



Differential approximation of MaxSAT and related problems

Bruno Escoffier, Vangelis Th. Paschos

Lamsade, Université Paris Dauphine

Workshop Computer Science and Decision Theory



Outline



- Introduction
 - $MaxSAT$ and related problems
 - Polynomial approximation
 - $MaxSAT$ and standard approximation
 - $MaxSAT$ and differential approximation
- Differential approximation of $MaxSAT$
- Conclusion



MaxSAT and related problems



- A set of n boolean variables (x_1, \dots, x_n)
- A set of m clauses C_1, \dots, C_m :
 $C_j = (l_{i_1} \vee \dots \vee l_{i_{|C_j|}})$, where $l_{i_k} \in \{x_{i_k}, \overline{x_{i_k}}\}$
- *SAT* : decide whether the formula is satisfiable or not.
- *MaxSAT* : find a truth assignment which maximizes the number of satisfied clauses.



MaxSAT and related problems

Variations :

- *MaxkSAT* : $|C_j| \leq k$
- *MaxEkSAT* : $|C_j| = k$.
- *MinSAT* : minimize the number of satisfied clauses
- *MinkSAT*, *MinEkSAT*, ...

Polynomial approximation

Find an algorithm which computes “good” solutions.

$(x_1 \vee x_2 \vee \overline{x_3}), (\overline{x_1} \vee x_2 \vee x_3), (x_1 \vee \overline{x_2} \vee x_3), (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$

Truth assignment $S = (1, 1, 1)$: satisfies 3 clauses.

Polynomial approximation

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Truth assignment $S = (1, 1, 1)$: satisfies 3 clauses.

$m(\phi, S)$ vs. $opt(\phi)$:

$$m(\phi, S) \geq \frac{3}{4}opt(\phi)$$

→ good solution

Polynomial approximation

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$(x_1 \vee x_2 \vee \overline{x_3}), (\overline{x_1} \vee x_2 \vee x_3), (x_1 \vee \overline{x_2} \vee x_3), (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$

Truth assignment $S = (1, 1, 1)$: satisfies 3 clauses.

$m(\phi, S)$ vs. $opt(\phi)$:
 $m(\phi, S) \geq \frac{3}{4}opt(\phi)$
→ good solution

4 solutions satisfy 3 clauses,
4 solutions satisfy 4 clauses.
→ S is the worst solution !

$m(\phi, S) - \omega(\phi)$ vs. $opt(\phi) - \omega(\phi)$

Polynomial approximation

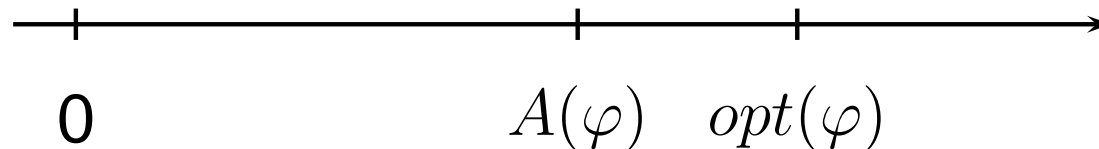
A ptaa (polynomial time approximation algorithm) A :

- produces a feasible solution $A(\varphi)$ on any instance φ .

Polynomial approximation

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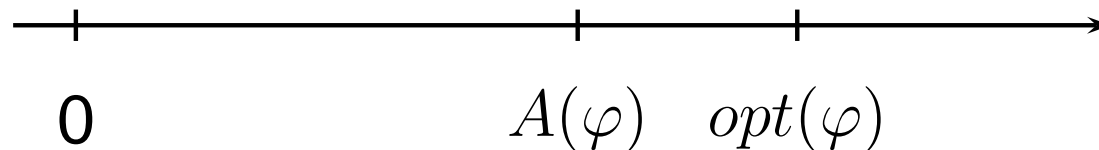
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- standard ratio ρ if $\forall \varphi, m(\varphi, A(\varphi)) \geq \rho \text{opt}(\varphi)$



Polynomial approximation

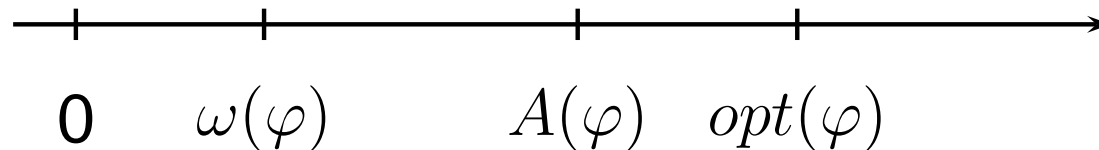
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- produces a feasible solution $A(\varphi)$ on any instance φ .
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- differential ratio δ if

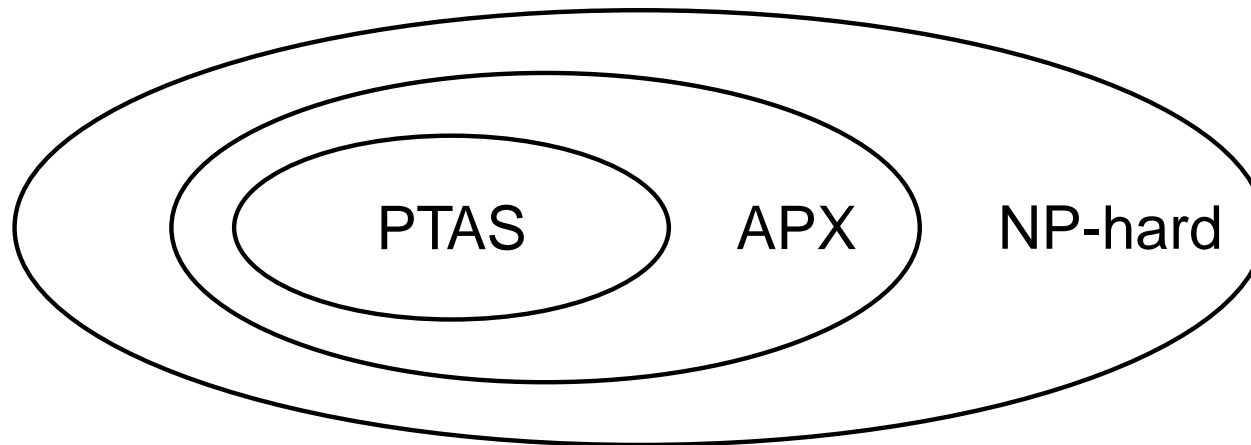
$$|m(\varphi, A(\varphi)) - \omega(\varphi)| \geq \delta |\text{opt}(\varphi) - \omega(\varphi)|$$



Polynomial approximation

Aim : classify the problems according to their approximability properties.

- Is there a constant c such that Π is c -approximable ?
→ Π is in APX
- Is this true for any constant $c < 1$?
→ Π is in PTAS.



MaxSAT and standard appx.

For all these problems, there exists a ptaa which guarantees approximation ratio c (c is a constant which depends on the problem).

problem	lower bound
<i>MaxSAT</i>	0.77 [Asano et al.,79]
<i>MaxE2SAT</i>	0.93 [Feige et al.,95]
<i>MaxE3SAT</i>	7/8 [Johnson,74]
<i>MinSAT</i>	2 [Bertsimas et al.,96]
<i>MinE2SAT</i>	1.11 [Avidor et al.,2002]
...	

MaxSAT and standard appx.

Inapproximability : if $P \neq NP$, then Π is not ρ -approximable

problem	lower bound	upper bound
<i>MaxSAT</i>	0.77 [Asano et al.,79]	7/8 [Hastad,97]
<i>MaxE2SAT</i>	0.93 [Feige et al.,95]	21/22 [Hastad,97]
<i>MaxE3SAT</i>	7/8 [Johnson,74]	7/8 [Hastad,97]
<i>MinSAT</i>	2 [Bertsimas et al.,96]	1.37 [Marathe et al.,96]
<i>MinE2SAT</i>	1.11 [Avidor et al.,2002]	15/14 [Avidor et al.,2002]
...		

MaxSAT and differential appx.



- Some negative results from the standard approximation theory
- One strong negative result [Bazgan Paschos, 2002] : *MinSAT* is not $m^{1-\varepsilon}$ differential approximable.

→ A lot of open questions :

- Is *MaxSAT* constant-approximable ?
- Is this true for *MaxkSAT* ?
- Could we find any positive result ?



Plan



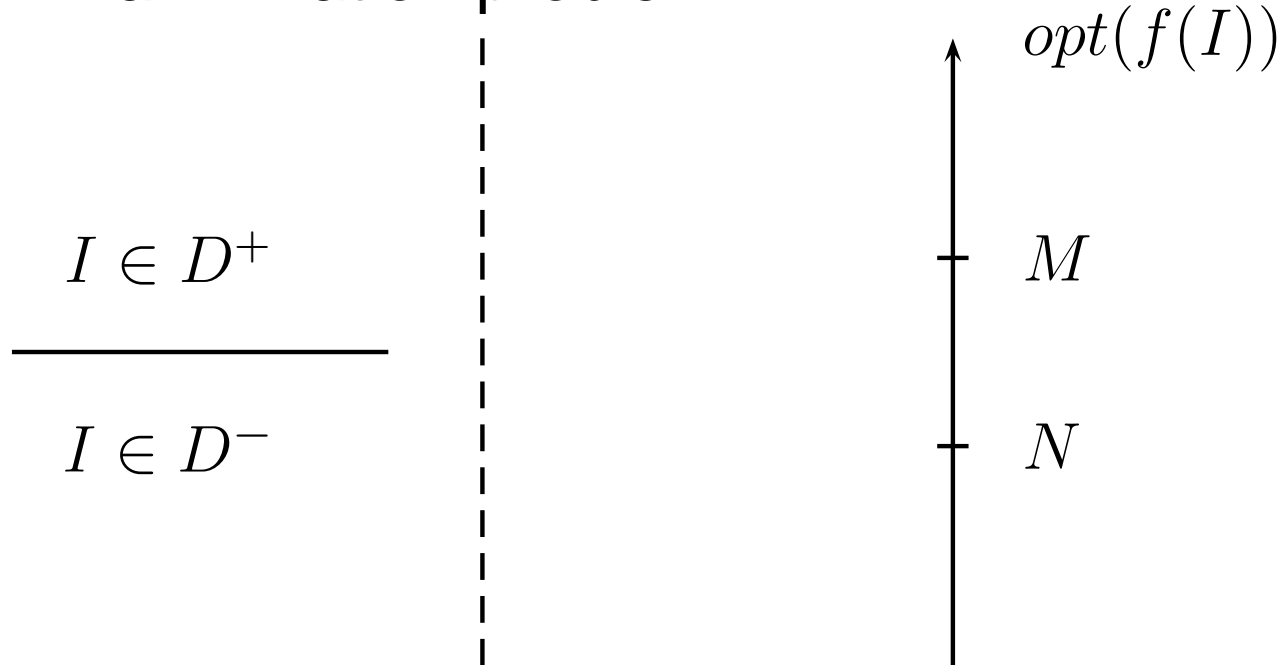
- Introduction
- Differential approximation of $MaxSAT$
 - $MaxSAT$ and GAP-reductions
 - Other inapproximability results
 - $MaxSAT$ and 0-DAPX
- Conclusion



MaxSAT and GAP-reductions

$D = (D^+, D^-)$: NP-complete decision problem.

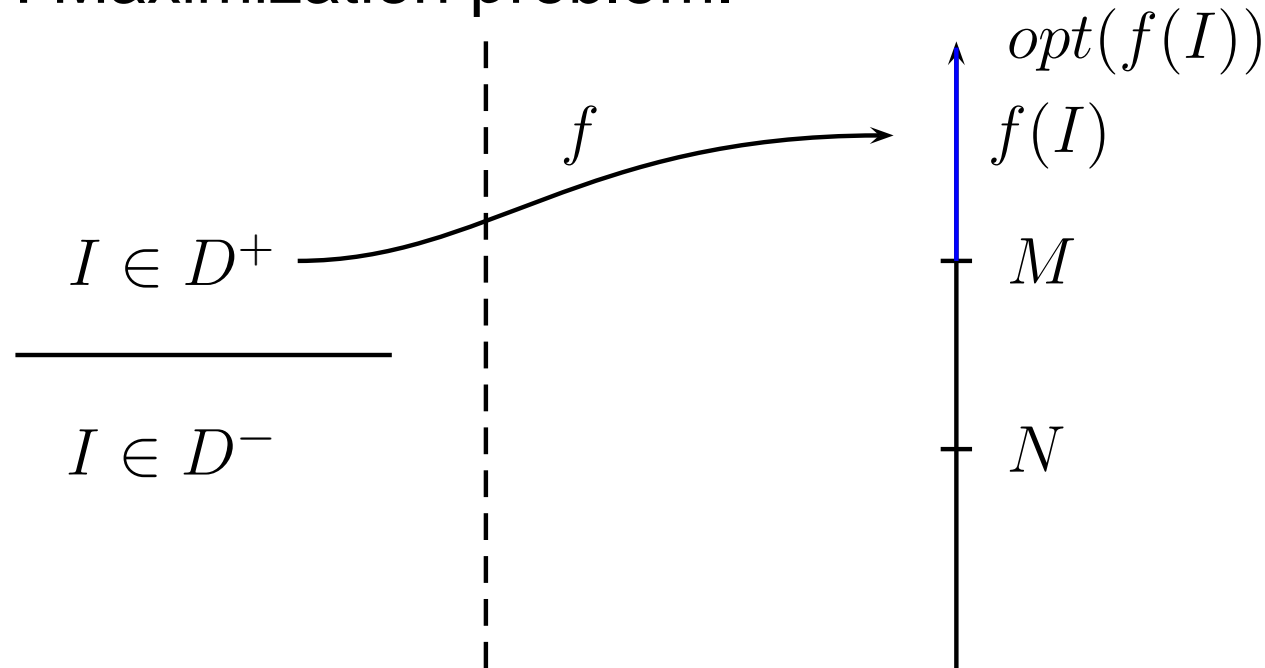
Π : Maximization problem.



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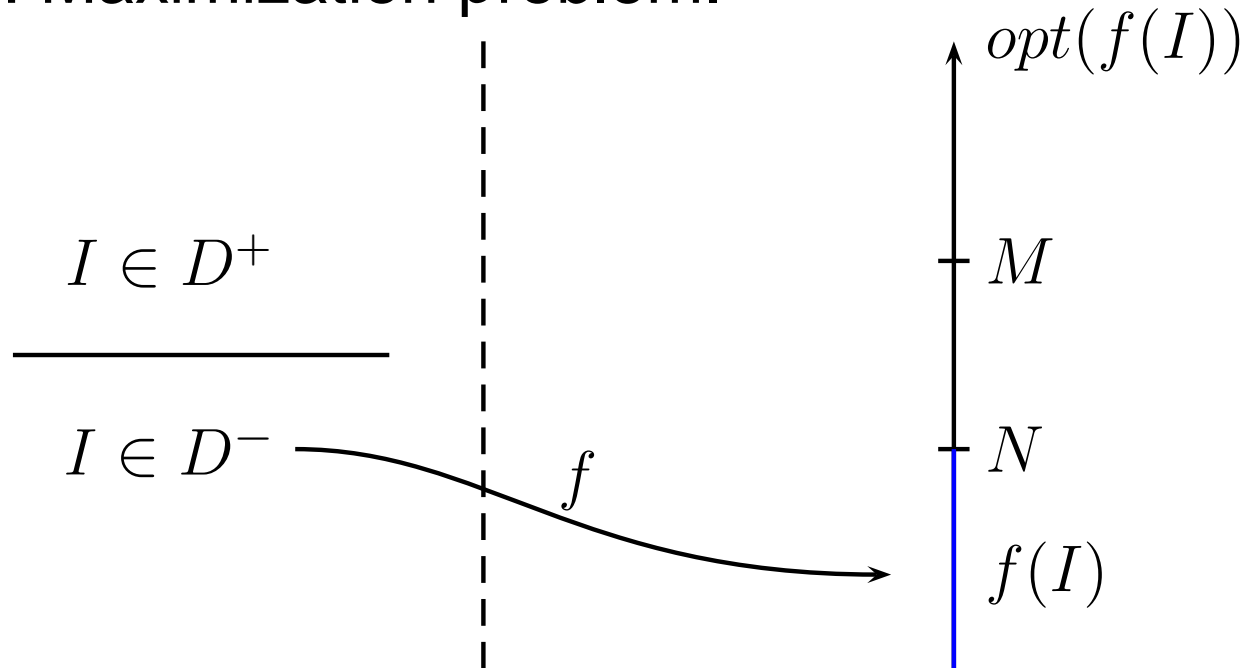
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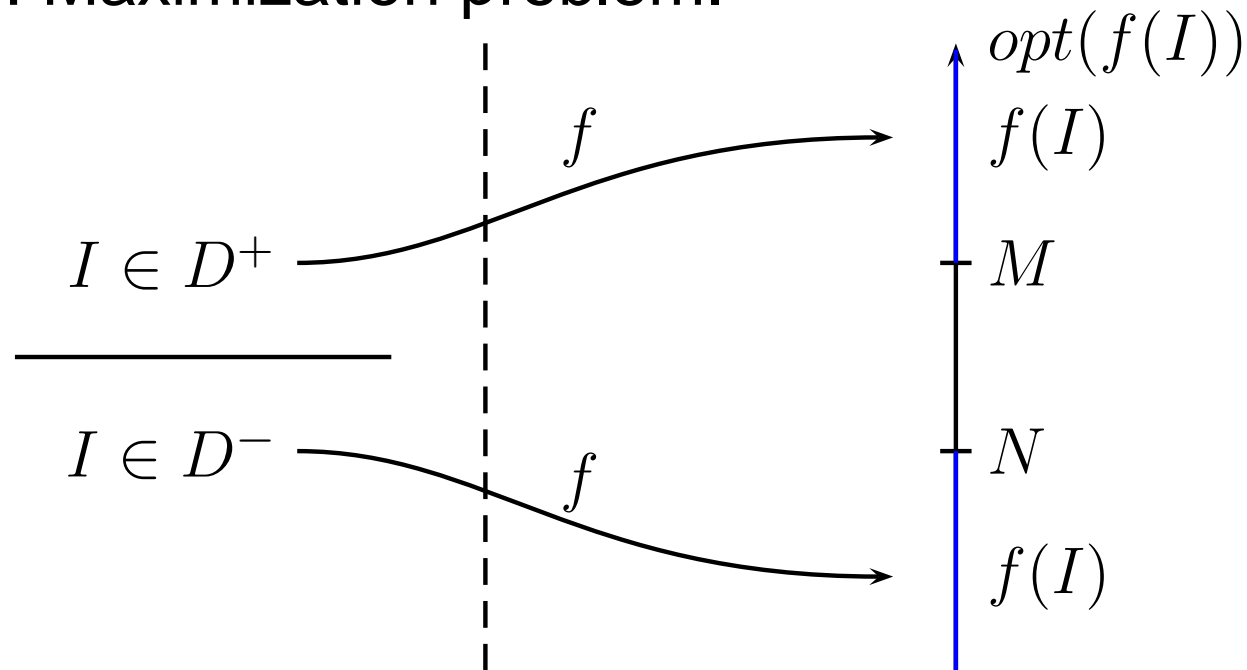
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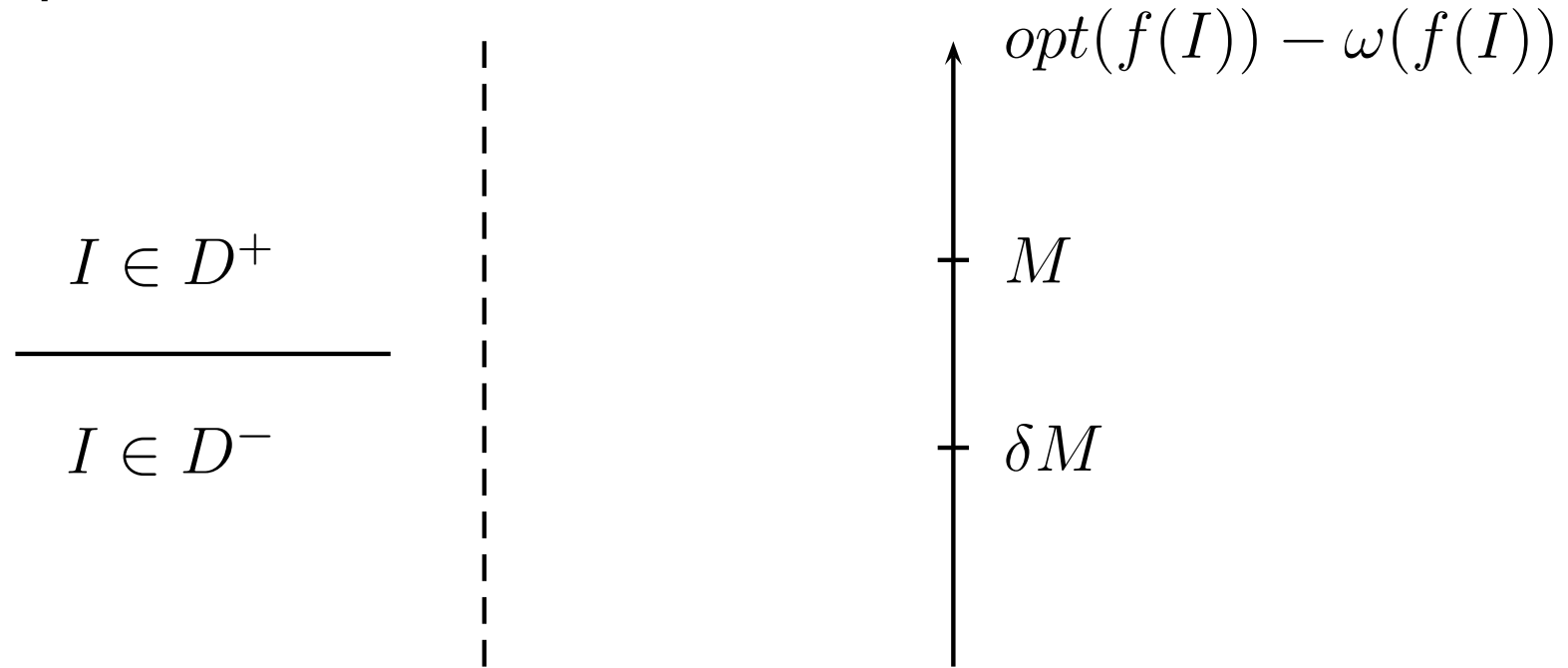
Π : Maximization problem.



Π is not standard N/M approximable.

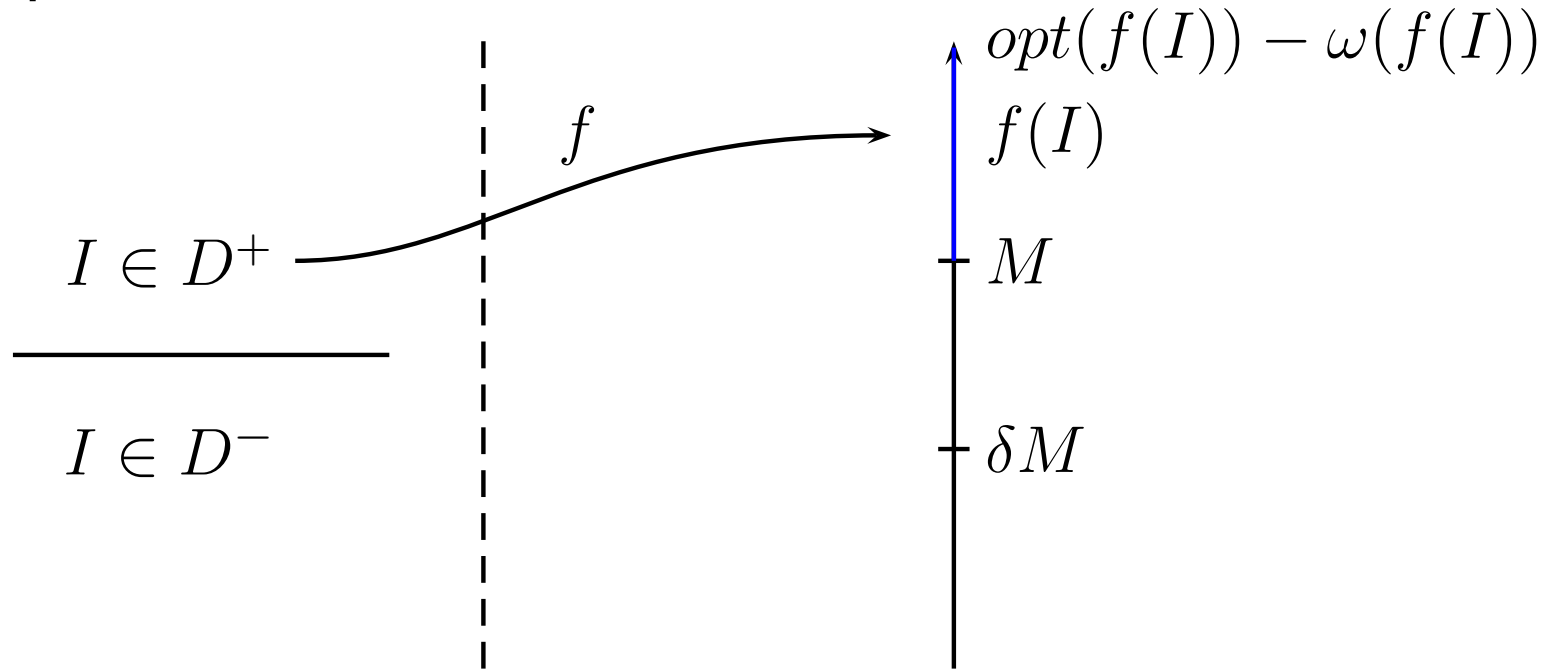
MaxSAT and GAP-reductions

Gap reduction to *MaxE3SAT*. For all $\delta > 0$:



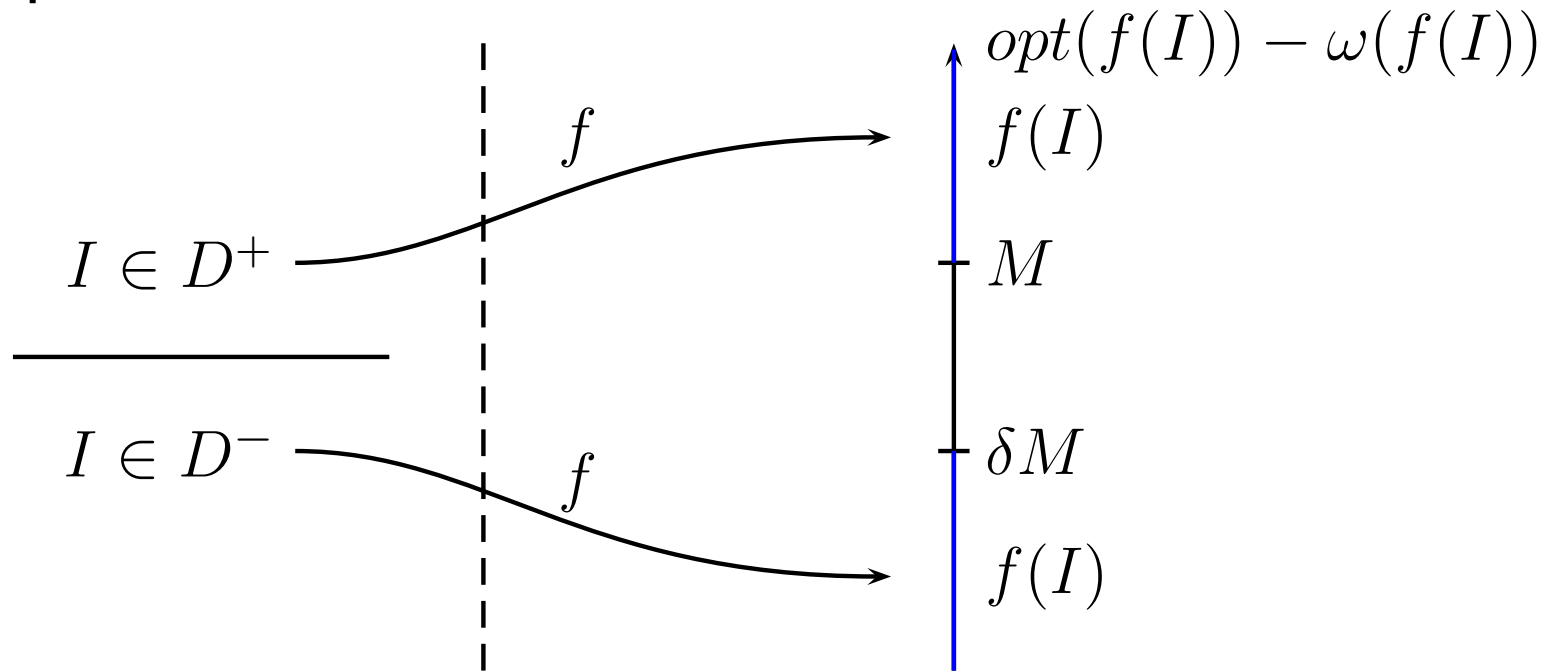
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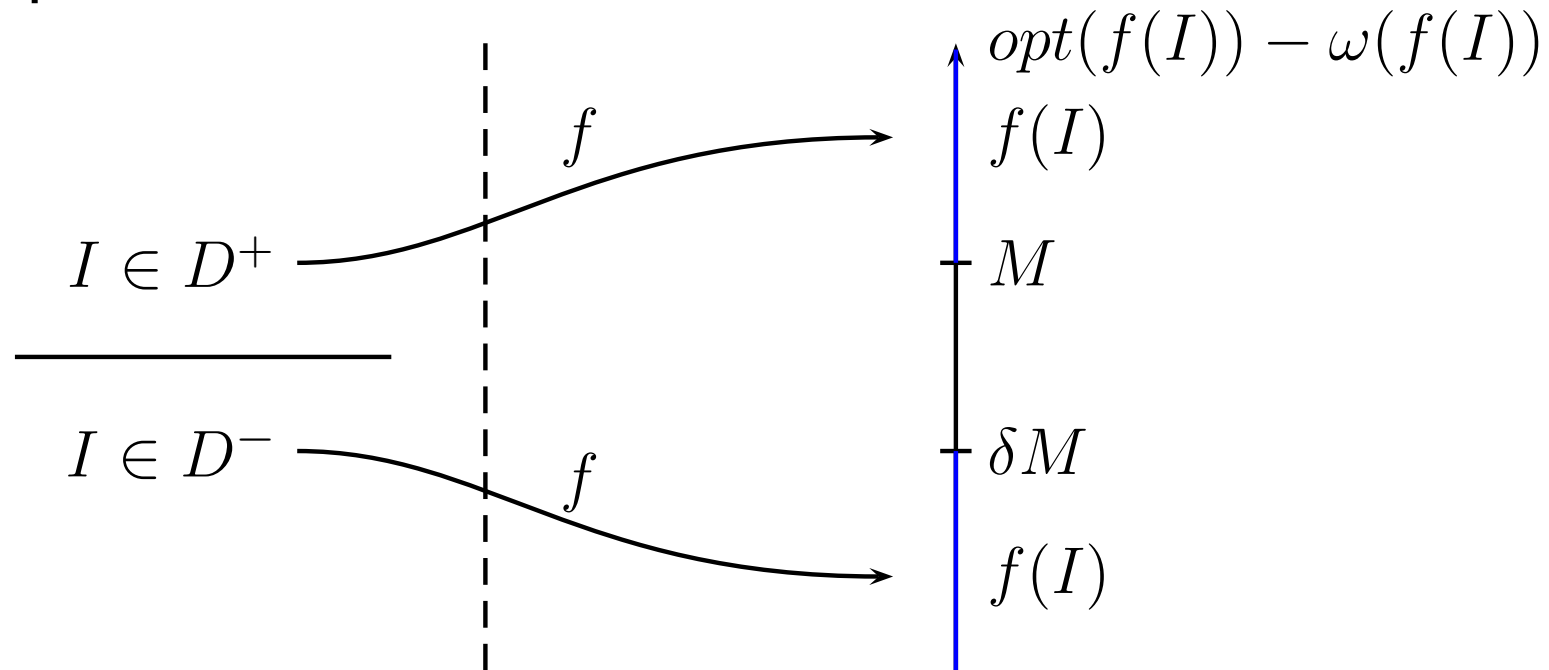
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MaxSAT and GAP-reductions

Gap reduction to *MaxE3SAT*. For all $\delta > 0$:



We cannot provide a constant approximation ratio of the value of $opt - \omega$.

→ Is *MaxE3SAT* not differential approximable within constant ratio?

MaxSAT and GAP-reductions

No, because we can't compute $\omega(f(I))$!

MaxE3SAT is not $1/2 + \varepsilon$ differential approximable, for all $\varepsilon > 0$.

MaxSAT and GAP-reductions

No, because we can't compute $\omega(f(I))$!

MaxE3SAT is not $1/2 + \varepsilon$ differential approximable, for all $\varepsilon > 0$.

→ the same bound holds for *MaxEkSAT*, $k \geq 3$. Could we strengthen this result?

General reduction to *MaxEkSAT*.

*For all k , *MaxEkSAT* is not $1/c(k)$ differential approximable, where $c(k) \rightarrow \infty$ for $k \rightarrow \infty$.*

Other inapproximability results

→ the same bound holds for $MaxkSAT$ and $Min(E)kSAT$.

→ $MaxSAT$ is not constant approximable!

→ $MaxNP \not\subseteq DAPX$

MaxSAT and the class **0-DAPX**

0 – *DAPX* : problems Π for which there is no ptaa A such that $\forall \phi, \delta(\phi, A(\phi)) > 0$, unless $P = NP$.

→ does *MaxSAT* \in 0 – *DAPX*, or can we devise an algorithm which never gives the worst solution ?

MaxSAT and the class **0-DAPX**

Randomized solution $R_s : x_i = TRUE$ with probability $1/2$.

$$w(\phi) < E(m(R_s, \phi)) < opt(\phi)$$

Derandomization : get in polynomial time a solution S_0
s.t. $m(S_0, \phi) \geq E(m(R_s, \phi))$

MaxSAT and the class **0-DAPX**

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$$m(S_0, \phi) \geq E(m(R_s, \phi)) > w(\phi)$$

$\rightarrow MaxSAT \notin 0 - DAPX$

The same holds for *MinSAT*, *MaxkSAT*, *Min(E)kSAT*.

Summary of results

	Approximation	Inapproximability
<i>MaxSAT</i>	$4.34/(m + 4.34)$	\notin DAPX
<i>MaxE2SAT</i>	$17.9/(m + 19.3)$	11/12
<i>Max3SAT</i>	$4.57/(m + 5.73)$	1/2
<i>MaxE3SAT</i>	$8/(m + 8)$	1/2
<i>MaxEkSAT</i>	$2^k/(m + 2^k)$	$1/c(k), c(k) \xrightarrow{k \rightarrow \infty} \infty$
<i>MinSAT</i>	$2/(m + 2)$	
<i>Min(E)kSAT</i>	$2^k/((2^{k-1} - 1)m + 2^k)$	$1/c(k), c(k) \xrightarrow{k \rightarrow \infty} \infty$
<i>Min2SAT</i>	$4/(m + 4)$	11/12

Conclusion



- Bridging the gaps :
 - $\exists k : \text{Max}k\text{SAT} \in \text{DAPX} ?$
 - $\forall k : \text{max}k\text{SAT} \in \text{DAPX} ?$
 - MaxSAT approximable within $\log(m) ? m^c ?$
- Links between GAP-reductions and differential approximation.



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