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Abstract

We consider a generalized Path Traveling Salesman Problem where the distances are defined by a 2-edge-connected graph metric and a constant number of salesmen have to cover all the destinations by traveling along paths of minimum total length. We show that for this problem there is a polynomial algorithm with asymptotic approximation ratio of $\frac{3}{2}$.

1 Introduction

The Traveling Salesman Problem is a paradigmatic, extensively studied difficult problem in combinatorial optimization. In its classic and most general formulation, we are given a complete bidirected graph with a non-negative integer cost c_{ij} associated to each arc ij, and we seek for a Hamiltonian circuit (or a Hamiltonian path, in the path version) in the graph, such that the sum of cost of its edges is minimum. Both versions (circuit and path, denoted MIN TSP and MIN PTSP, respectively) are **NP**-hard and hard to approximate within constant ratio (under usual complexity class hypotheses), even if the costs are symmetric [9, 12].

A natural variant of MIN TSP (resp., MIN PTSP) occurs when the costs induce a metric, ie in the case that, apart from being symmetric, they satisfy the triangular inequality: $\forall i \neq j \neq k \neq i$, $c_{ij} + c_{jk} \geq c_{ik}$.

This problem, denoted MIN ΔTSP (resp., MIN ΔPTSP) remains **NP**-hard and it is not approximable within a ratio better than 220/219 unless **P** = **NP** [16], while the best approximation ratio known to this day is still $\frac{3}{2}$, obtained by the simple algorithm shown in 1976 by N. Christofides[4].

For MIN ΔPTSP Hoogeveen[10] showed in 1991 polynomial approximation algorithms of ratios $\frac{3}{2}$ when there is at most one prespecified path endpoint and $\frac{5}{3}$ if both endpoints are given, respectively.

In the last years, the exploration of MIN ΔTSP and its path variant gained anew high interest thanks to some ground-breaking results for the special case of MIN $\mathcal{G}$TSP ie when the metric is defined through a graph G (every c_{ij} being defined respectively as the shortest path length between i and j in the graph). In this vein, Gamarnik *et al.* [6] have managed to provide a slightly better than Christofides' ratio ($\frac{3}{2} - \frac{5}{389}$) for the case of MIN $\mathcal{G}$TSP on 3edge-connected cubic graphs; their result has been improved to $\frac{4}{3}$ and generalized to all cubic or subcubic (with better ratio of $\frac{7}{5}$) graphs by Boyd *et al.* [2]. Mömke and Svensson have revisited the "spanning tree+parities correcting minimum matching" idea of Christofides, by redesigning in an ingenious way the matching step, so that the parity of odd degree vertices in the spanning tree can be corrected also by edge deletions; thus they obtained a ratio of $\frac{14(\sqrt{2}-1)}{12\sqrt{2}-13} \approx 1,461$ for MIN $\mathcal{G}$TSP and $3 - \sqrt{2} + \epsilon \leq 1,587$ for the path version (with prespecified endpoints) [14]. Mucha gave a refined analysis of their algorithm and showed that actually it guarantees a ratio of $\frac{13}{9}$ [15].

Quite recently, An *et al.* have improved upon the $\frac{5}{3}$ ratio for the path MIN ΔTSP problem where both endpoints are given; they used randomized rounding of LP values techniques to obtain a ratio of $\frac{1+\sqrt{5}}{2} \approx 1,625$ [1]. Finally, Sebő and Vygen [19] have proved a new approximation algorithm for MIN $\mathcal{G}$TSP (resp., MIN $\mathcal{G}$PTSP) on 2-edge connected graphs, with ratio $\frac{7}{5}$ (resp., $\frac{3}{2}$). Their algorithm make use of the techniques developed by Mömke and Svensson, but they start from a different spanning graph found by an appropriate ear decomposition of the graph.

1.1 Generalizations to multiple salesmen

MIN k-TSP is a generalization of MIN TSP , in which a minimum cost covering of the graph by k cycles is sought. Several variants of MIN k-TSP have been studied, though less extensively than the original MIN TSP problem; see for example [3, 20, 11]. We give below a succinct survey of results related to this topic.

Frieze [5] has been among the first to study a version of MIN k-TSP, in the metric case (denoted MIN k-ΔTSP). He has shown a $\frac{3}{2}$-approximation algorithm for a variant where the objective is to span all vertices by k non-trivial simple cycles having in common one given vertex, in such a way that the total cost of the cycles is minimum.

Recently, Xu et al [20] have developed a novel $\frac{3}{2}$-approximation algorithm for the variant of MIN k-ΔTSP where k vertices are given and the objective is to span all vertices by by k non-trivial simple cycles of minimum total cost, such that every cycle cycle contains one prespecified vertex.

Less is known for the path versions of MIN k-ΔTSP (denoted in a similar way by MIN k-ΔPTSP): for the path version of the Frieze's problem, ie when all paths have a common prespecified endpoint, Rathinam and Sengupta [3] have shown a $\frac{5}{3}$-approximation algorithm. For the case where for each of the sought k paths one endpoint is given, no polynomial approximation algorithm of ratio better than 2 is known: a straightforward 2-approximate algorithm is proved by Rathinam *et al.* [17]; in the same paper they present a $\frac{3}{2}$ approximation for the special case of $k = 2$.

In this paper, we study the multiple salesmen path problem when the distances are defined by a graph metric and no path endpoints are specified; the problem is formally defined in Section 2. We show that the construction of a connected T-join by Sebő et Vygen [19] yields directly a $\frac{3}{2} + o(1)$-approximation algorithm for this problem. The aforementioned result is established especially when the graph is 2-edge-connected.

2 Definitions and preliminaries

This paper deals with indirected simple graphs and multigraphs. As usually, such a graph G will be defined as an ordered pair (V, E) of its vertex set V and edge set E, respectively; when necessary multigraphs will be defined accordingly, by the ordered pair of their vertex set and edge list. In particular, the multigraph obtained by doubling the edges of another simple graph or multigraph G, will be denoted by $2G$. $|V|$, will be denoted by n if there is no risk of confusion. The size of a path is considered to be the number of edges in it. Let $V' \subseteq V$ ($E' \subseteq E$, respectively). The *induced* subgraph of G on V' (on E', respectively)), denoted by $G[V']$ (G[E'], respectively) is $(V', \{\{v, u\} \in E | v, u \in V'\})$ $((\{v \in V | \exists\{v, u\} \in E'\}, E')$, respectively). Let $G = (V, E)$ be a graph (simple or not) and E' a part of E. The degree of $v \in V$ with respect to E', denoted by $\delta_{E'}(v)$ (when $E' = E$, simply by $\delta(v)$) is the number of edges in E' having v as extremity. A graph G is *connected* if for any pair of its vertices v, u there is a path in the graph connecting v et u. $G = (V, E)$ is *2-edge connected* if for any of its edges e, $G(V, E \setminus e)$ remains connected.

2.1 T-joins

Let $G = (V, E)$ be a multigraph and $T \subseteq V$. A T-join is a list of edges J contained in E such that $\forall v \in T$, $\delta_J(v)$ is odd and $\forall v \in V \setminus T$, $\delta_J(v)$ is pair. The size of a T-join is $|J|$. Clearly, there can be no T-joins with $|T|$ impair.

It is also clear, that for any $T \subseteq V$ of pair size there is a minimum T-join in $2G$; moreover it can be computed in polynomial time [19]. In a similar manner, it is easy to see that if G is connected, for any $T \subseteq V$ of pair size there is a minimum *connected* T-join in $2G$; however finding it is not easy: notice that a minimum connected $\emptyset$-join of size n is a Hamiltonian cycle, while an $\{s, t\}$-connected join, for some $s, t \in V$, of size $n - 1$ is a Hamiltonian s-t path.

In [19], Sebő *et al.* prove a polynomial algorithm for finding connected T-joins in $2G$, where G is 2-edge-connected, with approximation ratio $\frac{3}{2}$. Their algorithm makes use of ear decomposition techniques and for the special cases of $T = \emptyset$, $|T| = 2$, its ratio becomes $\frac{7}{5}$ and $\frac{3}{2}$, respectively, thus yielding the best ratios for a vast family of instances of MIN $\mathcal{G}$TSP and MIN $\mathcal{G}$PTSP .

As already mentioned, in what follows we rely upon their approach to show approximation ratios for the k-path MIN ΔTSP problem for metrics given by 2-edge-connected graphs, which is formally defined below.

2.2 The problem

The multiple salesmen path problem when the distances are defined by a graph metric (without prespecified path endpoints), denoted by MIN k-$\mathcal{G}$PTSP is formally defined as follows:

Definition 1. [MIN k-$\mathcal{G}$PTSP] *Let $G = (V, E)$ be a connected graph. Find k paths spanning all vertices of G with the least possible total number of edges.*

Notice that, since paths need not to be simple, there is always a solution of size $\leq 2(n-k)$, that can be found for example by cutting into k paths an Eulerian cycle in a partial multi-graph of $2G$.

Rigorously speaking, MIN k-$\mathcal{G}$PTSP is rather a family of problems, as k is a positive integer constant independent of the input, but we will slightly abuse terms, referring to it as a single problem whenever there is no risk of confusion. It is easy to see that MIN k-$\mathcal{G}$PTSP is **NP**-hard.

Indeed, for any G and k, we can decide HAMILTONIAN PATH(G), simply by checking whether MIN k-$\mathcal{G}$PTSP(G') for the given k has a solution of size $nk+k+1$, for a suitably defined, G-dependent G' which is built in polynomial time as follows:

- Take k copies of G, denoted $G_1, \ldots, G_k$, respectively;

- add to each G_i vertices s_i, x_i, u_i, t_i and the edges $\{s_i, x_i\}, \{y_i, t_i\}$, $\{\{x_i, v\}, \{y_i, v\} | v \in V(G_i)\}$

- add a hub vertex h and the edges $\{\{s_i, h\}, \{t_i, h\} | 1 \leq i \leq k\}$

It is straightforward to see that if G is Hamiltonian, then MIN k-$\mathcal{G}$PTSP(G') for k paths has a solution of size $nk + k + 1$, consisting of the Hamiltonian path on each G_i extended from each endpoint to cover s_i and t_i and for G_1 extended further from, say s_1 to h. The inverse direction also holds: if there is a solution $P_1, \dots, P_k$ of size $nk + k + 1$ for MIN k-$\mathcal{G}$PTSP(G'), then the P_is have to be simple and disjoint since they have to span $nk + 2k + 1$ vertices in total. Call a path P *exclusive* if $\exists i : V(P) \subseteq V(G_i) \cup \{s_i, t_i\}$. By the construction a non-exclusive path must contain h, hence there will be exactly one such path; if it does not cover entirely any G_i, then $k-1$ exclusive paths have to cover $V(G_1), \dots, V(G_k)$, which is impossible by the pigeonhole principle.
Thus the following proposition holds:

Proposition 1. MIN k-$\mathcal{G}$PTSP *is* **NP**-*hard.*

Up to slight modifications of the construction of G', the above proof can also be used to show the **NP**-hardness of MIN k-$\mathcal{G}$PTSP even in the case where the sought paths have one or both of their endpoints fixed.

3 The algorithm

Lemma 1. *Let opt*(G) *be an optimal solution of* MIN k-$\mathcal{G}$PTSP *for some connected* G*. Then there is a connected T-join in 2G of size* $\leq |opt(G)| + 2(k-1)$

Proof. Let opt(G) $= \{P_1, P_2, ..., P_k\}$. Let us consider the multigraph $P = P_1 + P_2 + ... + P_k$ with vertex set the union of the vertex sets of the P_is and edge list the concatenation of their respective edge lists. Let $C_1, C_2, ..., P_l$ ($l \leq k$) be the connected components of P.
The connectedness of G implies that there are l-1 edges $e_1, e_2, ..., e_{l-1}$ in E that make $P_1 + P_2 + ... + P_k + e_1 + e_2, ... + e_l$ connected. Apart from being connected, in the multigraph $G' = P_1 + P_2 + ... + P_k + 2e_1 + 2e_2 + .. + 2e_l$ where $2e_i$ is a duplicate copy of e_i every vertex has the same degree parity as in P. It is also easy to see

that in P (hence, in G') there are at most $2k$ vertices of odd degree: it is

$$\forall v \in V(G),\ \delta_{E(P)}(v) = \sum_{1}^{k} \delta_{E(P_i)}(v) \Rightarrow$$

$$\delta_{E(P)}(v) \mod 2 = \sum_{1}^{k} \left\{\delta_{E(P_i)}(v) \mod 2\right\} \mod 2 \leq \sum_{1}^{k} \left\{\delta_{E(P_i)}(v) \mod 2\right\} \Rightarrow$$

$$\sum_{v \in V} \left\{\delta_{E(P)}(v) \mod 2\right\} \leq \sum_{v \in V} \left(\sum_{1}^{k} \left\{\delta_{E(P_i)}(v) \mod 2\right\} \right) = 2k$$

Let now G'' be a partial subgraph of G′ obtained in the following manner: for each multiedge of odd arity, leave a single copy and for each edge of even arity leave 2 copies. Then $G'' = (V, J)$ is a connected partial multigraph of 2G and at most $|\bigcup_{i=1}^{k}\{s_i, t_i\}|$ are of impair degree with respect to J. It is straightforward to see that this edge-copies merging cannot alter the degree parities of the vertices in G'' with respect to the ones in G'. Hence G'' is a connected T-join in 2G for some T with $|T| \leq 2k$ and its size is $\leq |\mathrm{opt}(G)| + 2l \leq |\mathrm{opt}(G)| + 2(k-1)$ □

The algorithm consists of the following steps:

1. Compute, using ear decomposition of G and edge completion as described in [19], a $\frac{3}{2}$-approximation of a minimum cost connected T-join for all $\binom{n}{0}, \binom{n}{2}, \ldots, \binom{n}{2k}$ possible $T \subset V$.
2. Find an eulerian k path-traversal for each T-join (this can be done simply, for example by adding a hub vertex connected to all vertices of T and by computing an Eulerian tour in the resulting graph; removing the occurences of the hub and further cutting if necessary, will give a solution of k paths corresponding to each join).
3. Return the best solution among the ones computed in Step 2.

k being a constant, this can be done in polynomial time. Moreover, the following theorem holds:

Theorem 1. [19] *Given a 2-edge-connected graph $T \subseteq V$, $|T| = 2k$ with k constant, a connected T-join of cost $\frac{3}{2}$ of the optimal can be found in $2G$ in polynomial time.*

Thus, we can prove the following:

Theorem 2. *There is a polynomial* $\frac{3}{2} + o(1)$*-approximation algorithm for* MIN k-$\mathcal{G}$PTSP *on 2-edge-connected graphs.*

Proof. Let $\tau(T, G)$ be the size of a T-join computed by the algorithm as in [19]. By Lemma 1, there is a T^*, $|T^*| \leq 2k$ such that there is a connected T^*-join of size $\sigma(T^*, G) \leq \text{opt}(G) + 2(k-1)$. By the construction, the above algorithm will always return a solution s of total cost less than or equal to $\tau(T^*, G)$ which by Theorem 1 is less than or equal to $\frac{3}{2}\sigma(T^*, G)$. Thus, $\text{cost}(s) \leq \frac{3}{2}\sigma(T^*, G) \leq \frac{3}{2}(\text{opt}(G) + 2(k-1)) \Rightarrow \text{cost}(s) \leq \frac{3}{2}\text{opt}(G) + O(1)$. □

4 Discussion

We conjecture that a similar result can be established for the case where the paths have to start from given vertices. The problem where both endpoints are prespecified for each path seems to be even more difficult.

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