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# An axiomatic approach to social ranking under coalitional power relations

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## Abstract

In the literature of coalitional games, power indices have been widely used to assess the influence that a player has in situations where coalitions may be winning or losing. However, in many cases things are not so simple as that: in some practical situations, all that we know about coalitions is a relative comparison of strength. For instance, we know that a football team is stronger than another team, a political party is more reliable than another party, an evaluation committee is more representative than another one, and so on, but we are not able to determine which teams, parties, or committees share the characteristics to be winning (or losing) in general. Still, in those situations we could be interested to rank single individuals according to their ability to influence the relative strength of coalitions. In this direction, we introduce a different coalitional framework where we analyse a new notion of *ordinal* power “index” or social ranking by associating to each total preorder on the set of all coalitions (representing the relative power of coalitions) a ranking over the player set. We study some properties for this class of social rankings, and we provide axiomatic characterizations of particular ones showing close affinities with the classical Banzhaf index of coalitional games.

## 1 Introduction

In cooperative game theory, classical measures of agents’ power, like the Shapley index [9] or the Banzhaf index [1], are computed on the characteristic function of a game and, if additional information is available about the identities of the players and their interactions, may depend on some combinatorial structures describing which coalitions are more likely to form [5, 6]. In practical situations, however, the information concerning both the power and the effective cooperation possibilities of coalitions, are not easily accessible, and may concern hardly quantifiable factors like bargaining abilities, moral and ethical codes and other “psychological” attributes [3].

For example, in addition to what it can gain by itself, a coalition may obtain some more “power” by threatening not to cooperate with other players and causing them losses [3]. This is what happened in Italy after the recent election which took place on 24-25 February 2013 for the determination of the 630 members of the Chamber of Deputies and the 315 members of the Senate of the Italian Republic. The outcome of the election was that the centre-left

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alliance, led by the Democratic Party, obtained a clear majority of seats in the Chamber of Deputies, while in the Senate, no political group or party won an outright majority. More precisely, in order to form a coalition government holding the majority of seats in the Senate of the Republic, the Center-left alliance needed to make a coalition with the Center-right alliance or, alternatively, with the anti-establishment Five Star Movement. Holding the majority of the Chamber of Deputies, the centre-left alliance was clearly more powerful of the other parties, while the Center-right alliance and the Five Star Movement seemed to share the same power. On the other hand, according to the electoral system of the Italian Parliament, no single party was a winning one in the Senate of the Republic, and there was no unambiguous way to quantify how much power each party effectively had. Eventually, a coalition government between the Centre left and Centre right was formed, because of the refuse of the Five Star Movement to make a coalition with the Center-left alliance, that in turn accused a big loss in terms of party's public support consequent to the agreement with its historical antagonist. But how much important was the threat of the Five Star Movement not to cooperate with the Center-left alliance? Were the Five Star Movement and the Center-right alliance really equally powerful?

Face to the practical difficulty of establishing a quantitative estimation of the strength of coalitions, as in the Italian political situation described above, the objective of this paper is to determine how to rank the agents of a coalitional situation according to their ability to form (or to threat) alliance, and when only the relative comparison of power between coalitions is considered.

In a similar direction, recently the authors of [8] have introduced a model of coalition formation where the relative strength of disjoint coalitions is represented by an exogenous binary relation (namely, a *power relation*), and where the players build the society (represented by a partition of the player set) driven by the maximization of their position in the *social ranking* (a linear order over the players). A social ranking in [8] is computed over each partition of the player set in the following way: player  $i$  is ranked higher than  $j$  if  $i$  belongs to a more powerful coalition than  $j$  in the partition or, if they belong to the same coalition, the singleton  $\{i\}$  is more powerful than the singleton coalition  $\{j\}$ , with respect to the original power relation.

In this paper, our goal is not to describe the process of coalition formation. Here we are interested in providing an analytical method to describe how the relative comparison of the strength of coalitions may influence the ranking of agents in the society. Therefore, we define a social ranking as a map assigning to each total preorder on the set of all coalitions, a total preorder on the set of players. Differently from the model of [8], where a power relation is defined as a binary relation over disjoint coalitions, our power relation allows for the comparison of each pair of coalitions, even if their intersection is nonempty. This is an important aspect, because it allows for considering the potential threats inside a group. Moreover, we axiomatically characterize social rankings by means of properties dealing with the ordinal structure of power relations. The first property is the *dominance axiom*, which states that a player  $i$  is ranked better than  $j$  if, for every coalition  $S$ , the number of coalitions more powerful

than  $S$  and containing  $i$  is higher than the number of those containing  $j$ . The interpretation of this property is clear: whatever coalition  $S$  is going to form, a player with more opportunities to form stronger coalitions should be ranked higher than another with less. Roughly speaking, players with a systematically larger power of threatening should be ranked higher. The second property, namely the *additivity axiom*, allows for the composition of power relations with opposite social ranking. This composition is ruled out by the “strength” of the opposite social rankings, that is a measure of the “average” capacity of players to threaten coalitions. Surprisingly, on the class of all power relations over coalitions, a social ranking that satisfies both the dominance and the additivity axioms coincides with the ranking provided by the Banzhaf value of particular coalitional games related to the numerical representation of the power relation.

In next section we recall some notations and definitions about total preorders and coalitional games. Section 3 deals with the notion of dominance for social rankings and provides some characterizations for special classes of power relations. Section 4 introduces the notion of additivity for power relations and the composition of social rankings, and provides the axiomatic characterization of a social ranking satisfying both properties. Section 5 concludes with some further directions.

## 2 Preliminaries

A *binary relation*  $R$  on a finite set  $N = \{1, \dots, n\}$  is a collection of ordered pairs of elements of  $N$  and is denoted by  $R \subseteq N \times N$ . For each  $x, y \in N$ , the more familiar notation  $xRy$  will be used instead of the more formal  $(x, y) \in R$ . The following are some standard properties for a binary relation  $R \subseteq N \times N$ . *Reflexivity*: for each  $x \in N$ ,  $xRx$ ; *transitivity*: for each  $x, y, z \in N$ ,  $xRy$  and  $yRz \Rightarrow xRz$ ; *totality*: for each  $x, y \in N$ ,  $x \neq y \Rightarrow xRy$  or  $yRx$ ; *antisymmetry*: for each  $x, y \in N$ ,  $xRy$  and  $yRx \Rightarrow x = y$ . A reflexive, transitive, total binary relation is called *total preorder*. A total preorder that also satisfies antisymmetry is called *linear order*. In the following we denote the fact that it is not true that  $xRy$  with the notation  $\neg(xRy)$ . We denote by  $\mathcal{T}^N$  and  $\mathcal{P}^N$  the set of all total binary relations and all total preorders, respectively, on a finite set  $N$ . A *total preorder* on  $2^N$  is a reflexive, transitive and total binary relation  $\succ \subseteq 2^N \times 2^N$ . Often we will use the standard notation  $S \succ T$  to denote the fact that  $S \succ T$  and  $\neg(T \succ S)$ . Given a set  $T \in 2^N$ , we denote by  $[T]$  the *indifference class* of  $T$ , i.e.  $[T] = \{S \in 2^N \mid S \succ T \text{ and } T \succ S\}$ , and by  $|[T]|$  its cardinality. Given a set  $T \in 2^N$  such that there exists  $B$  with  $T \succ B$ , we shall denote by  $T^\sigma$  an element of  $2^N$  such that  $T \succ T^\sigma$  and there is no  $C$  such that  $T \succ C \succ T^\sigma$ .

Given a total preorder  $\succ \in \mathcal{P}^{2^N}$ , consider a bijection  $\theta : \{1, \dots, 2^n\} \rightarrow 2^N$  such that

$$S \succ T \Rightarrow \theta^{-1}(S) < \theta^{-1}(T),$$

for every  $S, T \in 2^N$ . Now, for each  $i \in N$ , let  $\Gamma^i(\succ)$  be a  $2^n$ -vector of natural numbers such that the  $k$ -th component represents the number of coalitions

containing  $i$  which are in relation with  $\theta(k)$ , i.e.

$$\Gamma_k^i(\succ) = |\{S \in 2^{N \setminus \{i\}} : S \cup \{i\} \succ \theta(k)\}|$$

for each  $k = 1, \dots, 2^n$ . Note that vector  $\Gamma(\succ)$  does not depend on the choice of the bijection  $\theta$ , since  $\Gamma_k^i(\succ) = \Gamma_l^i(\succ)$  for every  $k, l$  such that  $\theta(k) \succ \theta(l)$  and  $\theta(l) \succ \theta(k)$ .

A total preorder is called *dichotomous* if there is a partition of  $2^N$  in two indifference classes, say  $\mathcal{G}$  and  $\mathcal{B}$ , such that each element in the class  $\mathcal{G}$  is preferred to each element in  $\mathcal{B}$ . Given a total preorder  $\succ \in \mathcal{P}^{2^N}$ , for each  $T \in 2^N$  we denote by  $\succ_T$  the dichotomous total preorder on  $2^N$  such that the class of most preferred elements is defined as  $\mathcal{G}_T := \{S \in 2^N : S \succ T\}$ , and the less preferred one as  $\mathcal{B}_T := 2^N \setminus \mathcal{G}_T$ . We shall say that  $\succ_T$  is a dichotomous total preorder associated to  $\succ$  on  $T$  for each  $T \in 2^N$ .

Now, we provide some basic definitions about coalitional games. A *coalitional game* on a finite set  $N$  of *players* is a pair  $(N, v)$ , or simply  $v$ , where the *characteristic function*  $v$  is a map  $v : 2^N \rightarrow \mathbb{R}$  assigning to each *coalition*  $S \subseteq N$  a real value, and with  $v(\emptyset) = 0$ . A total preorder  $\succ$  on  $2^N$  naturally induces a coalitional game for each utility function  $v$  *representing*  $\succ$  (such that  $v(\emptyset) = 0$ ), i.e.  $v(S) \geq v(T) \Leftrightarrow S \succ T$  for each  $S, T \in 2^N$ . We shall denote by  $V(\succ)$  the set of all  $v$  representing the total preorder  $\succ$ .

Let  $\succ \in \mathcal{P}^{2^N}$ . We define the *canonical game* representing  $\succ$  as the coalitional game  $\hat{v} \in V(\succ)$  such that  $\hat{v}(T) - \hat{v}(T^\sigma) = |[T]|$  for each  $T \in 2^N$  (with  $\hat{v}(\emptyset) = 0$ ). For each  $v \in V(\succ)$  and each  $S, T \in 2^N$ , consider the game  $v_T^*$  such that:

$$\text{if } T \succ \emptyset, \quad v_T^*(S) = \begin{cases} \frac{1}{|[T]|} (v(T) - v(T^\sigma)) & \text{if } S \succ T, \\ 0 & \text{otherwise,} \end{cases} \quad (1)$$

$$\text{if } \emptyset \succ T, \quad v_T^*(S) = \begin{cases} \frac{1}{|[T]|} (v(T^\sigma) - v(T)) & \text{if } T \succ S, \\ 0 & \text{otherwise.} \end{cases} \quad (2)$$

As also remarked in [2], it is easy to check that  $v_T^*$  induces the dichotomous total preorder  $\succ_T$  associated to  $\succ$  on  $T$ , for every  $T \in 2^N$ , and that  $v = \sum_{T \in 2^N} v_T^*$ . For example, according to relations (1) and (2), respectively, with the canonical game  $\hat{v}$  in the role of  $v$ , we have that  $\hat{v}_T^*(S) \in \{0, 1\}$  if  $T \succ \emptyset$ , and  $\hat{v}_T^*(S) \in \{0, -1\}$  if  $\emptyset \succ T$ , for each  $S, T \in 2^N$ .

The *Banzhaf value* [1] a coalitional game  $(N, v)$  is defined as the  $n$ -vector  $\beta(v) = (\beta_1(v), \dots, \beta_n(v))$ , such that for each  $i \in N$

$$\beta_i(v) = \frac{1}{2^{n-1}} \sum_{S \in 2^{N \setminus \{i\}}} (v(S \cup \{i\}) - v(S)). \quad (3)$$

Note that the difference between the Banzhaf value of two players  $i$  and  $j$  in  $N$  can be written as the following relation

$$\beta_i(v) - \beta_j(v) = \frac{1}{2^{n-2}} \sum_{S \in 2^{N \setminus \{i, j\}}} (v(S \cup \{i\}) - v(S \cup \{j\})). \quad (4)$$

for every  $i, j \in N$  and where  $2^{N \setminus \{i, j\}}$  is the set of all subsets of  $N$  which do not contain neither  $i$  nor  $j$ . For further details on how the Banzhaf value ranks the players of a coalitional game see [2, 4].

### 3 Social ranking and the dominance axiom

In the remaining of the paper, we interpret a total preorder  $\succsim$  on  $2^N$  as a *power relation* on  $2^N$ , that is, for each  $S, T \in 2^N$ ,  $S \succsim T$  stands for ‘ $S$  is considered at least as powerful as  $T$  according to  $\succsim$ ’. We call the map  $\rho : \mathcal{P}^{2^N} \rightarrow \mathcal{T}^N$ , assigning to each power relation on  $2^N$  a total binary relation on  $N$ , a *social ranking solution* or, simply, a *social ranking*. Then, given a power relation  $\succsim$ , we will interpret the total binary relation  $\rho(\succsim)$  associated to  $\succsim$  by the social ranking  $\rho$ , as the relative strength of players in the society represented by the power relation  $\succsim$ . Precisely, for each  $i, j \in N$ ,  $i\rho(\succsim)j$  stands for ‘ $i$  is considered at least as strong as  $j$  according to the social ranking  $\rho(\succsim)$ ’. Note that we require that  $\rho(\succsim)$  is a total binary relation, that is we always want to express the relative comparison of two agents, but we do not exclude *a priori* the possibility of cycles in the relative comparison of strength among agents.

In the following, for every  $i, j \in N$ , we will say that  $\Gamma^i(\succsim)$  *dominates*  $\Gamma^j(\succsim)$  (denoted by  $\Gamma^i(\succsim) \geq \Gamma^j(\succsim)$ ) iff  $\Gamma_k^i(\succsim) \geq \Gamma_k^j(\succsim)$  for each  $k = 1, \dots, 2^N$ . We can now introduce the first property for social rankings.

**Axiom 1 (DOM).** *A social ranking  $\rho$  satisfies the dominance property iff for each  $i, j \in N$  and  $\succsim \in \mathcal{P}^{2^N}$ ,*

$$\Gamma^i(\succsim) \geq \Gamma^j(\succsim) \Rightarrow i\rho(\succsim)j.$$

The DOM axiom states that if player  $i$  has more possibilities than  $j$  to form coalitions more powerful than  $S$ , for every possible coalition  $S \in 2^N$  which is going to form, than  $i$  should be ranked higher than  $j$  according to the social ranking. The intuition behind this property is that if  $\Gamma^i(\succsim)$  dominates  $\Gamma^j(\succsim)$ , for some  $i$  and  $j$  in  $N$ , than player  $i$  has a larger power of threatening  $S$  than  $j$ , whatever coalition  $S$  is currently formed, since  $i$  has more opportunities than  $j$  to create coalition more appealing than  $S$ .

**Example 1.** *Consider the power relation  $\succ$  such that  $\{1, 2, 3\} \succ \{2\} \succ \{1, 3\} \succ \{1, 2\} \succ \{3\} \succ \{1\} \succ \emptyset \succ \{2, 3\}$ . We have that  $\Gamma^1(\succ) = (1, 1, 2, 3, 3, 4, 4, 4)$ ,  $\Gamma^2(\succ) = (1, 2, 2, 3, 3, 3, 3, 4)$  and  $\Gamma^3(\succ) = (1, 1, 2, 2, 3, 3, 3, 4)$ . Note that both  $\Gamma^1$  and  $\Gamma^2$  dominate  $\Gamma^3$ , whereas neither  $\Gamma^1$  dominates  $\Gamma^2$  nor  $\Gamma^2$  dominates  $\Gamma^1$ . If a social ranking  $\rho$  satisfies DOM, then we have that both  $1\rho(\succ)3$  and  $2\rho(\succ)3$ , but we can say nothing about the relative comparison between 1 and 2 in the social ranking  $\rho(\succ)$ . Moreover, note that the dominance is purely ordinal, in the sense that no considerations about the “strength” of the dominance is made to determine the social ranking. So, for instance, the fact that the score  $\sum_{k=1}^{2^n} \Gamma_k^1(\succ) = 22$  is larger than the score  $\sum_{k=1}^{2^n} \Gamma_k^2(\succ) = 21$ , does not play any role in the relative comparison between 1 and 2 with respect to  $\rho(\succ)$ .*

**Remark 1.** *If  $\succ \in \mathcal{P}^{2^N}$  is such that for some  $i, j \in N$ ,  $\Gamma^i(\succ) \geq \Gamma^j(\succ)$ , then the relative score defined as  $s_{ij}(\succ) = \sum_{k=1}^{2^n} \Gamma_k^i(\succ) - \sum_{k=1}^{2^n} \Gamma_k^j(\succ)$  is not negative, and may be interpreted as an indication of the intensity of the dominance: so, the dominance of 1 over 3 seems to be stronger than the one of 2 over 3, according to  $s_{13}(\succ)$  and  $s_{23}(\succ)$ .*

The following lemma is useful to guarantee that  $s_{ij}(\succ) \geq 0$  implies the dominance of  $\Gamma^i(\succ)$  over  $\Gamma^j(\succ)$ .

**Lemma 1.** *Let  $\succ \in \mathcal{P}^{2^N}$ ,  $R \in \mathcal{T}^N$  and  $i, j \in N$  such that  $\Gamma^i(\succ) \geq \Gamma^j(\succ) \Leftrightarrow iRj$ . Let  $\rho$  be a social ranking which satisfies DOM. Then,*

$$i\rho(\succ)j \Leftrightarrow \Gamma^i(\succ) \geq \Gamma^j(\succ) \Leftrightarrow s_{ij}(\succ) \geq 0.$$

*Proof.* Since  $R$  is total, we have that either  $\Gamma^i(\succ) \geq \Gamma^j(\succ)$  or  $\Gamma^j(\succ) \geq \Gamma^i(\succ)$  for every  $i, j \in N$  then, by the DOM of  $\rho$ , we have that

$$\Gamma^i(\succ) \geq \Gamma^j(\succ) \Leftrightarrow i\rho(\succ)j,$$

and by Remark 1, we also have that

$$\Gamma^i(\succ) \geq \Gamma^j(\succ) \Leftrightarrow \sum_{k=1}^{2^n} \Gamma_k^i(\succ) \geq \sum_{k=1}^{2^n} \Gamma_k^j(\succ).$$

□

The following proposition shows that the DOM axiom is sufficient to characterize social rankings on dichotomous power relations, and the rankings are represented by the Banzhaf value.

**Proposition 1.** *Let  $\rho$  be a social ranking which satisfies DOM. Then, for each dichotomous total preorder  $\succ$  on  $2^N$  and each  $i, j \in N$ , we have that the following relations hold:*

- i)  $i\rho(\succ)j \Leftrightarrow |\{S \in 2^{N \setminus \{i,j\}} : S \cup \{i\} \in \mathcal{G}\}| \geq |\{S \in 2^{N \setminus \{i,j\}} : S \cup \{j\} \in \mathcal{G}\}|.$
- ii)  $i\rho(\succ)j \Leftrightarrow \beta_i(v) \geq \beta_j(v)$ , for every  $v \in V(\succ).$

where  $\mathcal{G}$  is the indifference class of most powerful coalitions according to  $\succ$ .

*Proof.* First, note that for each  $i \in N$ ,  $\Gamma_k^i(\succ) = 2^{n-1}$ , if  $k = |\mathcal{G}| + 1, \dots, 2^n$  (all coalitions containing  $i$  are at least as powerful as  $\theta(k)$ ), and  $\Gamma_k^i(\succ) = |\{S \in 2^N : S \not\ni i \text{ and } S \cup \{i\} \in \mathcal{G}\}|$ , otherwise. Then, it immediately follows that

$$\Gamma^i(\succ) \geq \Gamma^j(\succ) \Leftrightarrow |\{S \in 2^{N \setminus \{i\}} : S \cup \{i\} \in \mathcal{G}\}| \geq |\{S \in 2^{N \setminus \{j\}} : S \cup \{j\} \in \mathcal{G}\}|,$$

and since some elements in the sets on the right are in common, we can rewrite the previous relation as the following one:

$$\Gamma^i(\succ) \geq \Gamma^j(\succ) \Leftrightarrow |\{S \in 2^{N \setminus \{i,j\}} : S \cup \{i\} \in \mathcal{G}\}| \geq |\{S \in 2^{N \setminus \{i,j\}} : S \cup \{j\} \in \mathcal{G}\}|.$$

So, the binary relation  $R$  on  $N$  such that  $iRj \Leftrightarrow |\{S \in \Sigma_{ij} : S \cup \{i\} \in \mathcal{G}\}| \geq |\{S \in \Sigma_{ij} : S \cup \{j\} \in \mathcal{G}\}|$  is total and the statement (i) follows from Lemma 1.

To prove statement (ii), simply note that by relation (4), for game  $v$  such that  $v(S) = a$ , if  $S \in \mathcal{G}$ , and  $v(S) = b$ , otherwise, and  $a > b$ , we have that

$$\beta_i(v) - \beta_j(v) = \frac{a-b}{2^{n-2}} (|\{S \in \Sigma_{ij} : S \cup \{i\} \in \mathcal{G}\}| - |\{S \in \Sigma_{ij} : S \cup \{j\} \in \mathcal{G}\}|). \quad (5)$$

□

In order to introduce the connection between the Banzhaf value of coalitional games with social rankings that satisfy DOM, we need the following lemma.

**Lemma 2.** *Let  $\succsim$  be a total preorder on  $2^N$ . Then, for each  $i, j \in N$*

$$\Gamma^i(\succsim) \geq \Gamma^j(\succsim) \Leftrightarrow \Gamma^i(\succsim_T) \geq \Gamma^j(\succsim_T) \quad (6)$$

for every  $T \in 2^N$ .

*Proof.* First, note that for every  $i \in N$ ,  $T \in 2^N$  and  $k = |\mathcal{G}_T| + 1, \dots, 2^n$  we have that  $\Gamma_k^i(\succsim) = \Gamma_k^j(\succsim) = 2^{n-1}$  (since all sets of  $2^N$  are weakly preferred to elements in  $\mathcal{B}_T$  w.r.t.  $\succsim_T$ ). So,

$$\Gamma_k^i(\succsim_T) - \Gamma_k^j(\succsim_T) = 0$$

for every dichotomous preorder  $\succsim_T$  and every  $k = |\mathcal{G}_T| + 1, \dots, 2^n$ .

Then, note that for each  $i \in N$ ,

$$\Gamma_k^i(\succsim_T) = \Gamma_{|\mathcal{G}_T|}^i(\succsim_T) = \Gamma_{|\mathcal{G}_T|}^i(\succsim). \quad (7)$$

for every set  $T \in 2^N$  and for every  $k = 1, \dots, |\mathcal{G}_T|$ , and relations (6) remains proved.  $\square$

Now we can introduce the main result of this section, showing that the fact that player  $i$  dominates player  $j$  with respect to the power relation  $\succsim$  is equivalent to the fact that the Banzhaf value of player  $i$  is larger than the Banzhaf value of player  $j$  for every characteristic function  $v \in V(\succsim)$ .

**Theorem 1.** *Let  $\succsim \in \mathcal{P}^{2^N}$  and For each  $i, j \in N$*

$$\Gamma^i(\succsim) \geq \Gamma^j(\succsim) \Leftrightarrow [\beta_i(v) \geq \beta_j(v) \text{ for every } v \in V(\succsim)]. \quad (8)$$

*Proof.* First note that from Lemma 2 and Proposition 1 we have that

$$\Gamma^i(\succsim) \geq \Gamma^j(\succsim) \Leftrightarrow \beta_i(v_T) \geq \beta_j(v_T) \quad (9)$$

for every  $T \in 2^N$  and every  $v_T \in V(\succsim_T)$ . Moreover, we may write every game  $v \in V(\succsim)$  as the sum

$$v = \sum_{T \in 2^N} v_T^*,$$

for every  $v \in V(\succsim)$ , where  $v_T^*$ , for every  $T \in 2^N$ , is defined as in relations (1) and (2). Consequently, for every  $v \in V(\succsim)$  and every  $i \in N$ , for the additivity of the Banzhaf value we have that

$$\beta_i(v) = \sum_{T \in 2^N} \beta_i(v_T^*).$$

and thus, by relation (9),  $\Gamma^i(\succsim) \geq \Gamma^j(\succsim) \Rightarrow \beta_i(v) \geq \beta_j(v)$ .



Now, we want to prove the opposite implication; so, assume that  $\beta_i(v) \geq \beta_j(v)$  for each  $v \in V(\succneq)$ . First, we prove that  $\beta_i(v_T) \geq \beta_j(v_T)$  for every  $T \in 2^N$  and every  $v_T \in V(\succneq_T)$ . Suppose, on the contrary, that there exists some  $T \subset N$  and some  $v_T^* \in V(\succneq_T)$  such that  $\beta_j(v_T^*) - \beta_i(v_T^*) = \gamma > 0$ . Then take  $v' \in V(\succneq)$  and let  $\alpha := \beta_i(v') - \beta_j(v') \geq 0$ . Note that  $\alpha v_T^* + v'$  is still an element in  $V(\succneq)$ . On the other hand, by the additivity of the Banzhaf value

$$\begin{aligned} \beta_j(\alpha v_T^* + v') - \beta_i(\alpha v_T^* + v') &= \\ \alpha \beta_j(v_T^*) + \beta_j(v') - \alpha \beta_i(v_T^*) - \beta_i(v') &= \\ \alpha(\gamma - 1), & \end{aligned} \quad (10)$$

which yields a contradiction if  $\gamma > 1$ . Then, it remains proved that  $\beta_i(v_T) \geq \beta_j(v_T)$  for every  $T \in 2^N$  and every  $v_T \in V(\succneq_T)$ , and the proof follows by relation (9).  $\square$

**Corollary 1.** *Let  $\succneq \in \mathcal{P}^{2^N}$  and let  $\rho$  be a social ranking which satisfies DOM. Then, for each  $i, j \in N$*

$$[\beta_i(v) \geq \beta_j(v) \text{ for every } v \in V(\succneq)] \Rightarrow i\rho(\succneq)j. \quad (11)$$

## 4 An additive axiom for social rankings

In the previous section we have studied the effect of the DOM axiom in determining the social ranking on certain classes of power relations, and we have shown that a social ranking satisfying DOM is somehow related to the Banzhaf value of every game representing the power relation. However, the DOM axiom alone is not sufficient to unequivocally determine a social ranking on the class of all power relations  $\mathcal{P}^{2^N}$ , since in general (as illustrated in the previous section) the relation of dominance on  $\Gamma^i$  vectors is not total.

In this section we introduce a second property, namely the *additivity property*, that allows for the combination of social rankings with opposite relative comparison. In order to do that, we need to introduce some further notations. Given  $\succneq \in \mathcal{P}^{2^N}$ , we denote by

$$K^{\succneq} = \{\sqsubseteq \in \mathcal{P}^{2^N} : S \succneq T \Rightarrow S \sqsupseteq T, \text{ for all } S, T \in 2^N\}$$

the set of all power relations compatible with  $\succneq$ . On  $K^{\succneq}$  we define the binary operation  $\oplus : K^{\succneq} \times K^{\succneq} \rightarrow K^{\succneq}$  such that for each two elements  $\sqsupseteq, \sqsubseteq \in K^{\succneq}$  and all  $S, T \in 2^N$ ,

$$S \sqsupseteq T \text{ and } S \sqsubseteq T \Rightarrow S \oplus (\sqsupseteq, \sqsubseteq)T$$

and

$$\begin{aligned} (S \triangleright T \text{ and } S \sqsubseteq T) \\ \text{or} \\ (S \sqsupseteq T \text{ and } S \sqsubset T) \end{aligned} \Rightarrow (S \oplus (\sqsupseteq, \sqsubseteq)T) \text{ and } \neg(T \oplus (\sqsupseteq, \sqsubseteq)S),$$

(as usual, notation  $S \triangleright T$  and  $S \sqsubset T$  means, respectively,  $[S \sqsupseteq T \text{ and } \neg(T \sqsupseteq S)]$  and  $[S \sqsubseteq T \text{ and } \neg(T \sqsubseteq S)]$ ).

**Axiom 2 (ADD).** Let  $\succ \in \mathcal{P}^{2^N}$ . A social ranking  $\rho$  satisfies the additivity property iff for each  $i, j \in N$  and  $\succeq, \sqsupseteq \in K^\succ$  such that  $i\rho(\succeq)j$ , and  $j\rho(\sqsupseteq)i$ , we have that

$$s_{ij}(\succeq) \geq s_{ji}(\sqsupseteq) \Leftrightarrow i\rho(\oplus(\succeq, \sqsupseteq))j.$$

The ADD axiom states that if in two compatible power relations  $\succeq$  and  $\sqsupseteq$  the corresponding social rankings work in an opposite way in the comparison of agents  $i$  and  $j$ , then, in the power relation resulting from the sum  $\oplus(\succeq, \sqsupseteq)$ , the relative comparison of  $i$  and  $j$  is determined by the comparison of the relative score between  $i$  and  $j$  in the two original power relations  $\succeq$  and  $\sqsupseteq$ . As we already said in the previous section, the indication provided by the relative score is an indication of the intensity of the dominance relation on vectors  $\Gamma^i$ . Then, it seems natural to use the information provided by the relative score to measure the effect of the combination of opposite power relations on a social ranking that satisfies both the DOM and the ADD properties.

**Example 2.** Consider again the power relation presented in Example 1 and other two power relations  $\succeq, \sqsupseteq$ , where  $\succeq$  is such that  $\{1, 3\} \succeq \{2\}$  and  $\{2\} \succeq \{1, 3\}$  (i.e.,  $\{2\} \{1, 3\}$  belong to the same indifference class w.r.t. to  $\succeq$ ), and  $\{1, 2, 3\} \succ \{2\} \succ \{1, 2\} \succ \{3\} \succ \{1\} \succ \emptyset \succ \{2, 3\}$ , and where  $\sqsupseteq$  is a dichotomous power relation with the indifference class of preferred sets  $\mathcal{G} = \{\{1, 2, 3\}, \{2\}\}$ . Note that  $\succ \equiv \oplus(\succeq, \sqsupseteq)$ .

Consider a social ranking  $\rho$  that satisfies both DOM and ADD axioms. Note that,  $\Gamma^1(\succeq) = (1, 2, 2, 3, 3, 4, 4, 4)$  and  $\Gamma^2(\succeq) = (1, 2, 2, 3, 3, 3, 3, 4)$  so, by the DOM property,  $1\rho(\succeq)2$ , whereas  $\Gamma^1(\sqsupseteq) = (1, 1, 4, 4, 4, 4, 4, 4)$  and  $\Gamma^2(\sqsupseteq) = (2, 2, 4, 4, 4, 4, 4, 4)$ , so, again by the DOM property,  $2\rho(\sqsupseteq)1$ . Moreover the relative score  $s_{12}(\succeq) = 23 - 21 = 2$ , while  $s_{21}(\sqsupseteq) = 28 - 26 = 2$ . Then, by the ADD property,  $1\rho(\succ)2$  and  $2\rho(\succ)1$ .

The decomposition of a power relation provided in the previous example can be done in general, as illustrated in the following lemma.

**Lemma 3.** Let  $\succ \in \mathcal{P}^{2^N}$  and  $i, j \in N$ . There exist  $\succeq, \sqsupseteq \in K^\succ$  such that  $\Gamma^i(\succeq) \geq \Gamma^j(\succeq)$ ,  $\Gamma^j(\sqsupseteq) \geq \Gamma^i(\sqsupseteq)$  and  $\succ \equiv \oplus(\succeq, \sqsupseteq)$

*Proof.* Consider the sets  $\Theta^{ij} = \{T \in 2^N \mid \Gamma^i(\succ_T) \geq \Gamma^j(\succ_T)\}$  and  $\Theta^{ji} = \{T \in 2^N \mid \Gamma^j(\succ_T) \geq \Gamma^i(\succ_T)\}$ , where, as usual,  $\succ_T$  are the dichotomous preorders associated to  $\succ$  for all  $T \in 2^N$ . By Proposition 1, it is easy to check that  $\Theta^{ij} \cup \Theta^{ji} = 2^N$  and  $\Theta^{ij} \cap \Theta^{ji} = \{T \in 2^N \mid \Gamma^i(\succ_T) = \Gamma^j(\succ_T)\} \neq \emptyset$  (least powerful coalitions always belong to  $\Theta^{ij} \cap \Theta^{ji}$ ).

Now, let  $v_T^*$ , for each  $T \in 2^N$ , be the characteristic functions defined by relations (1) and (2) and for some  $v \in V(\succ)$ . Consider the characteristic functions  $v^{ij} = \sum_{T \in \Theta^{ij}} v_T^*$  and  $v^{ji} = \sum_{T \in \Theta^{ji}} v_T^*$ , and the corresponding total preorders  $\succeq, \sqsupseteq \in K^\succ$  defined as follows:

$$S \succeq T :\Leftrightarrow v^{ij}(S) \geq v^{ij}(T), \quad (12)$$

and

$$S \sqsupseteq T :\Leftrightarrow v^{ji}(S) \geq v^{ji}(T), \quad (13)$$

for each  $S, T \in 2^N$ . Note that  $v^{ij} + v^{ji}$  is also an element of  $V(\succ)$  and, by definition,  $v^{ij} + v^{ji}$  is also an element of  $V(\oplus(\triangleright, \triangleleft))$ . So,  $\succ \equiv \oplus(\triangleright, \triangleleft)$ .  $\square$

Now, we introduce some properties of canonical games.

**Lemma 4.** *Let  $\succ \in \mathcal{P}^{2^N}$  and  $i, j \in N$ . Then,*

$$\beta_i(\hat{v}) - \beta_j(\hat{v}) = s_{ij}(\succ)$$

where  $\hat{v} \in V(\succ)$  is the canonical game representing  $\succ$ .

*Proof.* As in Lemma 3, consider Let the  $v_T^*$ , for each  $T \in 2^N$ , be the characteristic functions defined by relations (1) and (2) with the canonical game  $\hat{v}$  in the role of  $v$ . We have that

$$\begin{aligned} \beta_i(\hat{v}) - \beta_j(\hat{v}) &= \\ \sum_{k=1}^{2^n} (\beta_i(v_{\theta(k)}^*) - \beta_j(v_{\theta(k)}^*)) &= \\ \frac{1}{2^{n-2}} \left( \sum_{k=1}^{2^n} |\{S \in 2^N \setminus \{i\} : S \cup \{i\} \in \mathcal{G}_{\theta(k)}\}| - \right. \\ \left. \sum_{k=1}^{2^n} |\{S \in 2^N \setminus \{j\} : S \cup \{j\} \in \mathcal{G}_{\theta(k)}\}| \right) &= \\ \frac{1}{2^{n-2}} \left( \sum_{k=1}^{2^n} \Gamma_k^i(\succ_{\theta(k)}) - \sum_{k=1}^{2^n} \Gamma_k^j(\succ_{\theta(k)}) \right) &= \\ \frac{1}{2^{n-2}} \left( \sum_{k=1}^{2^n} \Gamma_k^i(\succ) - \sum_{k=1}^{2^n} \Gamma_k^j(\succ) \right) &= s_{ij}(\succ). \end{aligned} \tag{14}$$

$\square$

We can now introduce the main result of this section.

**Theorem 2.** *Let  $\succ \in \mathcal{P}^{2^N}$  and  $i, j \in N$ . Let  $\rho$  be a social ranking which satisfies DOM and ADD. Then,*

$$i\rho(\succ)j \Leftrightarrow \beta_i(\hat{v}_{\triangleright}) - \beta_j(\hat{v}_{\triangleright}) \geq \beta_j(\hat{v}_{\triangleleft}) - \beta_i(\hat{v}_{\triangleleft}),$$

where  $\hat{v}_{\triangleright} \in V(\triangleright)$  and  $\hat{v}_{\triangleleft} \in V(\triangleleft)$  are the canonical games representing  $\triangleright$  and  $\triangleleft$ , respectively, and where  $\triangleright, \triangleleft \in K^\succ$  are such that  $i\rho(\triangleright)j$ ,  $j\rho(\triangleleft)i$  and  $\succ \equiv \oplus(\triangleright, \triangleleft)$ .

*Proof.* Consider the total preorders  $\triangleright, \triangleleft \in K^\succ$  as defined in Lemma 3.

If  $\Gamma^i(\succ) \geq \Gamma^j(\succ)$ , then by the DOM axiom we have that  $i\rho(\triangleright)j$ . Moreover,  $\beta_i(\hat{v}_{\triangleleft}) - \beta_j(\hat{v}_{\triangleleft}) = 0$  and  $\beta_i(\hat{v}_{\triangleright}) - \beta_j(\hat{v}_{\triangleright}) = \beta_i(\hat{v}_{\succ}) - \beta_j(\hat{v}_{\succ}) \geq 0$ , where the inequality follows by Proposition 1.

Otherwise, if  $\neg(\Gamma^i(\succ) \geq \Gamma^j(\succ))$  and  $\neg(\Gamma^j(\succ) \geq \Gamma^i(\succ))$ , by the DOM axiom we have that  $i\rho(\triangleright)j$  and  $j\rho(\triangleleft)i$ , and by Lemma 4, we have that  $\beta_i(\hat{v}_{\triangleright}) - \beta_j(\hat{v}_{\triangleright}) = s_{ij}(\triangleright)$  and  $\beta_j(\hat{v}_{\triangleleft}) - \beta_i(\hat{v}_{\triangleleft}) = s_{ji}(\triangleleft)$ , which concludes the proof by means of the ADD axiom.  $\square$

Table 1: Canonical games representing  $\succeq, \sqsupseteq$  and  $\succ$ , respectively.

	$\{1, 2, 3\}$	$\{2\}$	$\{1, 3\}$	$\{1, 2\}$	$\{3\}$	$\{1\}$	$\emptyset$	$\{2, 3\}$
$\hat{v}_{\succeq}$	6	5	5	3	2	1	0	-1
$\hat{v}_{\sqsupseteq}$	2	2	0	0	0	0	0	0
$\hat{v}$	6	5	4	3	2	1	0	-1

**Example 3.** Consider again the power relation of Example 1. The canonical game  $\hat{v}_{\succeq} \in V(\succeq)$ ,  $\hat{v}_{\sqsupseteq} \in V(\sqsupseteq)$  and  $\hat{v} \in V(\succ)$  are shown in Table 1.

According to relation (4), the difference in Banzhaf values are the following:

$$\beta_1(\hat{v}_{\succeq}) - \beta_2(\hat{v}_{\succeq}) = \frac{1}{2}(\hat{v}_{\succeq}(\{1\}) - \hat{v}_{\succeq}(\{2\})) + \frac{1}{2}(\hat{v}_{\succeq}(\{1, 3\}) - \hat{v}_{\succeq}(\{2, 3\})) = 1,$$

$$\beta_1(\hat{v}_{\sqsupseteq}) - \beta_2(\hat{v}_{\sqsupseteq}) = \frac{1}{2}(\hat{v}_{\sqsupseteq}(\{1\}) - \hat{v}_{\sqsupseteq}(\{2\})) + \frac{1}{2}(\hat{v}_{\sqsupseteq}(\{1, 3\}) - \hat{v}_{\sqsupseteq}(\{2, 3\})) = 1,$$

and then, by Theorem 2, we can argue again that, if  $\rho$  satisfies both ADD and DOM properties, then  $1\rho(\succ)2$  and  $2\rho(\succ)1$ .

We conclude with some considerations about alternative formulations of the ADD axiom, using different criteria to evaluate the intensity of the dominance in opposite power relations.

**Axiom 3 (ADD\*).** Let  $\succ \in \mathcal{P}^{2^N}$ . A social ranking  $\rho$  satisfies the additivity\* property iff for each  $i, j \in N$  and  $\succeq, \sqsupseteq \in K^{\succ}$  such that  $\succ \equiv \oplus(\succeq, \sqsupseteq)$ ,  $i\rho(\succeq)j$ , and  $j\rho(\sqsupseteq)i$ , we have that

$$\sum_{k=1}^{2^n} \min(\Gamma_k^i(\succeq), \Gamma_k^i(\sqsupseteq)) \geq \sum_{k=1}^{2^n} \min(\Gamma_k^j(\succeq), \Gamma_k^j(\sqsupseteq)) \Leftrightarrow i\rho(\succ)j.$$

Then, the following characterization holds.

**Theorem 3.** Let  $\succ \in \mathcal{P}^{2^N}$  and  $i, j \in N$ . Let  $\rho^*$  be a social ranking which satisfies DOM and ADD\*. Then,

$$i\rho(\succ)j \Leftrightarrow \beta_i(\hat{v}) \geq \beta_j(\hat{v})$$

where  $\hat{v} \in V(\succ)$  is the canonical games representing  $\succ$ .

*Proof.* We simply note that  $\Gamma_k^i(\succ) = \min(\Gamma_k^i(\succeq), \Gamma_k^i(\sqsupseteq))$  for each  $i \in N$  and  $k = 1, \dots, 2^n$ . The remaining of the proof is similar to the proof of Theorem 2.  $\square$

**Example 4.** Consider the canonical game  $\hat{v}$  in Table 1. Note that,

$$\beta_1(\hat{v}) - \beta_2(\hat{v}) = \frac{1}{2}(\hat{v}(\{1\}) - \hat{v}(\{2\})) + \frac{1}{2}(\hat{v}(\{1, 3\}) - \hat{v}(\{2, 3\})) = 1.$$

Then, a social ranking  $\rho^*(\succ)$  satisfying DOM and ADD\* is such that  $1\rho^*(\succ)2$ ,  $1\rho^*(\succ)3$  and  $2\rho^*(\succ)3$ .

## 5 Concluding remarks

In this paper we have presented a preliminary approach to the problem of ranking the strength of agents in a coalitional situation where only a qualitative information about the relative power of coalitions is given.

As noticed, different notions of “intensity” of dominance, other than the one of relative score, could apply. For instance, following the example illustrated along the paper, one could argue that the dominance of 2 on 3 is stronger than the one of 1 on 3 because  $\{2\} \succ \{3\}$ , but  $\{3\} \succ \{1\}$ . Alternatively, it would be interesting to investigate the concept of “stronger than” relation introduced in the domain of preference representation with intervals [7].

Another interesting direction, is the analysis of more realistic classes of power relations, where, for instance, certain relative comparisons of power are not possible (e.g., for lack of information, incomparability of the social or the political position of the coalitions, etc.). Assuming that two coalitions  $S$  and  $T$  are not comparable does not imply that  $S$  and  $T$  cannot form or cannot be compared to other coalitions. This aspect further characterizes our approach from the theories about coalitional games with restrictions in cooperation [5].

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