ are Θ-minimal in $P(X,\Theta)$. Since x_0 is the ideal point for $X \cap (x_0 + \Theta)$, then by Thm. 3.1. a least-distance element of this set is Θ-minimal in $X \cap (x_0 + \Theta)$, consequently it is Θ-minimal in X.

q.e.d.

One can see that the above Thm. may not be strenghtened by removing the constraints $x_0 \leq_\Theta x$, an example when a least-distance point x_0 to $y \in SD(X,\Theta)$ fails to be Θ-minimal can be found even in $\mathbb{R}^2$ as it is exemplified in Fig. 3.5.

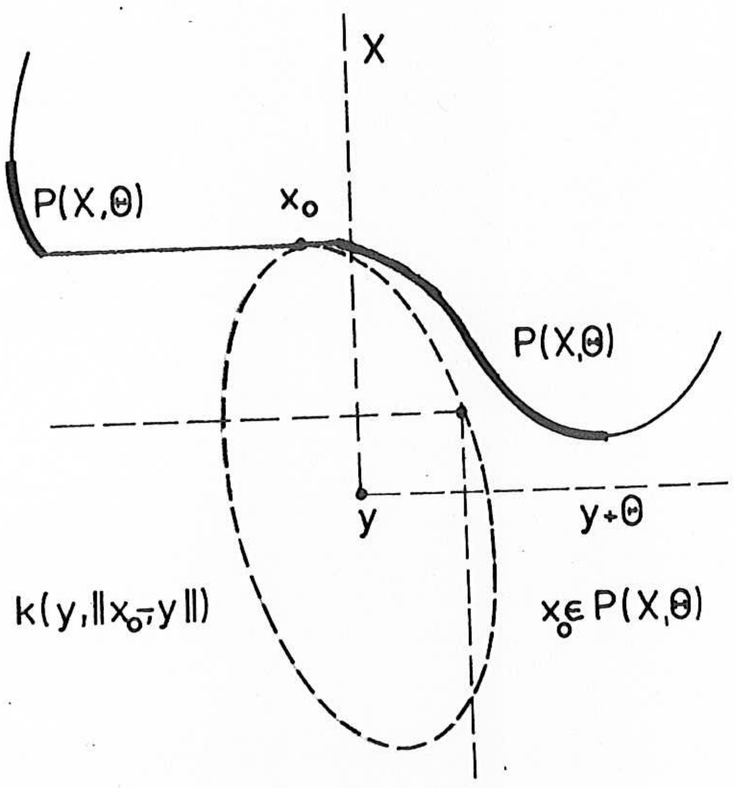


Fig. 3.5. The condition (9) is not sufficient for Θ-minimality when $y \in SD(X,\Theta)$ and X is non-convex.

4. Further properties of the set of strictly dominating points.

A general relation of the set $SD(X,\Theta)$ to totally and partly dominating points has been formulated as the inclusion (8) in Prop. 2.3. Now, we will study the properties of strictly dominating points in some special cases.

First we will answer the question whether the sets $PD(X,\Theta)$ and $SD(X,\Theta)$ coincide for convex X. It comes out that it is not true in a general case, however such a property may be proved for bicriteria problems.

<u>Theorem 4.1.</u> If $X \subset \mathbb{R}^2$ is Θ-closed and Θ-convex then

$$SD(X,\Theta) = PD(X,\Theta). \tag{13}$$

Proof: Let x be an arbitrary element of $PD(X,\Theta)$. By Prop. 2.3. it is sufficient to prove that $x \in SD(X,\Theta)$.

Suppose that it exists a point $y \in P(X \cap (x + \Theta),\Theta)$ which is not Θ-minimal in X. Then there exists $y_1 \in X$ such that $y \in y_1 + \Theta$. Let us consider the quadrangle with the vertices x, y, y_1 and x_1, where x_1 is an arbitrary element of $P(X,\Theta) \cap (x + \Theta)$ which exists since we assumed that x is partly dominating.
The points x_1 and y are non-comparable since both are elements of $P(X \cap (x + \Theta),\Theta)$. Taking into account that $y_1 \leq_\Theta y$, it follows that x_1, y and y_1 are not collinear - otherwize x_1 would belong to $y + \Theta$. Since X is Θ-convex, the triangle $[x_1, y_1, y]$ is contained in $X + \Theta$. On the other hand, x dominates x_1 and is non-comparable with y_1, hence we can similarly conclude that x_1, x, and y_1 form a non-degenerated triangle.

Since x may not be an element of $[x_1, y, y_1]$ which would imply that it belongs to $X + \Theta$, therefore the quadrangle $[x, x_1, y, y_1]$ is convex and non-degenerated. Consequently, the diagonal $[x_1, y_1]$ intersects the other one, $[x, y]$, at a point y_0 belonging to $X + \Theta$. However, $[x, y]$ is contained in $(x + \Theta) \cap (y - \Theta)$, therefore $y_0 \in y - \Theta$ and $y_0 \in X + \Theta$ since $[x_1, y, y_1] \subset X + \Theta$ which contradicts the assumption that y were Θ-minimal in $X \cap (x + \Theta)$ but dominated in X. Thus we conclude that each Θ-minimal point in $X \cap (x + \Theta)$ is Θ-minimal in X, i.e.

$$P(X \cap (x + \Theta), \Theta) = P(X, \Theta).$$

By definition, it means that $x \in SD(X, \Theta)$.

q.e.d.

Let us note that the above proof remains valid for such convex sets in E, dim $E > 2$, that there exist a subspace E_1, dim $E_1 = 2$, containing simultaneously x,y and the above defined points x_1 and y_1. Theorem 4.1. may not be true when X is convex but the dimension of the criteria space E is greater than 2.

An example of such situation for $E = \mathbb{R}^3$ and $\Theta = \mathbb{R}^3_+$ is given below.

<u>Example 4.1.</u> Let us consider the attainable set $X = [a, b, c]$, $a = (1, 0, 0)$, $b = (1, -1, 0)$, $c = (1/2, 1/2, 1/2)$ and a reference point $x = (0, 0, 0)$(cf. Fig. 4.1.). It is easy to see that $P(X, \Theta) = [c, b]$ and for $x = (0, 0, 0)$, $P(X, \Theta) \cap (x + \Theta) = [c, z]$, where $z = (2/3, 0, 1/3)$. However,

$R := P(X \cap (x + \Theta), \Theta) = [c, z] \cup [z, a]$, $a \notin P(X, \Theta)$.

Therefore $x \notin SD(X, \Theta)$, although it is an element of $PD(X, \Theta) = [c, b] - \mathbb{R}^3_+$. One can show that

$SD(X,\Theta) = PD(X,\Theta) \cap \{(y, y_2, y_3) \in \mathbb{R}^3 : y_2 \leq x_2^*\} \cup$
$\cup \{(z_1, z_2, z_3) \in \mathbb{R}^3 : z_1 \leq 1/2, z_2 \leq 1/2, z_3 = 1/2\} =$
$= \{(y_1, y_2, y_3) \in \mathbb{R}^3 : y_1 \leq 1, y_2 \leq -1, y_3 \leq y_1$ or
$y_1 = 1/2, y_2 = 1/2, y_3 = 1/2\}$.

where x_2^* is the second coordinate of the ideal point $x^*(X,\Theta) = (-1/2,-1, 0)$.

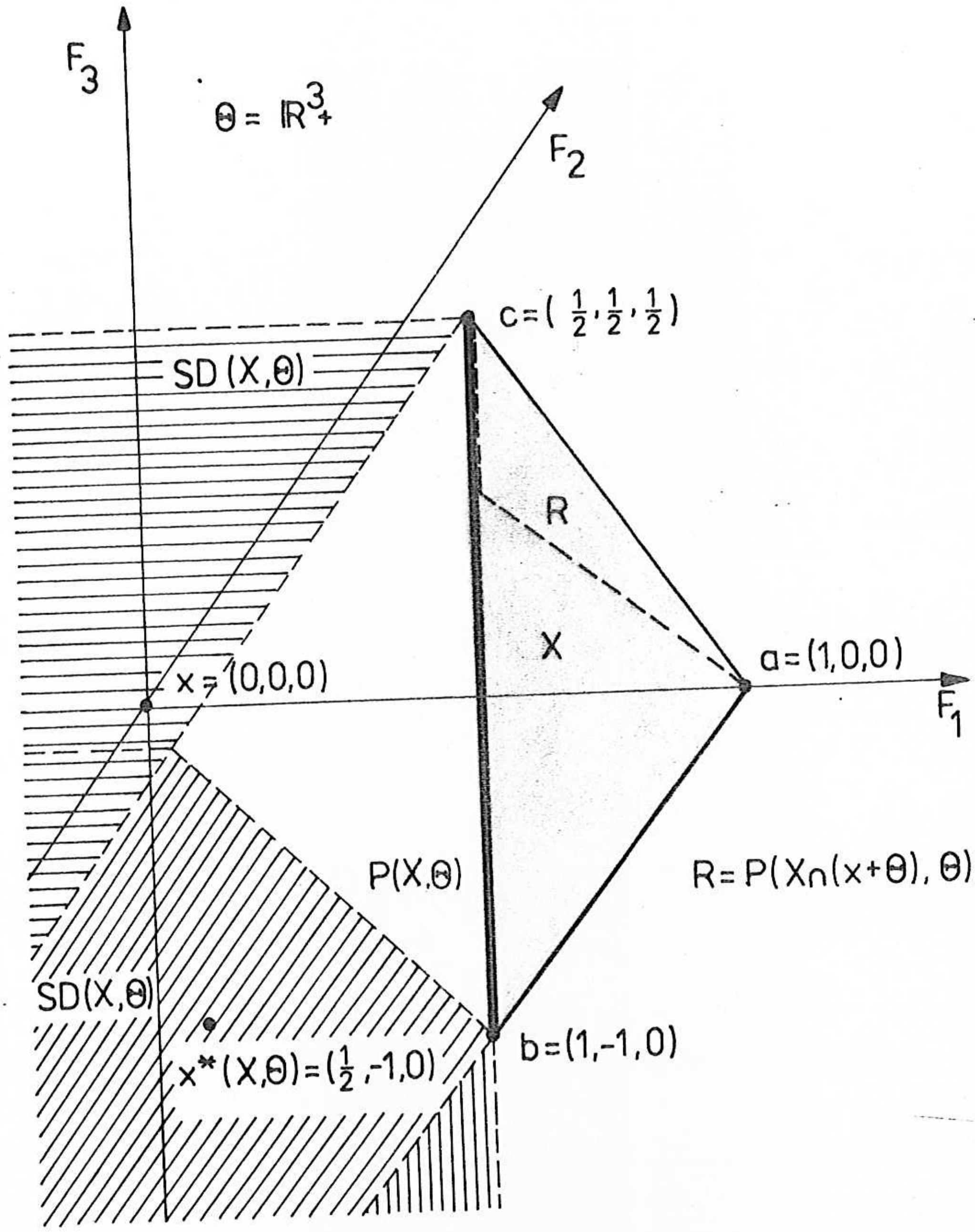


Fig. 4.1. An example when $P(X \cap (x + \Theta),\Theta) \neq P(X,\Theta) \cap (x + \Theta)$ for convex X.

Let us note that in general topological properties of the set $SD(X,\Theta)$ are determined by the properties of the whole set X, irrespectively of the properties of $P(X,\Theta)$ itself. For instance, if in the above presented example 4.1 we modify the set X by setting

$$X_1 := (X \setminus \{(z_1, z_2, z_3) \in \mathbb{R}^3 : r_1 < z_2 < r_2\}) \cup [q_1, q_2],$$

where

$b_2 \leq r_1 < r_2 \leq c_2$ and q_i are the intersections of planes $z_2 = r_i$ and the interval $[b, c]$, for $i = 1, 2$, then the set $SD(X_1,\Theta)$ fails to be closed, although so are X_1 and $P(X_1,\Theta)$.

However, one can prove that in $\mathbb{R}^2$ $SD(X,\Theta)$ is closed for closed $P(X,\Theta)$, and connected, for connected $P(X,\Theta)$.

Below we give a sketch of the procedure of evaluating $SD(X,\Theta)$ for $X \subset \mathbb{R}^2$.

4.1 Construction of $SD(X,\Theta)$ for $X \subset \mathbb{R}^2$.

Although the set $SD(X,\Theta)$ possesses interesting properties, its definition (cf. Def. 2.4.) does not suggest a constructive algorithm of finding $SD(X,\Theta)$, or even of verifying whether a given point $x_0 \in E$ is strictly dominating. Here we give a further characterisation of $SD(X,\Theta)$ which in some cases should be helpful in answering the above questions, especially when $E = \mathbb{R}^2$.

Lemma 4.1. If X is a closed and connected subset of $\mathbb{R}^2$ then

$$SD(X,\Theta) = \{y \in PD(X,\Theta) : P(\partial(y + \Theta) \cap X, \Theta) \subset P(X,\Theta)\}. \qquad (14)$$

Proof:

Observe first that if Θ degenerates to a half-line then for each $y \in \mathbb{R}^2$ $(y + \Theta) = \partial(y + \Theta)$, consequently

$$P((y + \Theta) \cap X, \Theta) = P((y + \Theta) \cap X, \Theta)$$

and $P((y + \Theta) \cap X, \Theta)$ contains at most one point, namely $P((y + \Theta) \cap X, \Theta) = \{x\}$, iff $y \in PD(X, \Theta)$ and is empty elsewhere. Therefore in this case

$$SD(X, \Theta) = PD(X, \Theta)$$

and (14) is trivially satisfied. Thus without a loss of generality we can suppose that Θ contains a base of $\mathbb{R}^2$ and the coordinates of points in $\mathbb{R}^2$ are related to this base.

Notice that from the connectedness of X it follows that if $\partial(y + \Theta) \cap X = \emptyset$ then $X \subsetneq y + \Theta$ and $y \in TD(X, \Theta)$. Thus we may assume that $\partial(y + \Theta) \cap (X + \Theta) \neq \emptyset$.

Each point $v \in \partial(y + \Theta) \cap X)$ maximizes one of coordinates of points from $X \cap (y + \Theta)$ therefore if $w \in P(\partial(y + \Theta) \cap X, \Theta)$ then w belongs also to $P((y + \Theta) \cap X, \Theta)$.

If y is strictly dominating then $w \in P(X, \Theta)$ which proves that $SD(X, \Theta)$ is contained in the set defined as the right-hand side of (14).

Suppose now that y is such that the inclusion

$$P(\partial(y + \Theta) \cap X, \Theta) \subset P(X, \Theta) \tag{15}$$

holds. Let us notice that $\partial(y + \Theta)$ consists of two half-lines beginning at y which allows us to distinguish the following subcases :

a) $P(\partial(y + \Theta) \cap X, \Theta)$ consists of two points x_1 and x_2

b) $P(\partial(y + \Theta) \cap X, \Theta)$ consists of one point .

Observe that in the case (a) it is sufficient to prove that if $x_1, x_2 \in P(X, \Theta)$ then an element z of $(y + \Theta) \cap X$ is Θ-optimal in X iff it is so in $(y + \Theta) \cap X$.

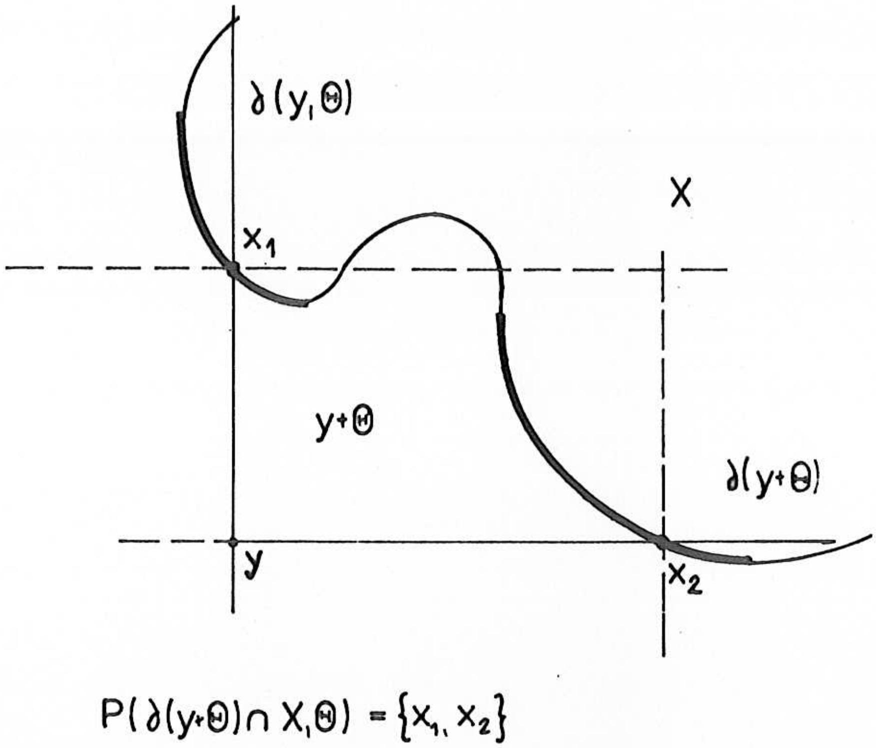


Fig. 4.2. An illustration of the proof of Lemma 4.1. - case (a)

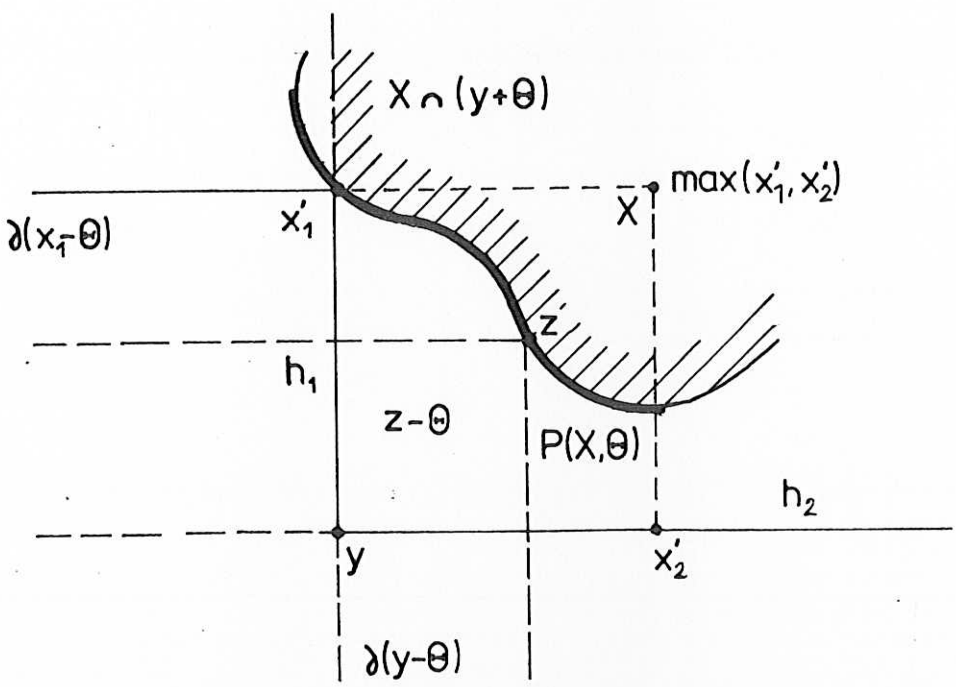


Fig. 4.3 An illustration of the proof of Lemma 4.1. - case (b)

Suppose that $z \in P((y + \Theta) \cap X, \Theta)$. From basic geometric properties of convex, nondegenerated cones in $\mathbb{R}^2$ (cf. Fig. 4.2.) it follows that if x_1 and x_2 belong to $P(X,\Theta)$ then any point z of $(y + \Theta) \cap X$ is either dominated by x_1 or x_2, or the intersection of X and $z - \Theta$ does not contain exterior points of $y + \Theta$. The latter property is implied by the inclusion

$$z - \Theta \subset (x_1 - \Theta) \cup (x_2 - \Theta) \cup (y + \Theta) \cap (\max(x_1, x_2) - \Theta)$$

for $z \in (y + \Theta) \cap (\max(x_1, x_2) - \Theta)$,

where

$$\max(x_1, x_2) = (\max(x_{11}, x_{21}), \max(x_{21}, x_{22})). \tag{16}$$

Consequently, if z is Θ-optimal in $(y + \Theta) \cap X$, it is also Θ-minimal for X, which ends the proof of the case (a). In case (b) let us take an arbitrary $z \in P((y + \Theta) \cap X, \Theta)$, and let x_1' be the Θ-minimal point of the intersection of $\partial(y + \Theta)$ and X. If we take as x_2' a point of $\partial(y + \Theta)$ such that

(i) x_2' belongs to the half-line h_2 different than that containing x_1' (recall that we assumed that Θ is non-degenerated and $\partial(y + \Theta)$ consists of two half-lines, h_1 and h_2)

(ii) $z \leq_\Theta \max(x_1', x_2')$, (cf. Fig. 4.3.),

then same arguments as applied for x_1 and x_2 in case (a) imply that $z \in P(X,\Theta)$. We only have to observe that if X is connected, $x_1' \in P(X,\Theta)$ and x_2' is defined as above, then the set $(\partial(x_1' - \Theta) \setminus \partial(y - \Theta)) \cup h_2$ divides $\mathbb{R}^2$ into two parts in such a way that X is contained in this one which does not contain y. Therefore x_2' may not be dominated by an element of X and $x_2' \in P(X \cup \{x_2'\}, \Theta)$ which allows to consider x_2' in the same way as in the proof of the case (a), with $x_1 := x_1'$, $x_2 := x_2'$ and $X' := X \cup \{x_2'\}$.

q.e.d.

Notice that to prove Lemma 4.1. in case (a) we did not apply the assumption that X is connected. This means that if $P(\partial(y + \Theta) \cap X, \Theta)$ consists of two different points then strict Θ-optimality of y can be tested irrespectively from the connectedness of X. However, in general, Lemma 4.1. is not true for disconnected X, as may be seen in Fig. 4.4. Nevertheless, it is possible to verify the strict Θ-optimality of a reference point in $\mathbb{R}^2$ basing on Lemma 4.2., namely in that case one shall determine the edge points of all connected components of $P(X,\Theta)$ which can be realized by studying the mutual locations of the local Pareto sets of the components of X. Remark that for disconnected X we cannot substitute the set $X + \Theta$, which is always connected, in place of X in the condition (14) since it may happen that $y \in SD(X,\Theta)$ but $y \notin SD(X + \Theta, \Theta)$ (cf. Fig.4.5).

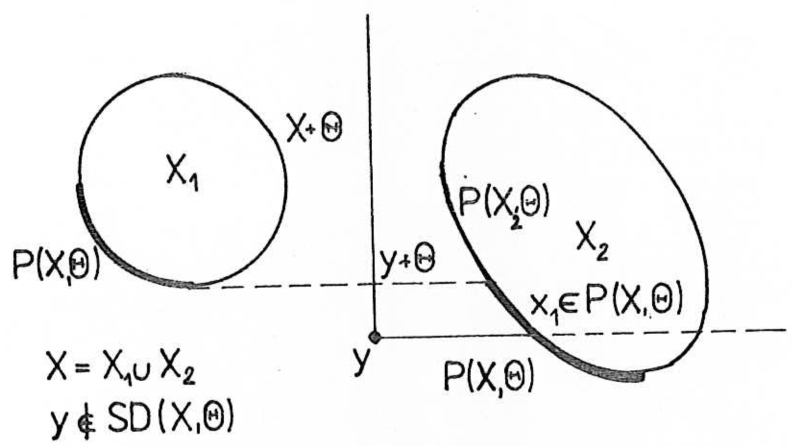


Fig. 4.4 For disconnected X the fulfillment of (14) is not sufficient for $y \in SD(X,\Theta)$

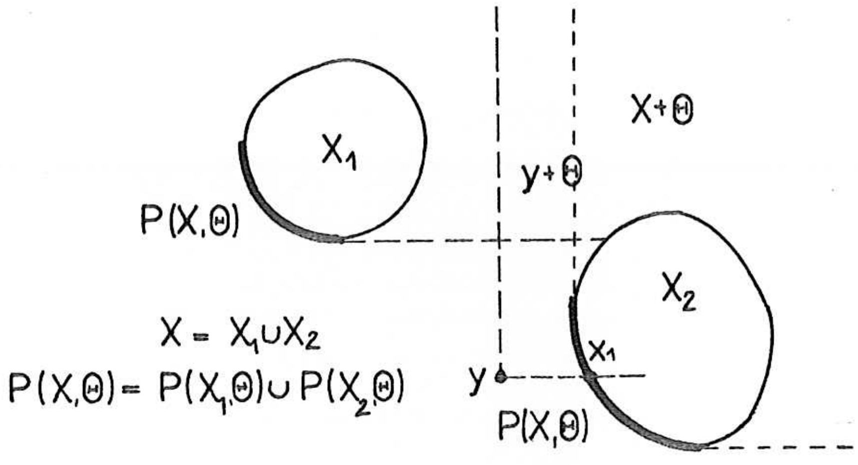


Fig. 4.5 The condition (14) may not be applied for $X + \Theta$ instead of X when X is disconnected.

From Lemma 4.1 and the above considerations it follows a characterization of strictly dominating points in $\mathbb{R}^2$:

Theorem 4.2. If X is a closed and connected subset of $\mathbb{R}^2$ then $SD(X,\Theta)$ can be represented as

$$\bigcup \{pr_1(S_i) \times pr_2(S_j) \; : \; i, \; j \in I \cup \{0\}\} \cap PD(X,\Theta), \qquad (17)$$

where $\{S_i\}_{i \in I}$ is the set of connected components of $P(X,\Theta)$, by definition $S_0 := TD(X,\Theta)$, r_i, $i = 1,2$, is a base of E, spanning the cone Θ, pr_i is the projection on the i-th axis in the system of coordinates (r_1, r_2).

This theorem implies the following procedure of verifying whether a reference point q in $\mathbb{R}^2$ is strictly dominating.

Step 0. Convert the coordinates of X and q to those of a base (r_1, r_2) spanning Θ.

Step 1. Calculate all local minima of F_1 and F_2.

Step 2. Find all local ideal points $x_i^* = (x_{1i}^*, x_{2i}^*)$ and edge points $e_{1i} = (x_{1i}, e_{12i})$ and $e_{2i} = (e_{21i}, x_{2i})$ associated to the connected components $\{S_i\}_{i \in I}$ of $P(X,\Theta)$, where $e_{12i} := \inf\{x_2 : (x_{1i}, x_2) \in X\}$ and

$$e_{21i} := \inf\{x_1 : (x_1, x_{2i}) \in X\} .$$

If the minimum of the j-th coordinate, $j \in \{1,2\}$, is not achieved then put $e_{kji} := \inf$, $k \in \{1,2\}$, where for each $x \in \mathbb{R}$ $x \leq \inf$.

Step 3. Find the global ideal point $x^* = (x_1^*, x_2^*)$ and corresponding edge points $e_1 := (x_1, e_{12})$ and $e_2 := (e_{21}, x_2)$.
If $q \leq x$ print "$q \in SD(X,\Theta)$" STOP.
If $q_1 \geq e_{21}$ or $q_2 \geq e_{12}$ print "$q \notin SD(X,\Theta)$" STOP.

Step 4. If $q_1 < x_1^*$ then go to Step 5.
Find i such that
$x_{1i}^* \leq q_1$ and for each $j \neq i$: $x_{1j}^* < x_{1i}^*$ or $x_{1j}^* > q_1$.
If $e_{21i} < q_1$ or $q_2 \geq e_{12i}$ then print "$q \notin SD(X,\Theta)$" STOP.
If ($e_{21i} > q_1$ or $e_{21i} = q_1$ and $e_{2i} \in S_i$) and $q_2 < x_2^*$ then print "$q \in SD(X,\Theta)$" STOP.

Step 5. Find n such that
$x_{2n}^* \leq q_2$ and for each $m \neq n$: $x_{2m}^* < x_{2n}^*$ or $x_{2n}^* > q_2$.
If $e_{12n} < q_2$ or $q_1 \geq e_{21n}$ then print "$q \notin SD(X,\Theta)$" STOP.
If $i = n$ then ~~verify~~ whether $q \in X$, if it is so then print "$x \notin SD(X,\Theta)$" STOP.
If $e_{12n} > q_2$ or $e_{12n} = q_2$ and $e_{1n} \in S_n$ then print "$q \in SD(X,\Theta)$" STOP.

Remark that to find all local ideal points in Step 2 for a non-convex vector optimization problem one has to determine all local minima of the objectives considered separately. Thus, in general, in this step one has to execute two global minimization procedures. Since they are usually based on randomized techniques, the evaluation of $SD(X,\Theta)$ may have an approximative character.

Applying the above algorithm for disconnected X, we may get an erroneus result of y is situated as in Fig.4.4 or 4.5. Namely, in this case the strict dominance of y will not be detected as Thm. 4.2 does not give a complete characterisation of $SD(X,\Theta)$ for disconnected X. However, as we will see in the following subsection only those elements of $SD(X,\Theta)$ which are of the form given in Thm.4.2 have the desired properties as reference points for distance scalarization procedures.

Let us note that an appropriate algorithm for verifying whether an element of $\mathbb{R}^n$, $n \geq 3$, is stricty dominating seems to be essentially more difficult as so far there are no constructive characterizations of $SD(X,\Theta)$ for $X \subset \mathbb{R}^n$, $n \geq 3$.

4.2. A sufficient condition for Θ-optimality without auxiliary constraints.

In Section 3 we have shown that a distance scalarization procedure with respect to a strictly dominating reference point may lead to a dominated least-distance point even in $E = \mathbb{R}^2$.

However, imposing additionally a regularity condition on the distance in E we can prove the following theorem 4.3. In its proof we will need some information about the structure of the set of strictly dominating points which is studied in a more detailed way later on.

<u>Theorem 4.3</u> If $\mathbb{E} = \mathbb{R}^2$, $X \subset E$ is connected, Θ satisfies (9), the distance in E is balanced and satisfies the condition

$$\arg\max \{ z_i : z \in k(x,r) \} \subset (x + \Theta) \cup (x - \Theta) \qquad (18)$$

for $r > 0$, $i = 1, 2$, where the coordinates of $z = (z_1, z_2)$ are related to a positively oriented basis (r_1, r_2) spanning a cone Θ_1 such that $\Theta_1 \supset \Theta$, then a least-distance element x_0 in X to a strictly dominating point y is Θ-optimal in X, irrespective whether the additional constraints $y \leq_\Theta x_0$ are fulfilled or not.

Proof: Let S be the set of connected components of $P(X,\Theta)$, $S = \{S_i\}_{i \in I}$, where I is an ordered set and let us associate to each S_i the local ideal point $x_i^* = (x_{i1}^*, x_{i2}^*)$ and the edge points $e_{1i} = (x_{i1}, e_{12i})$ and $e_{2i} = (e_{21i}, x_{i2})$ where

$$e_{12i} := \inf \{ x_2 \in \mathbb{R} : (x_{i1}, x_2) \in X \}$$

and

$$e_{21i} := \inf \{ x_1 \in \mathbb{R} : (x_1, x_{i2}) \in X \} .$$

We assume that the components S_i are ordered in such a way that if $i \prec j$ then $x_{i1} < x_{j1}$.

We will admit the convention $\inf \emptyset := +\infty$, but it may be shown that such situation may happen only if $i = \inf I$ for e_{12i} or $i = \sup I$ for e_{21i}.

Without a loss generality we can assume that Θ is non-degenerated and let r_1 and r_2 be a base spanning Θ.

In the following section we prove that $SD(X,\Theta)$ can be represented as a union of sets $Z(S_i,\Theta)$ (cf.(1.2)),

$$pr_1(TD(X,\Theta)) \times pr_2(S_i),\ pr_1(S_i) \times pr_2(TD(X,\Theta)$$

for $i \in I$, $pr_1(S\ )\times pr_2(S_j)$ for $i,j \in I$, $i \prec j$, and $TD(X,\Theta)$, where pr_k denotes the projection parallel to the k-th axis.

Therefore it is sufficient to prove this Thm. for each possible situation of a reference point $y = (y_1,y_2)$ within $SD(X,\Theta)$.

By Prop. 3.1 if $y \in Z(S_i,\Theta)$ then a least-distance element x to y in X situated within the set $(x_1 + \Theta)$ is Θ-optimal. However, the curve consisting of S_i and half-lines $h_1 := \{(t,\ e_{12i}) : t < x^*_{1i}\}$ and $h_2 := \{(e_{21i}, t) : t < x^*_{2i}\}$ separates the area of local partly dominating points of S, B_1, which interior does not contain any other point of X, since e_{1i} and e_{2i} are either Θ-optimal or are limits of Θ-optimal sequences, and the remaining part of $\mathbb{R}^2$, P_{i2}. The latter contains points of X and we will show that they are more distant from y than e_{11} or e_{21}. Suppose that $x = (x_1,x_2) \in S_i$ and consider the interval $[y,x]$ which in this case must intersect h_1 or h_2 in a point $z= (z_1,z_2)$. Without a loss of generality suppose that $z \in h_1$ and $z_2 = e_{12i}$, consequently $x_2 \geq e_{12i}$. According to (1.8), the maximum of the second coordinate over $k_{\|x - y\|}(y)$ must be achieved within $y + \Theta$. Simultaneously, following (1.8) and Prop.3.1, $k_{\|x - y\|}(y)$ has a common point with an element v of $P(X,\Theta) \cap (y + \Theta)$.

Observe now that for each Θ-optimal point $v = (v_1, v_2)$ it holds

$$x^*_{i2} \leq v_2 \leq e_{12i} ,$$

therefore

$$\max\{z_2 : z \in k_{\|x - y\|}(y)\} = e_{12i}$$

while no element of $X \setminus ((y + \Theta) \cap (y - \Theta))$ may have second coordinate greater or equal to this value. Hence we obtain a contradiction with the assumption that there exist a least distance element $x = (x_1, x_2) \in X \setminus (y + \Theta)$ such that $x_2 \geq e_{12i}$.
Similarly we will prove this theorem in the case when

$$y \in R_{ij} := pr_1(S_i) \times pr_2(S_j), \; i, \; j \in I, \; i \prec j,$$

where S_i and S_j are connected components of $P(X,\Theta)$ and pr_k denotes the projection parallel to the k - th axis.

Analogously as above we shall define

$$h_1 := \{ (t, e_{12i}) : t < x^*_{1i} \}$$

and

$$h_2 := \{ (e_{21j}, t) : t < x^*_{2j} \},$$

and let us suppose that a least-distance element $x = (x_1, x_2)$ does not belong to $(y + \Theta) \cap X$, so that the interval $[y, x]$ intersects h_1 at a point $z = (z_1, z_2)$. Now, it is sufficient to show that no element $v = (v_1, v_2)$ of $k_{\|x - y\|}(y) \cap (y + \Theta)$ has its second coordinate greater than e_{12i}.
If $v \in P(X,\Theta) \cap (y + \Theta)$ then

$$x^*_{j2} < v_2 < e_{12i} ,$$

hence we need only to investigate the case when $v \notin P(X,\Theta)$.
If $v \in y + \Theta$ then by the condition (9)

$$(v - \Theta) \cap (y + \Theta) \subset k_{\|y - v\|}(y) \subset k_{\|y - x\|}(y),$$

since $v \in k_{\|y-x\|}(y)$.

By definition of R_{ij},

$$y_2 \in pr_1(S_i),$$

and

$$\forall w = (w_1, w_2) \in S_i : w_2 < e_{12i},$$

therefore if $v_2 > e_{12i}$ then $(v - \Theta) \cap (y + \Theta)$ would have a non-empty intersection with S_i, consequently an element of S_i would belong to the interior of $k_{\|y-x\|}(y)$ which is impossible since we assumed that x is least-distance to y in X. Thus we get a contradiction with the assumption that $x \notin y + \Theta$. Observing that in the situation when $[y,x] \cap h_2 \neq \emptyset$ the proof is a repetition of that above let us end the proof of the case $y \in R_{ij}$.

Same arguments(only one separating half-line is needed) prove Θ-optimality of a least distame point x to a reference point y contained in the cartesian product of $TD(X,\Theta)$ and a projection of a connected component of $P(X,\Theta)$.Since Θ-optimality of x in the case $y \in TD(X,\Theta)$ we quoted as a classical result then all possible situation of y have been investigated and the proof of this Thm. is completed.

q.e.d.

Basing on the above Thm. 4.3 and considerations made in subsection 4.1 we can also formulate a criterion of θ-optimality for the disconnected case.

Observing that :

(i) if the cone θ is non-degenerated then the set $X + \theta$ is connected,

(ii) $A \subset B \Rightarrow SD(B, \theta) \subset SD(A, \theta)$,

(iii) $P(X, \theta) = P(X + \theta, \theta)$, i.e. X and $X + \theta$ have the same connected components of the set of θ-optimal points

let us conclude that the set defined by (17) is equal to $SD(X + \theta, \theta)$, and at the same time it is contained in $SD(X, \theta)$ and the following statement is true.

Corollary 4.1. If X is a closed subset of $\mathbf{R}^2$, y an element of $SD(X + \theta, \theta)$, and the distance and the ordering cone in $\mathbf{R}^2$ satisfy (9) and (18) then a least-distance solution to y in X is θ-optimal.

It is easy to see that if X is not connected and

$y \in SD(X, \theta) \setminus SD(X + \theta, \theta)$

then a least-distance element to y in X may not be θ-optimal even if (8) and (19) are satisfied, as an example may serve the situation presented in Fig. 4.4.

It is interesting to know which norms in $\mathbf{R}^n$ satisfy the condition (18). It turns out that a simultaneous satisfaction of (9) and (18) is equivalent to another, more intuitive property which may be called "symmetric monotonicity".

Proposition 4.2. Suppose that $\|.\|$ is a norm in $\mathbf{R}^n$. Then the following conditions are equivalent :

(i) $\|.\|$ satisfies (9) and (18) ;

(ii) $[\forall\, i \in \{1, \ldots, n\} \setminus \{j\} : y_i = z_i,\ |y_j| < |z_j|$, and $y_j z_j \geq 0] \Rightarrow \|y\| < \|z\|$ for all $y = (y_1, \ldots, y_n)$, $z = (z_1, \ldots, z_n) \in \mathbf{R}^n$ and $j \in \{1, \ldots, n\}$ (19)

Proof : Assume first that $\|.\|$ satisfies both (9) and (18). If y_j and z_j are non-negative then the inequality $|y| < |z|$ is implied by the fact that $\|.\|$ is strictly monotonically increasing which is equivalent to the assumed condition (9). If y_j and z_j are both negative then (19) can be expressed as

$$x_j < y_j \Rightarrow \|y\| < \|z\|.$$

Suppose the contrary, i.e. let $x \in \mathbb{R}^n$ be such that $z_i = x_i$ for $i = 1, \ldots, j - 1, j + 1, \ldots, n$ and $x_j < y_j < 0$ but $\|z\| \leq \|x\|$ and let us consider the closed ball $K_r(0)$, where $r := |z|$. From the theory of normed spaces it follows that $K_r(0)$ intersected with any affine subspace H of $\mathbb{R}^n$ such that $H \cap \theta$ is a convex cone in H, is a ball in H. Moreover, if the maximum of the j-th coordinate over $K_r(0)$ has been achieved at a point $v \in \theta$, then the maximum of j-th coordinate over $K_r(0) \cap H$ is achieved at $w \in H \cap \theta$.

Consider now the two dimensional affine subspace H_{ij} of $\mathbb{R}^n$ spanned by the i-th and j-th elements of the basis of $\mathbb{R}^n$ and passing through x and z. By (18) the maximum of i-th coordinate over the ball $K_r(0) \cap H_{ij}$, $i \in \{1, \ldots, n\}$, $i \neq j$, is achieved at a point v such that $v_j \geq 0$. Of course, $v_i \geq z_i$. since $z \notin \theta$, hence also $v_i \geq x_i$. Therefore x is contained in the triangle $T = [z, v, c_{ij}]$ where $c_{ij} := (x_1, x_{i-1}, 0, x_{i+1}, \ldots, x_{j-1}, 0, x_{j+1}, \ldots, x_n)$ which by the convexity of the norm is contained in $K_r(0) \cap H_{ij}$.

Consequently, x would belong to $K_r(0) \cap H_{ij}$, but we assumed that $|x| \geq r$, i.e. x must belong to the boundary of $K_r(0)$. However, in this case x must also belong to the boundary of T,

namely to the interval $[z, v]$, which is impossible since we assumed that $z_i = x_i < v_i$. This contradiction implies that the assumptions made were sufficient for the inequality $\|x\| < \|z\|$, which ends the first part of the proof.

To prove that (19) implies (9) and (18) it is sufficient to show that the maxima of coordinates over $K_r(0)$ are not achieved on $\mathbb{R}^n \setminus \theta$. The equivalence of (9) with a subcase of (19) has already been noted in the first part of the proof. Let us take $k \in \{1, \ldots, n\}$, two elements of $\mathbb{R}^n$, x and y such that $x_i = y_i$ for $i \in \{1, \ldots, j-1, j+1, \ldots, n\}$, $j \neq k$ and $x_j < y_j \leq 0$. By (19) $\|x\| > \|y\|$, consequently $y \in k_{\|x\|}(0)$, and in certain neighborhood of y there are points of $k_{\|x\|}(0)$ with the k-th coordinate greater than $y_i = x_i$. Since this schema is true for all $x \in \mathbb{R}^n \setminus \theta$ then it follows that the maximum of k-th coordinate cannot be achieved at such point.

q.e.d.

Let us remark that neither (9) implies (18) nor vice versa, an example of the distance satisfying (9) but not (18) is shown in Fig. 4.4, while the balls in a norm which satisfies (18) but fails to fulfill (9) are shown in Fig. 3.2.a and 3.2.b. Such norm can be defined e.g. by the formula :

$$n(x) := (0,5(x_1 + x_2)^2 + 2(x_1 - x_2)^2)^{1/2}.$$

Now it is easy to see that (19) is satisfied by distances generated by L_p norms, $1 \leq p < \infty$, i.e. functions of the form

$$L(x, p, w) := (\sum_{i=1}^{n} w_i |x_i|^p)^{1/p}, \tag{20}$$

where $w_i > 0$ for $1 \leq i \leq n$, x_i are the coordinates in $\mathbb{R}^n$ related to the basis spanning θ, and p is defined as above.

Therefore, we may formulate the following

<u>Corollary 4.2</u>. If the distance in $\mathbb{R}^2$ is generated by the norm which can be expressed by the formula (20) in the coordinates related to the cone θ, then a least-distance element in X to a reference point $y \in SD(X + \theta, \theta)$ is θ-optimal.

5. Final remarks.

Throughout this paper we have been emphasizing that the presented conditions for Θ-minimality are sufficient but not necessary. However, as it might be observed in conditions involving the use of strictly dominating points, the set of elements of the criteria space which may serve as the reference points in distance scalarization is strongly influenced by the shape of X which may not be assumed a priori known.

Therefore the results here presented may be classified as an attempt to approximate from below the set of potential reference points, assuming that the norm or a class of norms, and the partial order are fixed. Of course, the assumed condition (9) may also be, in some cases, relaxed.

There are still some open questions, such as a more constructive description of the set of potential reference points in $\mathbb{R}^k$, $k>2$, or the problem of removing the additional constraints occuring in Thm. 3.2 in the case where the dimension of criteria space is greater than 2.

Some questions such as proper or weak efficiency of the solutions obtained or the completness of characterization of $P(X,\Theta)$ by distance functions associated to different reference points were not studied for the brevity's sake.

A special class of scalarizing functions based on reference points which have been considered by Wierzbicki (1986) requires a separate treatment since they constitute an entirely different approach to scalarization problems. Those functions are defined in such a way that Θ-optimality of minima is implied by the form of the functions. In our case a distance function or a family of them is imposed as a value function by the decision situation concerned, and the only thing which remains is to verify whether the scalar minimization problem arisen has a θ-optimal solution, and to execute a distance-minimization procedure then.

However, such situation occur frequently while solving real-life problems, since the values of the distance functions often have an easy and intuitive interpretation to the decision-maker. Therefore, it is hoped that the results here obtained may be helpful in design of decision support systems based on reference points.

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