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Abstract:
This article deals with preference modeling. It concerns the concepts of discriminating thresholds as a tool to cope with the imperfect nature of knowledge in decision aiding. Such imperfect knowledge is related with the definition of each criterion as well as with the data we have to take into account. On the one hand, we shall present a useful theoretical synthesis for the analyst in his/her decision aiding activity, and, on the other hand, we shall provide some practical instructions concerning the approach to follow for assigning the values to these discriminating thresholds.

Keywords: Decision aiding process, Preference modeling, Imperfect Knowledge, Pseudo-criteria, Multi-criteria Analysis / Decision Support.

Les seuils de discrimination en tant qu'outils pour appréhender la connaissance imparfaite en aide multicritère à la décision : résultats théoriques et aspects pratiques

Résumé :
Cet article traite du concept de seuils de discrimination en tant qu'outils permettant de prendre en compte en aide multicritère à la décision le caractère imparfait (incertain, ambigu, mal déterminé) des connaissances. Ce caractère imparfait des connaissances affecte aussi bien la définition des critères que les données qu'ils doivent prendre en compte. On présente d'une part une synthèse des résultats théoriques utiles à l'analyste dans son travail de modélisation et d'autre part des indications pratiques concernant la démarche à suivre pour attribuer des valeurs à ces seuils.

Mots-clés: Processus d'Aide à la Décision, Modélisation des Préférences, Connaissance Imparfaite, Pseudo-Critères, Analyse Multicritères, Systèmes d'Aide à la Décision.

Contents

1. Introduction

Research on ordered structures requiring the definition of one or several thresholds gave birth to a wide range of theoretical works, as for instance, Krantz (1967); Luce (1973); Cozzens and Roberts (1982); Suppes et al. (1989); Vincke (1988); Abbas and Vincke (1993); Pirlot and Vincke (1997); Tsoukiàs and Vincke (2003); Ngo The and Tsoukiàs (2005).

The ordered structures with one or two thresholds are of a particular interest in decision aiding for modeling the imperfect knowledge (Jacquet-Lagrèze, 1975; Roy and Vincke, 1984, 1987; Bouyssou, 1989; Roy, 1989; Smets, 1991; Dubois and Guyonnet, 2011).

Preference modeling in decision aiding needs to take adequately into account the imperfect knowledge, especially in the case of multiple criteria methods (see, for instance, and concerning only Omega Journal Beynon and Wells, 2008; Bollinger and Pictet, 2008; Levanon and Passy, 1980; Wiecek et al., 2008). Indeed, the definition of each criterion frequently comprises some part of arbitrariness, and the data used to built criteria are also very often imprecise, ill-determined, and uncertain. This is why, for instance:

i) In the definition of a net present value, the elements to be taken into account (the amortization period and the discount rate) lead to make some choices, which comprise a part of *arbitrariness.*

ii) A criterion may be built from data obtained after a survey (through the application of questionnaires), which comprises inevitably an *imprecision* margin.

iii) As soon as certain data (parameters), to take into account in a given criterion, are represented by the values these parameters will possess in a more or less distant future, we are in presence of an *uncertainty*, which may be important.

iv) Certain types of consequences or outcomes that must be taken into account by a given criterion are difficult to define. They are *ill-determined.* This is the particular case of the market share conquered by a company, the quality of a product, the degree of inconvenience of a population due to a noise nuisance. Provide precise definitions for these concepts is a very hard and frequently impossible task.

There are several decision aiding models and method that make use of the concept of thresholds for modeling this imperfect knowledge; they may use one or two thresholds, called *discriminating thresholds* (Roy, 1985; Bouyssou and Roy, 1987; Maystre et al., 1994; Vincke, 1992; Roy, 1996; Rogers et al., 2000).

After bringing to light, in Section 2, the interest and the role of the concept of discriminating thresholds in decision aiding, we shall define formally, in Section 3, the concept of pseudo-criterion by pointing out the existence of a double definition of the thresholds (direct and inverse) and by giving the relation between both (see Theorem 1). Then, we shall present, in Section 4, a synthesis of the main theoretical results in decision aiding. We shall devote an extended section, Section 5, to the way the analyst should proceed in practice to assign adequate values to these thresholds and this for a variety of possible contexts.

Our main concern in this article is to call the attention of the reader to the pitfalls that can come from the difference between direct and inverse thresholds with respect to a criterion to be maximized or a criterion to be minimized, or from the discrete or continuous nature of the scale, especially within the framework of the Electre methods (Figueira and Roy, 2009; Figueira et al., 2013; Wang and Triantaphyllou, 2008).

2. Discriminating thresholds in decision aiding: For what purpose?

In this section we present some preliminary concepts and illustrate the purpose of making use of discriminating thresholds in decision aiding through four pedagogical examples.

2.1. Preliminary concepts

In what follows A denotes a set of *potential actions.* Each action, $a \in A$, can be defined by a brief descriptive phrase or term, say a label, corresponding to an extensive description. In such a case, A can be defined as follows, $A = \{a_1, a_2, \ldots, a_i, \ldots\}$. This set can be completely known *a priori* or it may also appear progressively during the decision aiding process. The actions a can also be elements of $\mathbb{R}^m$; they may represent solutions of a feasible set defined through mathematical constraints. In such a case, A is a set containing elements a of $\mathbb{R}^m$. Let g denote a given *criterion*, built for characterizing and comparing potential actions according to a considered point of view. This characterization of an action $a \in A$, denoted by $g(a)$, usually represents the *performance* of action a according to the point of view considered.

Let E_g denote the set of all possible performances, which can be assigned to actions $a \in A$ according to criterion g. Each element of E_g can be characterized by a pictorial object, a verbal statement, or more generally by a number. As for defining a preference model, E_g must be a completely ordered set: $>_g$ will be used to denote this order. When $>_g$ corresponds to the direction in which preferences increase, we say that g is a criterion to be *maximized*; in the opposite case, g is to be *minimized.* The completely ordered set E_g is called the *scale* associated with criterion g. The elements of the scale E_g are called *scale levels* or simply *levels.* The scale can be defined either by a sequence of *ordered levels* (discrete scales, see Examples 1 and 2, below) or by an interval of real numbers $[e_*, e^*]$ (continuous scales, see Examples 3 and 4, below). In practice, the scale is never really continuous since only certain rational numbers of the above interval are used to define a performance. The levels of a continuous scale are necessarily characterized by numerical values, while the different levels of a discrete scale can also be characterized by verbal statements. In such a case and since E_g is a completely ordered set, each level can again be characterized by a numerical value: its position or rank in the scale. In such conditions, $e_* = 1$ is the lowest level, while $e^* = |E_g| = n$ represents the highest level on the scale E_g.

Defining a criterion g is to build and to choose an *operational instruction* able to associate with any action $a \in A$ a performance $g(a) = e \in E_g$ judged appropriate to compare any ordered pair of actions from the point of view of the considered criterion. This operational instruction can be, depending on the circumstances or cases, based on expert judgements, questionnaires, forecasting techniques, several measurement tools, mathematical expressions, or even more complex algorithms using multiple data. If this operational instruction is, in its very nature, enough devoid of ambiguity, subjectivity, and arbitrariness and if the data that it makes use of are enough reliable, then the criterion g, thus built, is a preference model, which can be considered legitimate to lead to the following conclusions:

i) the *indifference* between two actions a and a' (aI_ga') is established if and only if $g(a) = g(a')$;

ii) the *preference* in favor of a over a' (aP_ga') is established without ambiguity if and only if $g(a) > g(a')$ when the criterion is to be maximized and $g(a) < g(a')$ when the criterion is to be minimized (this is valid even for a very small performance difference separating $g(a)$ from $g(a')$).

The above preference model, defined by *i)* and *ii)*, is called the *true-criterion* model. Very often, this model is not realistic. This missing of realism may come, as it will be explained through the four examples in next subsection, from different reasons: the operational instructions can incorporate some part of ambiguity, subjectivity, and arbitrariness. It can be supported by poor or fragile working hypotheses due to an imperfect knowledge of what we want to evaluate. These operational instructions can also make use of data obtained from imprecise measures or based on less rigorous definitions, or even data obtained from the application of less reliable procedures.

2.2. Some examples

In this subsection, four examples are presented aiming to illustrate the different concepts needed in the rest of the paper.

Example 1: Implementation time in number of months.

$$E_g = \{6, 7, \ldots, 35, 36\} \qquad (g \text{ is a criterion to be minimized})$$

Here, an action a is an investment project: the time we are interested in is the one that was estimated for being necessary to implement a project (viewed as a set of tasks). This estimation can neither be made with a precision of one month nor it can even probably be made with a precision of two months. This leads to suppose that:

i) if two actions a and a' are such that $|g(a) - g(a')| = 1$, then this performance difference is not significant;

ii) to be able to conclude that the implementation time of a is significantly shorter than the implementation time of a', it is necessary to consider $g(a) < g(a') + 2$.

In such conditions criterion g is a preference model that seems legitimate to support the following conclusions:

i) the indifference aI_ga' is established if and only if $|g(a) - g(a')| \leqslant 1$;

ii) the preference aP_ga' is established without ambiguity if and only if $g(a) < g(a')$ and $|g(a) - g(a')| > 2$.

These conclusions are different from those provided by a true-criterion model. Moreover, they should be completed: what should we conclude in the case where $g(a) = g(a') - 2$? This performance difference is clearly incompatible with $a'P_ga$. Nevertheless, this difference is considered very weak to lead us to unquestionably suppose that the implementation time of a is significantly lower than the implementation time of a'. In other words, we are in presence of an ambiguity situation corresponding to a hesitation between the two conclusions, aI_ja' and aP_ga'. If there is a preference it should be in favor of a over a', but such a preference is very weakly established to exclude by itself the possibility of an indifference between the two actions. This situation corresponds to what is called in decision aiding *weak preference* (*i.e.*, a weakly established preference) and denoted by aQ_ga'.

Example 2: Fitness with respect to an objective.

$$E_g = \{\text{opposing, neutral, possibly favorable but questionable, unquestionable but weak, significant but partial, complete}\} \quad (g \text{ is a criterion to be maximized})$$

Criterion g should take into account the way different projects $a \in A$ make their contribution to an objective we assume well defined. An expert is in charge of evaluating each project on the six levels above E_g scale.

We suppose that the instructions provided to the expert:

i) will not allow him/her to give extreme evaluations (opposing and complete), unless he/she is strongly convinced that these extreme evaluations are justified;

ii) could lead him/her, quite frequently, to hesitate between two consecutive levels, among the four intermediate levels, given the way these levels are characterized from the example provided to him/her.

It justifies, according to such a kind of criteria, that two projects a and a' could be seen as indifferent when they have, for example, the following performances: $g(a)$ = "significant but partial" and $g(a')$ = "unquestionable but weak". A project a is strictly preferred to a project a'' only when $g(a'')$ is at most evaluated "possibly favorable but questionable".

Again the true-criterion model is not adequate to model such a situation.

Example 3: Costs in K€ for renovating a set of buildings.

$$E_g = [6\,000, 50\,000] \quad (g \text{ is a criterion to be minimized})$$

It is well-known that the forecasts, which can be established to evaluate such costs are most often supported by poor data (the precise nature of jobs, the volume of workload, the costs of materials, ...).

First, let us consider the less ambitious renovation projects with cost relatively modest. The lack of accurate data can lead to consider a project a' with cost $g(a') = 16\,000$ as not significantly more expensive than a project a with cost $g(a) = 15\,000$. But, a project a' with cost strictly greater than 17 000 should be considered significantly more expensive than a project a with cost $g(a) = 15\,000$.

Now, let us consider the projects with high cost, for example, a project b where $g(b) = 40\,000$. It could lead us to assume that a project b' with $g(b') = 42\,000$ is not significantly more expensive than project b. But, a project b' with cost strictly greater than 45 000 should be considered as significantly more expensive than project b.

Example 4: The expected market share.

$$E_g = [0, 100] \quad (g \text{ is a criterion to be maximized})$$

The action a is a new product that will be possibly launched in the market. A marketing survey allowed us to assess the market share that can be conquered one year after launching this new product. We assume that the forecasting analysts are able to provide an expected value $g(a)$ within a range bounded by a pessimistic value $g^-(a)$ and an optimistic value $g^+(a)$.

The range $[g^-(a), g^+(a)]$ defines an indetermination margin, which is desirable to take into account in the preference model that must constitute the criterion g. The comparisons of the market shares of two products, a and a', must not be only based on the comparison of the percentages $g(a)$ and $g(a')$, but also on the indetermination margin $[g^-(a), g^+(a)]$.

The four examples show, in a clear way, the need of taking into account the impact imperfect knowledge may have on the manner of defining a criterion for modeling the preferences of a decision-maker, according to a certain point of view. In the next subsection, we shall show how this impact can be taken into account through the concepts of discriminating thresholds. For such a purpose the true-criterion model should be replaced by a more rich and realistic one: the pseudo-criterion model. Despite we are not dealing with a new model (see the references cited in the Introduction), it seemed to us necessary to reformulate its definition (see Section 3) in a more rigorous way, by making more clear certain distinctions, which have led in practice to some sources of confusion and mistakes. Then, we gathered and made a synthesis of the results on this pseudo-criterion model; such results were scattered in the scientific French and English literature of this domain over a large number of works (see Section 4), some of a no easy access. Finally, we shall examine (see Section 5) different ways of assigning adequate values to the discriminating thresholds for applying the pseudo-criterion model in decision aiding, with some realism. This lead us again to return to the four examples of this section.

2.3. Discriminating thresholds

The purpose of considering discriminating thresholds is to tackle, in a realistic way, the preference situations established on unquestionable bases (strict preference) and situations which can be considered compatible with an indifference.

Assumption 1 (Basic Assumption). *We assume that the sources of arbitrariness, imprecision, uncertainty, and/or ill-determination, the discriminating thresholds have been designed to take into account and modeling, affect, in the same way, the operational instruction and the nature of the data, whatever the action a under consideration. In other words, two actions a and a' susceptible to be characterized with the same performance, will be affected in the same way by such sources of imperfect knowledge. Consequently, the discriminating thresholds defined hereafter do not depend on the considered actions, but only on the performances that would be assigned to them.*

Definition 1 (Preference threshold). *The preference threshold, p, between two performances, is the smallest performance difference that when exceeded is judged significant of a strict preference in favor of the action with the best performance. This difference (which is by definition non-negative) can be equal to zero (which corresponds to the case of the true-criterion model).*

Definition 2 (Indifference threshold). *The indifference threshold, q, between two performances, is the largest performance difference that is judged compatible with an indifference situation between two actions with different performances. This difference (which is by definition non-negative) can be equal to zero and it is at most equal to the preference threshold.*

As for the use of continuous scales or more generally when all the levels are defined by numbers, the difference that allows to define the above thresholds is the smallest or the largest difference between two performances, say $g(a)$ and $g(a')$. Thus, in Example 1, it is clear that $p = 2$ and $q = 1$. Example 3 shows in a clear way the fact that the performance difference can depend on the place occupied in the scale by the performances $g(a)$ and $g(a')$. In this case we are in presence of *variable thresholds*. Precise definitions as well as some results on variable thresholds are provided in the next section.

When the levels of an original scale are not defined by numbers (but, for instance, by verbal statements) the question on how to define the thresholds is always present; it is important to know and modeling in which terms the differences of performances must be taken into account to define the thresholds. In such a case, the scale is necessarily defined through an ordered list of levels as in Example 2. The order allows to associate a position (rank) with each level. The difference can thus be measured by the rank difference. Also in this case the thresholds can vary along the range of the scale. In the Example 2 (for a rigorous justification, see subsection 5.2.1), in rank 1 (the lowest one), the two thresholds p and q are equal to zero. In the next ranks, 2, 3, and 4, p and q are both equal to 1. In rank 5, again p and q are equal to zero. Let us observe that the thresholds defined in this way take into account the performance differences with an increasing preference direction. In other words, the thresholds are defined from the worst of the two performances. The thresholds thus defined are called *direct thresholds*. The reverse definition leads to talk about *inverse thresholds*. This distinction between direct and inverse thresholds has no object when in presence of constant thresholds. When thresholds vary along the range of the scale, this distinction has very important effects, as it will be shown in the next subsections; it can lead to several pitfalls.

When the indifference and preference thresholds are not equal, there is “room” for performance differences that are at the same time strictly greater than the indifference thresholds and at most equal to the preference thresholds. Such differences are not large enough to characterize a strict preference in favor of one of the two actions, but nevertheless they are very large to be compatible with the indifference situation. They reflect a very realistic situation (see Examples 1 and 3) that corresponds to a hesitation between indifference and strict preference in favor of the action with the best performance, and excluding a strict preference in favor of the action with the least good performance. The discriminating thresholds allow thus, in case where $p > q$, to delimit an ambiguity zone in which it is not possible to “cut” between a preference or an indifference in favor of the action with the best performance; this preference being thus very weakly established. The *pseudo-criterion* model (*c.f.* Section 3) allows to take into account this important aspect.

To end this subsection, let us call the attention of the reader on the fact that we should not make any kind of confusion between the discriminating thresholds and other thresholds, called *dispersion thresholds* (see Roy, 1985, 1996). It is not rare (see Example 4) that a plausible or reasonable performance, $g(a)$, of an action a, should be associated with two margins $\eta^-(a)$ and $\eta^+(a)$ leading to fix two bounds, which allows to define the performance $g(a)$: $g^-(a) = g(a) - \eta^-(a)$ and $g^+(a) = g(a) + \eta^+(a)$. These margins, which can represent according to the different cases, the imprecision of a measure, the possible impact of uncontrolled phenomena, the part of arbitrariness related to the way $g(a)$ is computed, ..., are called *dispersion thresholds*. They correspond to the way performances should be viewed as ill-defined over certain values ranges.

When $\eta^-(a)$ and $\eta^+(a)$ do not depend on the particular considered action a, but only on the performance $g(a)$ (*cf.* Assumption 1) we shall show in subsection 4.3, that discriminating thresholds can be deduced in a very rigorous way from the data that correspond to the dispersion thresholds.

3. The pseudo-criterion model: Definitions and preliminary results

This model generalizes the true-criterion model recalled in the previous section.

Definition 3 (Pseudo-criterion, g, with direct thresholds)**.** *A pseudo-criterion, g, with direct thresholds is a real-valued function, g, defined for all $a \in A$, associated with two real-valued threshold functions, $p(g(a))$ and $q(g(a))$, verifying the following conditions:*

i) $p(g(a)) \geqslant q(g(a)) \geqslant 0$*;*

ii) $g(a) + p(g(a))$ *and* $g(a) + q(g(a))$ *are monotone non-decreasing functions of* $g(a)$*, if* g *is a criterion to be maximized;*

iii) $g(a) - p(g(a))$ *and* $g(a) - q(g(a))$ *are monotone non-decreasing functions of* $g(a)$*, if* g *is a criterion to be minimized.*

Let a and a' denote two actions to be compared, where a has a performance $g(a)$, at least as good as the performance of a', $g(a')$. The following conditions hold, for such ordered pairs $(a, a') \in A \times A$:

$$aP_ga' \quad \Leftrightarrow \quad |g(a) - g(a')| > p(g(a')) \tag{3.1}$$
$$aI_ga' \quad \Leftrightarrow \quad |g(a) - g(a')| \leqslant q(g(a')) \tag{3.2}$$
$$aQ_ga' \quad \Leftrightarrow \quad q(g(a')) < |g(a) - g(a')| \leqslant p(g(a')) \tag{3.3}$$

Definition 4 (Pseudo-criterion, g, with inverse thresholds)**.** *A pseudo-criterion, g, with inverse thresholds is a real-valued function, g, defined for all $a \in A$, associated with two real-valued threshold functions, $p'(g(a))$ and $q'(g(a))$, verifying the following conditions:*

i) $p'(g(a)) \geqslant q'(g(a)) \geqslant 0$*;*

ii) $g(a) - p'(g(a))$ *and* $g(a) - q'(g(a))$ *are monotone non-decreasing functions of* $g(a)$*, if* g *is a criterion to be maximized;*

iii) $g(a) + p'(g(a))$ *and* $g(a) + q'(g(a))$ *are monotone non-decreasing functions of* $g(a)$*, if* g *is a criterion to be minimized.*

Let a and a' denote two actions to be compared, where a has a performance $g(a)$, at least as good as the performance of a', $g(a')$. The following conditions hold, for such ordered pairs $(a, a') \in A \times A$:

$$aP_ga' \quad \Leftrightarrow \quad |g(a) - g(a')| > p'(g(a)) \tag{3.4}$$
$$aI_ga' \quad \Leftrightarrow \quad |g(a) - g(a')| \leqslant q'(g(a)) \tag{3.5}$$
$$aQ_ga' \quad \Leftrightarrow \quad q'(g(a)) < |g(a) - g(a')| \leqslant p'(g(a)) \tag{3.6}$$

The reasons that lead to impose the monotonicity conditions will be explained in subsection 4.1.

Figures 1 and 2 illustrate the relations between direct and inverse thresholds for a criterion to be maximized (Figure 1) and for a criterion to be minimized (Figure 2), respectively. Let us remark that in these figures the performance $g(a)$ varies, while $g(a')$ is fixed.

From these figures, the reader can easily verify the following two relations:

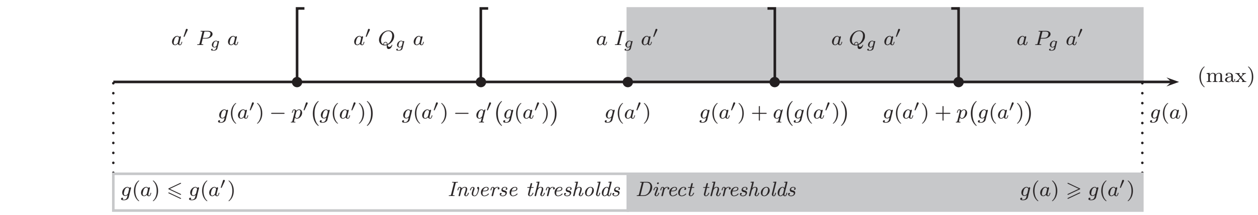

Figure 1: Representation of I_g, Q_g, and P_g (max)

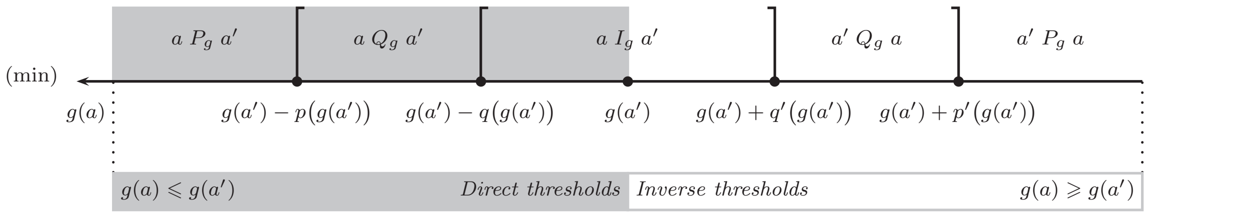

Figure 2: Representation of I_g, Q_g, and P_g (min)

$a)$ if g is a criterion to be maximized:

$$aI_ga' \Leftrightarrow -q'(g(a')) \leqslant g(a) - g(a') \leqslant q(g(a')) \tag{3.7}$$

$b)$ if g is a criterion to be minimized:

$$aI_ga' \Leftrightarrow -q(g(a')) \leqslant g(a) - g(a') \leqslant q'(g(a')) \tag{3.8}$$

As it was mentioned in subsection 2.3, in case of constant thresholds, direct and inverse thresholds are equal. For whatever the performance x of an action a, direct and inverse thresholds are functionally linked by the following relations:

As for the preference thresholds:

$a)$ if g is a criterion to be maximized:

$$p\big(x - p'(x)\big) = p'(x) \tag{3.9}$$

$b)$ if g is a criterion to be minimized:

$$p\big(x + p'(x)\big) = p'(x) \tag{3.10}$$

As for the indifference thresholds:

$a)$ if g is a criterion to be maximized:

$$q\big(x - q'(x)\big) = q'(x) \tag{3.11}$$

b) if g is a criterion to be minimized:

$$q\big(x + q'(x)\big) = q'(x) \tag{3.12}$$

The relation (3.11) can be proved as follows. Let $g(a) = x$. The smallest performance of an action a' such that $a' I_g a$ is $g(a') = x - q'(x)$. In such conditions, according to the definition of the direct threshold, we have $q(g(a')) = q'(x)$. The same kind of reasoning allow us to prove relations (3.9), (3.10), and (3.12).

4. Theoretical results

We start this section by analyzing and justifying the coherency conditions that must be fulfilled by the discriminating thresholds. Then, we present a study of variable discriminating thresholds as affine functions. Finally, we show how to build discriminating thresholds from dispersion thresholds.

4.1. Coherency conditions

In Definitions 3 and 4 (see Section 3), several conditions have been imposed on the discriminating thresholds. The first ones, $p(g(a)) \geqslant q(g(a)) \geqslant 0$, are inherent to the way thresholds are defined.

We justify hereafter *la raison d'être* of the monotonicity conditions for the case of direct preference thresholds and for a criterion to be maximized. The reasoning for all the other cases of Definitions 3 and 4 is analogous.

Let a and a' denote two actions such that:

i) $g(a) > g(a')$,

ii) $g(a) + p(g(a)) = x$,

iii) $g(a') + p(g(a')) = y$.

Suppose $x < y$. From the Assumption 1 and the Definition 1 (*cf.* subsection 2.3), it follows that, in such conditions, the smallest performance difference that could be surpassed, from $g(a')$, to have a strict preference situation would be at most equal to $x - g(a')$. This implies, $y - g(a') \leqslant x - g(a')$. Then, $y \leqslant x$, which contradicts the hypothesis. This proves the monotonicity condition.

The following Conditions 1 and 2 provide an equivalent formulation (but, frequently more useful in practice) of the monotonicity conditions as they were introduced in Definitions 3 and 4.

Condition 1 (Direct discriminating thresholds coherency)**.** *Let a and a' denote two actions.*

a) *if g is a criterion to be maximized and $g(a) > g(a')$:*

$$\frac{p\big(g(a)\big) - p\big(g(a')\big)}{g(a) - g(a')} \geqslant -1, \tag{4.1}$$

b) *if g is a criterion to be minimized and $g(a) < g(a')$:*

$$\frac{p\big(g(a')\big) - p\big(g(a)\big)}{g(a') - g(a)} \leqslant 1. \tag{4.2}$$

To obtain the coherency conditions that applies to direct indifference discriminating thresholds, it is only necessary to replace p by q in the above two conditions.

It is obvious that these conditions are equivalent to the monotonicity conditions written as follows:

As for Condition 4.1:

$$g(a') + p(g(a')) \leqslant g(a) + p(g(a)).$$

As for Condition 4.2:

$$g(a') - p(g(a')) \geqslant g(a) - p(g(a)).$$

Condition 2 (Inverse discriminating thresholds coherency)**.** *Let a and a' denote two actions.*

a) if g is a criterion to be maximized and $g(a) > g(a')$:

$$\frac{p'\big(g(a)\big) - p'\big(g(a')\big)}{g(a) - g(a')} \leqslant 1, \tag{4.3}$$

b) if g is a criterion to be minimized and $g(a) < g(a')$:

$$\frac{p'\big(g(a')\big) - p'\big(g(a)\big)}{g(a') - g(a)} \geqslant -1. \tag{4.4}$$

To obtain the coherency conditions that applies to inverse indifference discriminating thresholds, it is only necessary to replace p by q in the above two conditions.

As in the previous case, here also the equivalence between these conditions and the monotonicity conditions is obvious.

Remark 1. *Let g denote a criterion to be maximized and M denote a value at least equal to e^* ($M \geqslant e^*$). Minimizing a criterion, f, such that $f(a) = M - g(a)$ is equivalent to maximize g. In such conditions the direct (inverse) thresholds of $g(a)$ become the inverse (direct) thresholds of $f(a)$.*

Remark 2. *In practical situations E_g is always bounded from below by e_* and from above by e^*. Let g denote a criterion to be maximized. Suppose that the direct preference threshold is constant and equal to p whenever $g(a) \leqslant e^* - p$; otherwise the value of this thresholds is $e^* - g(a)$. To be more rigorous and precise, constant thresholds are not really constant in the upper part of the scale: when $g > e^* - p$ the monotonicity condition leads to put $p(g(a)) = e^* - g(a)$. However, in this upper part of the scale (under the condition that $g(a)+p$ is defined) there is nothing against to keep $p(g(a)) = p$ since according to this new definition of the preference threshold the strict preference situation remains unchanged. The same kind of situation occurs in the lower part of the scale when considering a criterion to be minimized. (This remark is also valid for inverse preference thresholds and direct preference and indifference thresholds.)*

Remark 3. *Let us notice that in real-world situations it is rare that relations 4.1. to 4.4 lead to an equality. Indeed, with an equality, there is no possibility of having a strict preference situation. Consider, for example, Condition 4.1. The equality leads to:*

$$p(g(a)) - p(g(a')) = g(a') - g(a) \quad \Leftrightarrow \quad g(a) = g(a') + p(g(a')) - p(g(a)).$$

We can deduce:

$$g(a) \leqslant g(a') + p(g(a')).$$

Thus, there is no ordered pair $(a, a') \in A \times A$ *fulfilling the relation:*

$$g(a) > g(a') + p(g(a')) \quad \Leftrightarrow \quad aP_g a'.$$

We can obtain the same type of result with relations 4.2, 4.3, and 4.4. With the same kind of reasoning we can show that when replacing p by q in relations 4.1 to 4.4, the equality leads to $aI_g a'$*, for whatever the performances of the actions a and* a'*. In multiple criteria decision aiding, when in presence of a criterion leading to these conclusions it means that such a criterion can be discarded from the family of criteria.*

In the next subsection a particular class of variable thresholds is analyzed and some useful relations between direct and inverse thresholds are precisely and clearly expressed.

4.2. Variable thresholds as affine functions

In practical situations, variable thresholds can often be modeled as affine functions:

i) as for the case of direct thresholds:

$$p\big(g(a)\big) = \alpha_p\, g(a) + \beta_p, \tag{4.5}$$

$$q\big(g(a)\big) = \alpha_q\, g(a) + \beta_q, \tag{4.6}$$

ii) as for the case of inverse thresholds:

$$p'\big(g(a)\big) = \alpha'_p\, g(a) + \beta'_p, \tag{4.7}$$

$$q\prime\big(g(a)\big) = \alpha'_q\, g(a) + \beta'_q. \tag{4.8}$$

Conditions 4.1 to 4.4. lead to impose:

$\alpha_p \geqslant -1$ (g is a criterion to be maximized),

$\alpha_p \leqslant \ \ 1$ (g is a criterion to be minimized),

$\alpha'_p \leqslant \ \ 1$ (g is a criterion to be maximized),

$\alpha'_p \geqslant -1$ (g is a criterion to be minimized).

To obtain similar conditions that applies to indifference discriminating thresholds, it is sufficient to replace p by q in the above two conditions.

Remark 3 (*cf.* subsection 4.1) shows that in real-world situations, the above inequalities are, in general, strict inequalities.

Theorem 1 shows the functional relationship between direct and inverse thresholds taking into account this particular type of threshold functions.

Theorem 1. *When the variable preference thresholds are defined by the relations (4.5) and (4.7), we have:*

a) *If g is a criterion to be maximized, then*

$$\alpha'_p = \frac{\alpha_p}{1+\alpha_p} \quad \textit{and} \quad \beta'_p = \frac{\beta_p}{1+\alpha_p}, \quad \textit{with} \quad \alpha_p > -1,$$

b) *If g is a criterion to be minimized, then*

$$\alpha'_p = \frac{\alpha_p}{1-\alpha_p} \quad \textit{and} \quad \beta'_p = \frac{\beta_p}{1-\alpha_p}, \quad \textit{with} \quad \alpha_p < 1.$$

The same type of relations can be obtained from relations (4.6) and (4.8) for variable indifference thresholds.

Proof (for an alternative proof see also Roy, 1985, pp. 258-263):

a) Let $p'(x) = \alpha'_p\ x + \beta'_p$ and $p(y) = \alpha_p\ y + \beta_p$, such that $y = x - p'(x)$. From relation (3.9), $p(x - p'(x)) = p'(x)$ is equivalent to $\alpha_p \left(x - (\alpha'_p\ x + \beta'_p)\right) + \beta_p = \alpha'_p\ x + \beta'_p$. This equality is also equivalent to $\alpha_p\ (1-\alpha'_p)x - \alpha_p\ \beta'_p + \beta_p = \alpha'_p\ x + \beta'_p$. By equivalence, the latter equality is also true, for all x, if and only if: $\alpha_p\ (1-\alpha'_p) = \alpha'_p$ and $-\alpha_p\ \beta'_p + \beta_p = \beta'_p$. These two equalities imply that: $\alpha'_p = \frac{\alpha_p}{1+\alpha_p}$ and $\beta'_p = \frac{\beta_p}{1+\alpha_p}$, with $\alpha_p \neq -1$ (*cf.* Remark 3).

b) The proof of this case is similar. □

Remark 4. *We considered the case where an affine function, which characterizes the threshold, is the same over the range of the scale E_g, $[e_*, e^*]$. We can also model a threshold through the use of a piecewise linear function (see Back to Example 3 in subsection 5.3.)*

In Section 5, some practical comments for assigning numerical values to the thresholds are provided, taking into account all the results presented in the previous sections.

4.3. Building discriminating thresholds from dispersion thresholds

Let us consider the case where dispersion thresholds, $\eta^-(a)$ and $\eta^+(a)$ (see the end of subsection 2.3) are introduced (see also Example 4 in subsection 2.2). When $\eta^-(a)$ and $\eta^+(a)$ only depend on the performance $g(a)$ it is possible to derive preference and indifference thresholds from dispersion thresholds.

In what follows we only consider g as a criterion to be maximized. Let us start with the case where the dispersion thresholds do not depend on $g(a)$. Let a and a' denote two actions such that $g(a) \geqslant g(a')$. It is very natural to consider that there is a strict preference in favor of a if and only if,

$$g(a') + \eta^+ \leqslant g(a) - \eta^-.$$

Thus, according to Definition 1, we have,

$$p(g(a')) = p'(g(a)) = \eta^+ + \eta^-. \tag{4.9}$$

It is also natural to consider that there is an indifference between the two actions if and only if,

$$g(a) - \eta^- \leqslant g(a') \leqslant g(a) + \eta^+$$

and

$$g(a') - \eta^- \leqslant g(a) \leqslant g(a') + \eta^+.$$

Thus, according to Definition 2, we have,

$$q(g(a')) = q'(g(a)) = \min\{\eta^-, \eta^+\}. \tag{4.10}$$

Let us consider now the case where the dispersion thresholds are not constant, but they are modeled as follows.

$$\eta^-(g(a)) = \alpha^- g(a) + \beta^- \quad \text{and} \quad \eta^+(g(a)) = \alpha^+ g(a) + \beta^+, \quad \text{with} \quad \alpha^- < 1 \ \text{ and } \ \alpha^+ > -1.$$

Following the same kind of reasoning as for the constant dispersion thresholds, it is possible to show (see subsection 9.3.4 in Roy, 1985) that,

$$p(g(a)) = \frac{(\alpha^+ + \alpha^-)g(a) + \beta^+ + \beta^-}{1 - \alpha^-}, \tag{4.11}$$

and,

$$q(g(a)) = \min\left\{\alpha^+ g(a) + \beta^+, \frac{\alpha^- g(a) + \beta^-}{1 - \alpha^-}\right\}. \tag{4.12}$$

5. Practical aspects

The way of dealing, in practice, with the definition or construction of the discriminating thresholds (*i.e.*, the way of determining the values, constant or variable, which should be associated with the criteria) is strongly constrained by the nature of each criterion. We shall examine successively the case of discrete scales, then the case of continuous scales, notably by coming back to the examples of subsection 2.2. This way of proceeding requires the cooperation of both the analyst and the decision-maker. The respective role of the former and the latter depends much more on the manner a criterion is defined and the data associated with it, than from the distinction between discrete and continuous scales. That is why we begin by tackling this question, *i.e.*, the respective role of each one of the two actions.

5.1. Respective role of the analyst and the decision-maker

The discriminating thresholds were introduced in Section 2, as two concepts designed to take into account the fact that a criterion g being defined, the knowledge of the performances $g(a)$ and $g(a')$, of two actions a and a', respectively, it is not sufficient to conclude that, according to criterion g, one of these two actions is strictly preferred to the other one, as soon as the difference $g(a) - g(a')$ is not equal to zero.

The indifference and preference thresholds must be defined in order to discriminate the cases where the difference of performances can be considered enough convincing of a strict preference from those cases where this non-zero difference of performances can be considered compatible with the indifference between the two actions, a and a'. The fact that a non-zero difference cannot be enough convincing of a strict preference difference comes (as it was already mentioned in Section 2) from the existence of some arbitrariness, imprecision, uncertainty, or ill-determination, which is frequently present in:

i) the operational instruction chosen to define criterion g: expert, questionnaires, forecasting techniques, mathematical expressions, or even more complex algorithms.

ii) the sources from which data (quantitative or qualitative) come from; data that are then used by the chosen operational instruction to define the performances of every criterion.

Except for certain situations or circumstances, it is the decision-maker or his/her representative, the person who defines the points of view to take into account in a decision aiding process. The analyst and possibly some experts in close collaboration with him/her work together to model the points of view by a family of criteria. It is the person who built the criteria the best qualified one to define the part of arbitrariness, imprecision, ill-determination, or uncertainty contained in the operational instruction and in the different sources of data. In general, it is the analyst, frequently in a close collaboration with the decision-maker and sometimes with certain experts, who must decide about the most adequate way in decision aiding to take into account the part of arbitrariness, imprecision, ill-determination, and/or of uncertainty in the definition of the performance $g(a)$, according to the considered point of view. The analyst can do it by introducing probability distributions or fuzzy numbers (in Omega Journal see, for instance, Hatami-Marbini and Tavana, 2011), but providing a meaningful definition of such distributions or such numbers can prove to be more difficult than defining discriminating thresholds. The latter manner of proceeding is worthy; it avoids to give more meaning to the performances than they really mean. Moreover, this allows avoiding, at least in certain aggregation models (as for instance the Electre type methods), that an important negative performance difference on a given criterion can be compensated by the presence of several not very significant positive differences on other criteria.

As for the role of the decision-maker in the definition of the thresholds, it depends on his/her own personality (and/or the personality of his/her representative). The analyst must cooperate with the decision-maker (or his/her representative) in one or some particular aspects, in a more or less close way, given his/her familiarity about the sources of certain data used in the computation of the performance $g(a)$ as well as the operational instruction that have been used to define criterion g, taking into account the considered point of view. It is not rare that the decision-maker (and also his/her representative) is a very occupied person or has not enough background to actively participate in this aspect of the modeling process. The decision-maker must thus trust the analyst and other available experts or better qualified people.

Finally, we would like to call the attention of the reader for the following point. The previous considerations show, in an obvious way, that the discriminating (indifference and preference) thresholds cannot be assimilated or considered as preference parameters. This expression applies the parameters which serve for characterizing the respective role which play the different criteria in the aggregation model (weights, veto thresholds, acceptation, rejection, and/or aspiration levels).

In some cases, the decision-maker certainly may consider that the differences modeled by the thresholds represent preference value judgments because he/she considers that small performance differences are not enough significant of strict preferences. In such cases, the analyst must ask him/her if it would not be because he/she considers this small difference to be less credible or not very realistic, taking into account the way the corresponding performances were built. If his/her answer is negative then, in order to make sure the decision-maker definitely understands the question, the analyst may ask him/her to consider the case where a 1€ difference between 100€ of benefit and a benefit of 99€ is established with certainty. In such a situation, what could justify that the decision-maker does not prefer to earn 100€ rather than 99? The only possible decision-maker's answer is because his/her preference is very weak. It is certainly not excluded to use the indifference threshold for modeling an intensity of preference considered nil, and the preference threshold for modeling an intensity of preference considered weak. But, proceeding in this way is to forget that thresholds were not designed for such a purpose and in particular in Electre methods the thresholds do not intervene with this meaning (Figueira et al., 2013). If we want to make use of the thresholds to play a role of a model of a certain form of intensity of preferences, it requires precaution (Roy and Słowiński, 2008).

It is, above all, by examining the operational instruction that leads to the definition of criterion g and the data that are used to compute the performance $g(a)$ that the analyst in collaboration with the decision-maker determine the values to be assigned to the indifference and the preference thresholds.

Let us point out that a large majority of the authors who are interested in ordered structures with thresholds (*cf.* references cited at the beginning of the Introduction) assume that these thresholds are part of the definition of the scale they are consequently intrinsically linked. This hypothesis can lead to look at the thresholds as being preference parameters. It is a kind of temptation to adopt this point of view after numerical values are assigned to these thresholds. In other words, we should not assume that the values assigned to these thresholds make part of the definition of the scale. This assumption leads to forget what allowed the assignment of numerical values to such thresholds. First of all, the assignment of numerical values to the thresholds is mainly due to the way criterion g was defined as well as to the imperfect nature of data, which are used to determine the performance $g(a)$ of an action a as a level of the scale E_g. It is the discriminating power of this criterion g that indifference and preference thresholds have as the object to take into account: the indifference and preferences which follow from it do not belong to the definition of the scale, but to the way the criterion applies actions of A to the scale.

In the remaining of this section, except in the case of an opposite mention, the considerations and the results concern the case of direct thresholds for criteria to be maximized. The reader will be able to transpose them, without difficulties, to the case of inverse thresholds as well as to the case when criteria are to be minimized.

5.2. Discrete scales

We start this section by examining the case where the analyst wishes to determine the values to assign to the discriminating thresholds in order to preserve the concrete meaning of the performances, he/she wants to keep unchanged the definition of the original scale associated with the considered criterion. Then, we examine the case where a codification of the original scale must be introduced. Finally, we give some words about the particular situation of weak preferences when in presence of discrete scales.

5.2.1. The criterion scale is used as in its original version

Here (as in the other cases) for determining the values to be assigned to the discriminating thresholds, the analyst must use Definitions 1 and 2. A reference level, e_i, being chosen (for instance, in the neighborhood of the middle of the scale), the analyst must try to determine (considering the operational instruction and the sources of data, which leads to define the performance of an action a on this scale):

i) The level e_k ($k \geqslant i$), the closest to e_i, such that there is strict preference between e_{k+1} and e_i; according to Definition 1, $p(e_i) = k - i$. The level e_k will differ from e_i whenever the part of arbitrariness, imprecision, ill-determination, or uncertainty, which impacts in the definition of the criterion, justifies the existence of reasonable doubts leading to consider that an action a verifying $g(a) = e_k$ is not strictly preferred to an action a' verifying $g(a') = e_i$.

ii) The level e_j ($j \geqslant i$), the most distant from e_i, which remains compatible with an indifference between e_i and e_j; according to Definition 2, $q(e_j) = j - i$. The level e_j will differ from e_i whenever the part of arbitrariness, imprecision, ill-determination, or uncertainty, which impacts in the definition of the criterion, allows to think that, instead of assigning to an action a the performance $g(a) = e_j$, one could have just assign the performance e_i and *vice versa*, *i.e.*, instead of assigning to an action a' the performance $g(a') = e_i$, one could have just assign the performance e_j.

The analyst, in a more or less close collaboration with the decision-maker and/or some experts (see subsection 5.1) must wonder whether $p(e_i)$ (as well as $q(e_i)$) depends on the chosen reference level e_i. For such a purpose he/she can take back the same approach with the top and the bottom levels of the scale, to check if it leads to the same results. If it is the case, the analyst can assign constant values to the thresholds. Otherwise, the analyst should think about the possibility of considering at least one thresholds (possibly even the two) as a variable threshold. In the latter case, the threshold must be defined for each one of the levels of the scale.

Let us recall that (*cf.* Remark 2 in subsection 4.1) the hypothesis of constant thresholds is compatible with the values of $p(e_i)$ and $q(e_i)$, which decrease when e_i gets closer to e^*. To examine whether the hypothesis of constant thresholds is an acceptable assumption, it is necessary, as a result, to avoid choosing reference levels very much near to the upper bound of the scale. In the case of a criterion to be minimized some caution should be taken into account with respect to the lower bound of the scale.

Let us illustrate the previous considerations on some examples.

Back to Example 1 (cf. subsection 2.2.). In this example the criterion is to be minimized. It follows from the presentation of this example that between two implementation times, the strict preference, in favor of the action with the shortest implementation time appears to be entirely justified only if the difference between the two implementation times is strictly greater than two months (therefore, it means at least 3 months, considering the discrete nature of the scale); this is valid for whatever the position in the scale of the chosen reference level. The analyst is, therefore, led to adopt a constant threshold equal to 2 months. It means that a difference of two months is not considered enough convincing of a strict preference. It does not mean, therefore, that such difference can be considered as no significant for every case. That is why the analyst can reasonably put $q = 1$ month.

Back to Example 2 (cf. subsection 2.2.). In this example the criterion is to be maximized. As we pointed out in subsection 2.2., we shall label the successive levels of this verbal scale by their ranks, which lead to put: e_1 = opposing, e_2 = neutral, e_3 = possibly favorable but questionable, e_4 = unquestionable but weak, e_5 = significant but partial, and e_6 = complete.

Let us assume that the analyst chooses e_3 as the reference level. On the one hand, considering the instructions given to the expert, so that a project a is strictly preferred to a project having as performance e_3, one must have $g(a) > e_4$. Consequently, $p(e_3) = 1$. For the same reasons, it will lead to put $p(e_2) = p(e_4) = 1$. On the other hand, as for the expert having as instructions to only keep the extreme performances, e_1 and e_6, when in presence of unquestionable cases, the analyst must put $p(e_1) = p(e_5) = 0$, to take into account such advice. Given this situation, the expert can frequently come back and hesitate between two successive levels, among the four intermediate levels; the analyst must put $q(e_i) = 1$, for all $i = 2, 3, 4, 5$. On the contrary, his/her absence of hesitation to assign the extreme levels must lead to put: $q(e_1) = q(e_5) = 0$.

New Example cf. Roy et al. (1986). In this example we are interested in a very large number of facilities (for instance, the subway stations of the Paris region), where we should assess the more or less degree of degradation. For such an assessment, we asked several experts in charge of visiting all the facilities and assign a score to each one of them, within the discrete range from 0 to 20. In a preliminary step some typical facilities were shown to the experts:

i) Facilities coming from a recent renewing process, for which it is necessary to assign the degree of degradation 0.

ii) Facilities judged to be in a high dilapidated state, for which it is necessary to assign the degree of degradation 20.

iii) Facilities judged to be in a "medium" state of degradation, for which it is necessary to assign the degree of degradation 10.

Every facility was visited by at least one expert; some were visited by two, and a few by three. The scores assigned by the experts to a given facility, when this facility was visited by several experts, differ very often by 1 point, sometimes by 2 points, but very seldom by 3 points. The analyst must carefully analyze this case of divergence of scoring. If, as it was the case in our example, he/she observed that these divergences appear with similar frequencies in the bottom, in middle, and at the top of the scale he/she is led to keep constant thresholds. The analyst can put $p = 2$ and $q = 1$.

5.2.2. Coding the original criterion scale

Whatever the way the levels of a scale are initially characterized (a numerical value as in Example 1 or a verbal expression as in Example 2), coding a scale consists of associate to each level e_i of the original scale, E_g, with a numerical value, $\chi(e_i)$, that can be used instead of the original characterization of e_i. It is quite obvious that this coding must be such that $\chi(e_i) < \chi(e_{i+1})$, for $i = 1, \ldots, n-1$. A coding χ being defined, the discriminating thresholds $p(\chi(e_i))$ and $q(\chi(e_i))$ can be derived from the preference and indifference thresholds defined over the original scale by applying the following relations:

$$p(\chi(e_i)) = \chi(e_i^p) - \chi(e_i), \tag{5.1}$$

$$q(\chi(e_i)) = \chi(e_i^q) - \chi(e_i), \tag{5.2}$$

where e_i^p and e_i^q are the levels of scale E_g defined as follows:

i) If e_i is characterized by a numerical value $v(e_i)$ (see Example 1), then e_i^p and e_i^q are defined, respectively, by the following relations:

$$v(e_i^p) = v(e_i) + p(e_i), \tag{5.3}$$

$$v(e_i^q) = v(e_i) + q(e_i). \tag{5.4}$$

ii) If e_i is characterized by its rank i (see Example 2), then e_i^p and e_i^q are defined by the relations (5.3) and (5.4), respectively, with $v(e_i) = i$.

Let us observe that, despite the discrete nature of scales, the levels e_i^p and e_i^q are perfectly defined by the relations (5.3) and (5.4).

Several reasons may led the analyst to substitute or replace the original characterization levels by new ones resulting from a coding χ. We shall present only the two most current. Whatever the reasons, which lead the analyst to use a coding χ, he/she should start, for obtaining the values of $p(\chi(e_i))$ and $q(\chi(e_i))$, by determining appropriate values for the thresholds in the original scale, as it was mentioned in subsection 5.2.1., in order to use in a further step, the relations (5.1) and (5.2).

1^{st} *Case: An original numerical scale $v(e_i)$ must be normalized.* Let $v(e_1)$ and $v(e_n)$ denote the numerical values, which characterize the extreme levels of the original numerical scale. The normalization is defined as follows:

$$\chi(e_1) = 0,$$

$$\chi(e_n) = 1,$$

$$\chi(e_i) = \frac{v(e_i) - v(e_1)}{v(e_n) - v(e_1)}, \quad i = 2, \ldots, n-1.$$

After applying the relations (5.1) and (5.2) we obtain:

$$p(\chi(e_i)) = \frac{p(e_i)}{v(e_n) - v(e_1)},$$

and,

$$q(\chi(e_i)) = \frac{q(e_i)}{v(e_n) - v(e_1)}.$$

2^{nd} *Case A (verbal or numerical) original scale must be coded in such a way that at least one of the two thresholds becomes constant.* Let us recall that it is always possible to find such a kind of coding (see, for example, Roubens and Vincke 1985). When a coding χ is adopted, which renders constant the values of one of the two thresholds, the values for the other one are perfectly defined by the relations (5.1) or (5.2). Consequently, there is no fundamental reason to fix these thresholds as constants. There exist, however, two important cases in practice when we fix one of the thresholds as a constant, the other becomes automatically as a constant too:

i) in the original scale, the analyst put $p(e_i) = q(e_i)$, for all $e_i \in E_g$;

ii) in the original scale, the analyst put $q(e_i) = 0$, for all $e_i \in E_g$, and the coding χ is designed to render constant the preference threshold.

In practice, adopting a coding for making at least one of two thresholds as a constant allows, in a large number of cases, to render more understandable the role that thresholds play in the decision aiding process. This advantage is particularly important when in presence of verbal scales (*cf.* hereafter *Back to Example 2*).

Associating a numerical code with each one of the verbal expressions is frequently necessary, not only from a computational point of view, but also for facilitating the interaction with the decision-maker or his/her representative.

Coding each level by its rank looses every piece of information concerning the more or less discriminating nature of the difference between two levels. It will be differently when dealing with a coding that allows to take into account the thresholds in such a way that at least one of the two will become constant.

Back to Example 2. As it was mentioned in subsection 5.2.1 in this example we have:

$$p(e_1) = q(e_1) = 0,$$

$$p(e_2) = q(e_2) = p(e_3) = q(e_3) = p(e_4) = q(e_4) = 1,$$

$$p(e_5) = q(e_5) = 0.$$

It is sufficient to put:

$$\chi(e_1) = 0,\ \chi(e_2) = 2,\ \chi(e_3) = 3,\ \chi(e_4) = 4,\ \chi(e_5) = 5,\ \chi(e_6) = 7,$$

for getting the definition of a coding fulfilling the following relation:

$$p(\chi(e_i)) = q(\chi(e_i)) = 1, \quad \text{for all} \quad i = 1, \ldots 5.$$

This coding shows in a very obvious way the particular role (say, a very discriminating role) that extreme levels play.

When in presence of a scale E_g with a small number of levels (as it is the case in this example), intuition and possibly some trial-and-error easily allows to find a coding in which the chosen threshold is constant. When in presence of a scale with a large number of levels, this task becomes much more harder. Such cases are rare in practice. We shall provide only in this article the main guidelines of a procedure allowing to obtain a solution. The procedure described hereafter deals with the case where it is the preference threshold that must become constant. As for a constant indifference threshold we only need to replace p by q.

Let G denote a graph defined from a the set of vertices E_g and a set of valued arcs U. The arcs, $u = (e_i, e_j)$, as well as the values or lengths, $d(e_i, e_j)$, are defined as follows:

i) $(e_i, e_{i+1}) \in U$ and $d(e_i, e_{i+1}) = 1$ for all $i = 1, \ldots, n-1$.

ii) $(e_i^p, e_i) \in U$ and $d(e_i^p, e_i) = -p$ for all $i = 1, \ldots, n-1$.

iii) $(e_i, s(e_i^p)) \in U$, where $s(e_i^p)$ is the level immediately after e_i^p, and $d(e_i, s(e_i^p)) = p + 1$, for all i such that $s(e_i^p)$ is defined.

If we assign to p a value such that G has no circuit of strictly positive length, then $\chi(e_1) = 0$ and $\chi(e_i) =$ length of the longest path from e_1 to e_i, for all $i = 2, \ldots, n$, is an adequate coding with a constant preference threshold p (see Roubens and Vincke, 1985, p. 37 and Roy, 1985, p.162).

It is quite obvious that the chosen threshold p must fulfill the condition: $p \geqslant p^* =$ largest rank difference between levels e_i^p and e_i. With such a minimum, the graph G may contain, in certain cases, circuits of strictly positive length. It should be noticed that, this will not occur when the scale E_g does not contain any level e_i such that $e^p_{i-2} = e^p_{i-1} = e_i$ and $e^p_i > e_i$.

As for the proof of this result (which has no room in the scope of this article) the reader can consult the work by Roy (2013).

5.2.3. The weak preference situation in the case of discrete scales

Let e_i denote a level of a discrete or continuous scale E_g (g being a criterion to be maximized). Let e_j the highest level such that $e_i I e_j$ ($e_j = e^q$). The set K_i containing the levels e_k such that $e_k Q e_i$ is a non-empty set if and only if $q(e_i) < p(e_i)$. If the scale if a discrete one, then set K_i contains a minimal element, e_{j+1}. On the contrary, when E_g is a continuous scale, K_i does not possess a minimal element.

As a consequence, we should be very carefully when dealing with weak preference situations in the definition of the concordance index, $c(a, a')$, of Electre methods (see Figueira et al., 2013). This led us to call the attention of the reader on what follows.

Let us recall that $c(a, a')$ (roughly meaning a degree of outranking of a over a') takes into account the weights of criteria which help to validate the assertion, "a is at least as good as a'" denoted by aSa'. Every criterion leading to aPa', aQa', and aIa' is taken into account with its overall weight. It is obvious that a criterion leading to $a'Pa$ must not be taken into account for validating such an assertion. On the contrary, a criterion leading to $a'Qa$ must not be completely discarded with respect to its contribution to the assertion aSa'. This weak preference situation represents a hesitation between $a'Ia$ and $a'Pa$. The criterion is thus taken into account by a fraction, ψ, of its weight. This fraction can be interpreted as the proportion of voters (the weight corresponds to the voting power of the criterion) in favor of the assertion aSa'. This proportion should be as closed as possible to 1 when the hesitation is more in favor of the indifference. It should be zero when we reach the strict preference situation in favor of a'.

In the case of a continuous scale, this leads to the following formula:

$$\psi = \frac{p(g(a)) - (g(a') - g(a))}{p(g(a)) - q(g(a))},$$

with,

$$q(g(a)) < g(a') - g(a) \leqslant -p(g(a)), \text{ for } p(g(a)) \neq q(g(a)). \tag{5.5}$$

This relation leads effectively to:

i) $\psi = 1$ iff $g(a') = g(a) + q(g(a))$: the only situation that validates $a'Ia$ without hesitation.

ii) $\psi = 0$ iff $g(a') = g(a) + p(g(a))$: situation that, due to the continuous nature of the scale, only leads to the absence of the hesitation between $a'Ia$ and $a'Pa$; the latter imposes thus its power.

Things will be different when in presence of a discrete scale since $g(a') = g(a) + p(g(a))$ stills continue to correspond to a hesitation situation. In such a case the possibility of $a'Ia$ cannot be discarded. Formula (5.5) does not correctly reflect the situation. The value $\psi = 0$ only can be reached by a value of $g(a')$, which corresponds to the level immediately greater than to the level characterized by $g(a) + p(g(a))$. In such conditions it is necessary to replace $p(g(a))$, in formula (5.5), by:

i) $p(g(a)) + 1$, if the scale levels are characterized by their ranks.

ii) $p(g(a)) + \delta(a)$, when the scales are characterized by a function v; $\delta(a)$ being defined by the difference between the value that characterizes the levels that immediately follows $g(a) + p(g(a))$ and the value $g(a) + p(g(a))$.

When in presence of constant thresholds, Formula (5.5) can thus be adapted to the case of a discrete scale and it becomes as follows:

$$\psi = \frac{p + 1 - (g(a') - g(a))}{p + 1 - q},$$

with,

$$q \leqslant g(a') - g(a) \leqslant p, \quad \text{for} \quad p \neq q. \tag{5.6}$$

Let us observe that this formula is still valid when $p = q$, which corresponds to a situation of absence of weak preference. When $p = q + 1$, which corresponds to a unique situation of real hesitation ($g(a') = g(a) + p$), formula (5.6) leads to $\psi = 1/2$ (which seems a very adequate value). Similarly, if $p = q + 2$, each one of the two hesitation situations leads to $\psi = 2/3$ and $\psi = 1/3$, respectively.

5.3. Continuous scales

Once again, the analyst must support his/her activity on Definitions 1 and 2, for assigning values to the discriminating thresholds.

In the case where the original scale has been coded, the analyst must start by assigning values to the thresholds of the original scale. Indeed, it is only on this scale that he/she will be able to appreciate what there is imprecise, ill-determined, and uncertain in the definition of the performance $g(a)$, considering the operational instruction of the criterion as well as the data, which are taken into account in the computation of this performance. He/she must then use relations (5.1) and (5.2) (such as they are defined in case *i)*) to derive the thresholds on the coded scale.

The analyst must start by trying to know if it is reasonable to assign, to each one of the two thresholds, a constant value, or if he/she must make use of affine functions (*cf.* subsection 4.2), or possibly others.

Examples 3 and 4 (*cf.* subsection 2.2) will serve us to illustrate what can be the analyst way of proceed.

Back to Example 3 (cf. subsection 2.2). In such a context, the analyst must examine, with the person who knows the weak points in the details of the costs forecasts, to apprehend the part of uncertainty, or even arbitrary, which have an impact on the overall amount of the expected costs. This impact depends, in general, on the overall amount of the expected costs. That is why the hypothesis of constant thresholds is not adequate. The hypothesis of thresholds functions of the form $\alpha g(a) + \beta$ seems *a priori* to be adequate. To check if it is the case and then assign the values to the α and β parameters, for each one of the thresholds functions, it is enough to get some information of the type we pointed out in the presentation of this example. This information is relative to two points in the scale, one rather in the bottom, the other rather at the top of the scale, but not very closed to the extreme bounds of the scale to avoid collateral consequences.

Let us start to show how this type of information allows to define these thresholds functions. We shall then see how to check whether these thresholds are adequate, and if they are not, how the analyst can try to change them.

Cost is a criterion to be minimized. Let us consider the information provided about a cost of 15 000 €. To say that only the projects with a cost greater than or equal to 17 000 € will be considered as significantly more expensive, means that with respect to a reference cost of 15 000 €, the inverse threshold is 2 000 €. The use of an affine function leads to:

$$17\ 000 - 15\ 000 = 15\ 000\alpha'_p + \beta'_p \tag{5.7}$$

To say that only the projects with a cost greater than or equal to 45 000 € will be able considered as significantly more expensive than projects with a cost of 40 000 €, leads to put, for the same reasons as in the previous situation:

$$45\ 000 - 40\ 000 = 40\ 000\alpha'_p + \beta'_p \tag{5.8}$$

Relations (5.7) and (5.8) constitute a system of two equations with two unknowns, which resolution leads to consider, as an inverse preference threshold function, the following function:

$$p'(g(a)) = \frac{12}{100}g(a) + 200 \tag{5.9}$$

By exploiting other information provided in the description of Example 3 (*cf.* subsection 2.2), the analyst will be led to the following function, after the resolution of a new system of two equations with two unknowns:

$$q'(g(a)) = \frac{4}{100}g(a) + 400 \tag{5.10}$$

The analyst must now check that the thresholds functions (5.9) and (5.10), resulting from such computations, really meet all the conditions required for these thresholds functions (*cf.* subsection 4.2).

Finally, the analyst must be sure that these functions are also adequate to model the thresholds in the middle of the scale. Formula (5.9) when applied to an expected cost of 25 000 € leads to an inverse preference threshold of 3 200 €. Let us consider that this value of the preference threshold is judged very large and that a value of 3 000 € would be preferable. It leads to put:

$$3\ 000 = 25\ 000\alpha'_p + \beta'_p \tag{5.11}$$

This would lead to adopt a piecewise linear affine function to build the inverse preference thresholds, defined as follows:

i) For $g(a) \leqslant 25\ 000$, by the resolution of the two equations (5.7) and (5.11):

$$p'(g(a)) = \frac{10}{100}g(a) + 500 \text{ for } g(a) \leqslant 25\ 000.$$

ii) For $g(a) \geqslant 25\ 000$, by the resolution of the two equations (5.7) and (5.11):

$$p'(g(a)) = \frac{2}{15}g(a) - \frac{5\ 000}{15} \text{ for } g(a) \geqslant 25\ 000.$$

Let us end this example by calling the attention of the reader to the following point. If, instead of a cost we consider a profit, for which the same information would have been provided, the affine functions (5.9) and (5.10) would define, in such conditions, the direct thresholds. Theorem 1 provides the appropriate formula for obtaining the inverse thresholds.

Back to Example 4 (cf. subsection 2.2). In this example the criterion represents a market share $g(a)$, which could be conquered at the end of one year if the product a is lunched in the market. We assumed that the the forecasting analyst could bound his/her forecasts by an optimistic value $g^+(a)$ and by a pessimistic value $g^-(a)$. To construct a pseudo-criterion, from such data, it is necessary to assume (*cf.* subsection 2.3) that the amplitude of the provided range is not specific to action a, but that it can only depend on the position of the expected performance $g(a)$ on the scale $[0, 100]$. The analyst must ask the forecasting analyst if he/she, considering the market survey, has serious reasons to suppose that the differences, $g^+(a) - g(a) = \eta^+ g(a)$ and $g(a) - g^-(a) = \eta^- g(a)$, really depend on the place of $g(a)$ on the scale $[1, 100]$. Considering the quality of the information gathered, in the market survey, we should be able to suppose, in a large number of cases, that in fact these dispersion thresholds are constant. The most likely value $g(a)$ being often already optimistic, the constant values kept, for these thresholds lead to put $\eta^+ < \eta^-$: for example $\eta^+ = 5$, $\eta^- = 10$. In such conditions, the analyst will be able to adopt the following values (*cf.* relations 4.9 and 4.10) :

$$p = 10 + 5 = 15, \quad q = \min\{5, 10\} = 5.$$

If serious reasons led to reject the hypothesis of constant thresholds, the analyst in interaction with the forecasting analyst could consider the use of affine functions (*cf.* subsection 4.3.) to assign values to the coefficients, α^-, α^+, β^-, and β^+. He/she will only have then to solve two systems of two equations with two unknowns. Let us take into account the values of the dispersion thresholds characterized by a pessimistic and an optimistic value associated with the performances as follows:

i) one in the bottom of the scale, for instance, $g(a) = 25 \ : \ [17, 30]$;

ii) the other one at the top of the scale, for instance, $g(a) = 75 \ : \ [65, 83]$.

The two systems are the following:

$$25\alpha^- + \beta^- = 8 \quad \text{and} \quad 75\alpha^- + \beta^- = 10$$

and

$$25\alpha^+ + \beta^+ = 5 \quad \text{and} \quad 75\alpha^+ + \beta^+ = 8$$

thus, we obtain

$$\alpha^- = \frac{4}{100}, \ \beta^- = 7 \quad \text{and} \quad \alpha^+ = \frac{6}{100}, \ \beta^+ = \frac{7}{2}.$$

By using relations (4.11) and (4.12), then it will lead to obtain:

$$p(g(a)) = \frac{10}{96} g(a) + \frac{105}{960},$$

and

$$q(g(a)) = \min\left\{\frac{6}{100} g(a) + \frac{7}{2}, \ \frac{4}{96} g(a) + \frac{700}{96}\right\}.$$

6. Conclusions

In this article we pointed out the relevance of a decision aiding tool, the discriminating thresholds, for modeling the preferences of a criterion. Then, we gathered, gave a new shape, and illustrated, in a succinct way, the set of theoretical results that were scattered over diverse theoretical works. In this synthesis, we mainly devoted our work to call the attention of the reader to the pitfalls, which it is necessary to watch out, whenever we are using direct or inverse thresholds and criteria to be maximized or minimized. Finally, we dealt with the way of assigning precise values to the discriminating thresholds in some applications: respective role of the analyst(s), the decision-maker(s), and (possibly) the expert(s); how to proceed when in presence of a discrete scale or when in presence of a continuous scale; some concrete case-studies were used to illustrate the results and the practical aspects throughout the article. In each one of these concrete cases we dealt with a criterion of a particular type. In a real-world decision aiding problem, these types of criteria may be present conjointly. The concept of discriminating thresholds allows thus for modeling, in a coherent way, the imperfect knowledge as it occurs for each one of the criteria. The recent developments on threshold based methods renders this concept of the utmost importance in decision aiding. Recent advances and challenges in this kind of methods include the extension of outranking approaches to be able to accommodate and to deal with a hierarchical structure of criteria (Corrente et al., 2013), the extension of outranking methods to group decision making problems (Valadares-Tavares, 2012), and the hybridization of outranking approaches with preference disaggregation and machine learning for preference modeling based approaches (Kadziński et al., 2012).

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