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Abstract

Outranking relations such as produced by the Electre I or II or the Tactic methods are based on a concordance and non-discordance principle that leads to declaring that an alternative is "superior" to another, if the coalition of attributes supporting this proposition is "sufficiently important" (concordance condition) and if there is no attribute that "strongly rejects" it (non-discordance condition). Such a way of comparing alternatives is rather natural and does not require a detailed analysis of tradeoffs between the various attributes. However, it is well known that it may produce binary relations that do not possess any remarkable property of transitivity or completeness. The axiomatic foundations of outranking relations have recently received attention. Within a conjoint measurement framework, characterizations of reflexive *concordance-discordance* relations have been obtained. These relations encompass those generated by the Electre I and II methods, which are non-strict (reflexive) relations. A different characterization has been provided for strict (asymmetric) preference relations such as produced by Tactic. The goal of this paper is to analyze the relationships between reflexive and asymmetric outranking relations. Co-duality plays an essential rôle in our analysis. It allows to understand the correspondence between the previous characterizations. Making a step further, we provide a common axiomatic characterization for both types of relations. Applying the co-duality operator to concordance-discordance relations also yields a new and interesting type of preference relation that we call concordance relation with bonus. The axiomatic characterization of such relations results directly from co-duality arguments.

Keywords: Multiple criteria analysis, Concordance, Discordance, Outranking methods, Conjoint measurement, Nontransitive preferences, Veto, Bonus, Co-duality

Contents

1 Introduction

Most outranking methods, including the well known Electre methods (Roy, 1968, Roy and Bertier, 1973), base the comparison of alternatives on the so-called *concordance & non-discordance* principle. It leads to accepting the proposition that an alternative is "superior" to another if the following two conditions are fulfilled:

- *concordance condition*: the coalition of attributes supporting this assertion is "sufficiently important",
- *non-discordance condition*: there is no attribute that "strongly rejects" this assertion.

Stating that an alternative is "superior" to another may have two different meanings. In Electre methods, "superior" means "not worse" i.e., "at least as good as". Such methods aim at building a *reflexive* preference relation that is interpreted as an "at least as good as" relation. In general, such relations may lack nice transitivity or completeness properties (on these issues, see Bouyssou, 1992, 1996). In previous work (BP09b)[1], we have characterized the reflexive binary relations that can be obtained on the basis of the concordance-discordance principle like in the Electre I (Roy, 1968) and Electre II methods (Roy and Bertier, 1973).

In other outranking methods, like the Tactic method (Vansnick, 1986), "superior" means "strictly better than". Such methods build an asymmetric relation that is interpreted as strict preference. As in the reflexive case, the obtained relations are not necessarily transitive and they may have cycles. Asymmetric relations satisfying the concordance/non-discordance condition have been previously characterized in BP02c, BP05b, BP06.

Common sense and usage suggest a simple relationship between strict and non-strict preference relations: alternative x is at least as good as alternative y if y is not better than x and *vice versa*. In terms of binary relations, this amounts saying that the "at least as good as" and "better than" relations are the *co-dual* of each other, i.e., one is the complement of the inverse of the other. This intuition should be questioned. Indeed, starting from an asymmetric preference relation and taking its co-dual leads to a complete preference, while, usually, non-strict outranking relations are incomplete preference relations. Conversely, the co-dual of an incomplete relation, such as a non-strict outranking relation, is not asymmetric, hence it hardly can be interpreted as a strict preference. Should we consider the

[1]In the sequel, previous papers by the present authors will be referred to by the authors' initials, BP, followed by the year of publication and possibly a letter to distinguish different papers published the same year.

asymmetric part of the co-dual? As we shall see, such a relation is not a strict outranking relation as they can be defined in the spirit of the TACTIC method.

The purpose of this paper is to examine the correspondence between strict and non-strict outranking relations. After notation is presented in the next section, we discuss this question in an informal way in Section 3, using definitions of strict and non-strict outranking relations that respectively encompass the relations yielded by the TACTIC and ELECTRE methods. In Section 4, we recall what is needed from our previous axiomatic work and analyze in depth the relationship between strict and non-strict outranking relations mainly using co-duality. This analysis leads us to the definition and characterization of a new model for preference relations (strict and non-strict concordance relations with "bonus"). Finally, we draw some conclusions and present perspectives for future research.

2 Notation and definitions

In this section we set the notation and recall some elementary definitions that will be used throughout the paper.

A preference relation on a set X is, in general, denoted by $\mathcal{R}$. A binary relation $\mathcal{R}$ on X is said to be *reflexive* if $a \mathcal{R} a$, for all $a \in X$. It is *complete* if $a \mathcal{R} b$ or $b \mathcal{R} a$, for all $a, b \in X$. Relation $\mathcal{R}$ is *asymmetric* if $a \mathcal{R} b \Rightarrow Not[b \mathcal{R} a]$, for all $a, b \in X$. It is *transitive* if $(a \mathcal{R} b$ and $b \mathcal{R} c) \Rightarrow a \mathcal{R} c$, for all $a, b, c \in X$. It is *Ferrers* if $(a \mathcal{R} b$ and $c \mathcal{R} d) \Rightarrow (a \mathcal{R} d$ or $c \mathcal{R} b)$, for all $a, b, c, d \in X$. It is *semi-transitive* if $(a \mathcal{R} b$ and $b \mathcal{R} c) \Rightarrow (a \mathcal{R} d$ or $d \mathcal{R} c)$, for all $a, b, c, d \in X$. A weak order is a complete transitive relation. A semiorder is a reflexive Ferrers and semi-transitive relation. A pair of semiorders $(\mathcal{R}_1, \mathcal{R}_2)$ on X form a *homogeneous chain of semiorders* (Doignon et al., 1988) if $\mathcal{R}_1 \subseteq \mathcal{R}_2$ and there is a weak order $\mathcal{T}$ on X such that, for $i = 1, 2$, we have

$$x \mathcal{T} y \Rightarrow \text{for all } z \in X,\ [y \mathcal{R}_i z \Rightarrow x \mathcal{R}_i z] \text{ and } [z \mathcal{R}_i x \Rightarrow z \mathcal{R}_i y]. \tag{1}$$

When a pair of alternatives (a, b) belongs to a relation $\mathcal{R}$, we write indifferently $(a, b) \in \mathcal{R}$ or $a \mathcal{R} b$. Starting from a relation $\mathcal{R}$, we can derive several other relations by using appropriate operators. For all $a, b \in X$, we define:

- the *dual* (or inverse or reciprocal) $\mathcal{R}^d$: $a \mathcal{R}^d b$ if $b \mathcal{R} a$
- the *complement* $\mathcal{R}^c$: $a \mathcal{R}^c b$ if $Not[a \mathcal{R} b]$
- the *co-dual* $\mathcal{R}^{cd}$: $a \mathcal{R}^{cd} b$ if $Not[b \mathcal{R} a]$
- the *asymmetric part* $\mathcal{R}^\alpha$: $a \mathcal{R}^\alpha b$ if $a \mathcal{R} b$ and $Not[b \mathcal{R} a]$
- the *symmetric part* $\mathcal{R}^\sigma$: $a \mathcal{R}^\sigma b$ if $a \mathcal{R} b$ and $b \mathcal{R} a$

- the *symmetric complement* $\mathcal{R}^\rho$: $a\ \mathcal{R}^\rho\ b$ if $Not[a\ \mathcal{R}\ b]$ and $Not[b\ \mathcal{R}\ a]$.

We recall a few straightforward properties of the co-dual operator, for they will be used in the sequel. The co-dual of a complete relation is its asymmetric part. The co-dual of an asymmetric relation is complete. The co-dual operator is an involution between the set of complete relations and the set of asymmetric ones. It also establishes an involution between reflexive and irreflexive relations (see Monjardet (1978) for proofs and many more results).

The set of alternatives will be denoted by X. As is usual in conjoint measurement this set will be identified with the Cartesian product $\prod_{i=1}^n X_i$ of n sets X_i. The latter are interpreted as the range of values of n attributes ($n \geq 2$) that completely describe the alternatives in the decision problem at hand. These sets X_i are not assumed to be sets of numbers, not even to be ordered sets. The set $\{1, 2, \dots, n\}$ will be denoted by N. We use X_{-i} to denote the cartesian product $\prod_{j \in N, j \neq i} X_j$. Assuming that x is an element of X, x_{-i} is the element of X_{-i} obtained by removing the ith coordinate of vector x, which describes x on attribute i. Assuming that a belongs to X and $x_i \in X_i$, (x_i, a_{-i}) is the element of X which has the same description as a on all attributes but one: the description of (x_i, a_{-i}) on the ith attribute is x_i.

3 Variants of outranking relations

We start with briefly recalling the definition of a reflexive outranking relation as used in the Electre I method and show that such relations fit in with a slightly more general and abstract definition. Such relations are interpreted as non-strict preferences. We do the same with the asymmetric outranking relation of the Tactic method. Such relations are interpreted as strict preferences. On the basis of these general definitions, we investigate the relationship between reflexive and asymmetric outranking relations, mainly using co-duality.

In order to avoid unnecessary minor complications, we restrict our attention to relations $\mathcal{R}$ on $X = \prod_{i=1}^n X_i$ for which each attribute is *influential.* This requirement is a sort of non-triviality condition for attributes. We say that attribute $i \in N$ is *influential* (for $\mathcal{R}$) if there are $x_i, y_i, z_i, w_i \in X_i$ and $x_{-i}, y_{-i} \in X_{-i}$ such that $(x_i, x_{-i})\ \mathcal{R}\ (y_i, y_{-i})$ and $Not[(z_i, x_{-i})\ \mathcal{R}\ (w_i, y_{-i})]$ and *degenerate* otherwise. A degenerate attribute has no influence whatsoever on the comparison of the elements of X and may be suppressed from N.

3.1 Electre I

We describe how a reflexive outranking relation, interpreted as a non-strict preference, is built according to the Electre I method[2]. Using a real-valued function u_i defined on X_i, and a pair of non-negative thresholds pt_i and vt_i, with $pt_i \leq vt_i$, we define the semiorders S_i and U_i as follows[3]: for all $x_i, y_i \in X_i$,

$$x_i \; S_i \; y_i \Leftrightarrow u_i(x_i) \geq u_i(y_i) - pt_i \tag{2}$$

$$x_i \; U_i \; y_i \Leftrightarrow u_i(x_i) \geq u_i(y_i) - vt_i. \tag{3}$$

The pair of relations (S_i, U_i) on X_i form a homogeneous nested chain of semiorders as defined in Section 2 (with the underlying weak order T_i such that $x_i \; T_i \; y_i \Leftrightarrow u_i(x_i) \geq u_i(y_i)$). Relation S_i interprets as the "at least as good" relation on attribute i. The relation P_i, the asymmetric part of S_i, is interpreted as a "better than" relation. pt_i is the preference threshold on attribute i. We read "$x_i \; U_i \; y_i$" as "level x_i is not unacceptably bad with respect to level y_i". The relation U_i is a *non-veto* relation on attribute i. In contrast, the co-dual of U_i is the veto relation V_i. It is defined as follows: for all $x_i, y_i \in X_i$,

$$\begin{aligned} y_i \; V_i \; x_i &\Leftrightarrow Not[x_i \; U_i \; y_i] \\ &\Leftrightarrow u_i(x_i) < u_i(y_i) - vt_i. \end{aligned} \tag{4}$$

Hence $y_i \; V_i \; x_i$ means that y_i is *far* better than x_i.

In Electre I, the outranking relation $\mathcal{R}$ is determined using positive weights w_i attached to each attribute and a threshold s with $(1/2 \leq s \leq 1)$, such that, for all $x, y \in X$,

$$x \; \mathcal{R} \; y \Leftrightarrow \frac{\sum_{i \in S(x,y)} w_i}{\sum_{j \in N} w_j} \geq s \text{ and } V(y,x) = \varnothing, \tag{5}$$

where $S(x,y) = \{i \in N : x_i \; S_i \; y_i\}$, the set of attributes on which x is at least as good as y, and $V(y,x) = \{i \in N : y_i \; V_i \; x_i\}$, the set of attributes on which x is unacceptably bad as compared to y.

Outranking relations such as $\mathcal{R}$ are reflexive, need not be complete and do not in general enjoy nice transitivity properties (Bouyssou, 1996). As a consequence, deriving a recommendation to the decision maker on the basis of such relations is not straightforward. In order to do that, the analyst may use one of the so-called *exploitation* procedures (see Roy and Bouyssou (1993, Ch. 6), or Bouyssou et al. (2006, Ch. 7)).

[2] This version of the Electre I method is not the historical one (Roy, 1968), but a more "modern" version as presented in Roy and Bouyssou (1993, p. 251).

[3] Assuming the existence of constant threshold representations for these semiorders is not restrictive for finite X_i (Aleskerov et al., 2007, p.222).

Remark 1
Note that we do not consider *valued* outranking relations such as those obtained by the ELECTRE III (see Roy (1978), Roy and Bouyssou (1993, pp. 284–289)) and the PROMETHEE (Brans and Vincke, 1985) methods. This is due to the fact that the tools currently developed in the framework of conjoint measurement theory only deal with crisp relations. In contrast, our analysis does encompass the crisp outranking relations produced by the ELECTRE II method. We do not develop this point further for the sake of conciseness. •

3.2 Outranking relations

A general definition of a reflexive outranking relation was given in BP09b, where such a relation is referred to as a *reflexive concordance-discordance relation* (R-CDR). The same paper established a characterization of such relations by a system of independent axioms. Since it turns out that reflexivity plays no rôle in the analysis, we restate this definition below, dropping the assumption that the relation is reflexive. In the sequel, the expression *outranking relation* will be used as exact synonymous of *concordance-discordance relation* (CDR).

Definition 2 (Concordance-discordance relation (CDR))
A binary relation $\mathcal{R}$ on $X = \prod_{i=1}^{n} X_i$ is a concordance-discordance relation *(CDR) if there are:*

1. *a* complete *binary relation S_i on each X_i $(i = 1, 2, \ldots, n)$ (with asymmetric part P_i and symmetric part I_i),*

2. *an* asymmetric *binary relation V_i on each X_i $(i = 1, 2, \ldots, n)$ such that $V_i \subseteq P_i$,*

3. *a binary relation $\trianglerighteq$ between subsets of N having N for union that is* monotonic *w.r.t. inclusion, i.e., for all $A, B, C, D \subseteq N$ with $A \cup B = N$ and $C \cup D = N$,*
$$[A \trianglerighteq B, C \supseteq A, B \supseteq D] \Rightarrow C \trianglerighteq D, \tag{6}$$

such that, for all $x, y \in X$,

$$x \mathcal{R} y \Leftrightarrow [S(x, y) \trianglerighteq S(y, x) \text{ and } V(y, x) = \varnothing], \tag{7}$$

where $S(x, y) = \{i \in N : x_i \; S_i \; y_i\}$ and $V(y, x) = \{i \in N : y_i \; V_i \; x_i\}$.

We say that $\langle \trianglerighteq, S_i, V_i \rangle$ is a type I representation *of $\mathcal{R}$ as a CDR.*

A concordance relation *(CR) is a CDR in which the V_i relations are all empty*[4]*.*

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As explained in BP09b, the type I representation $\langle \trianglerighteq, S_i, V_i \rangle$ of a CDR may not be unique. This is true even if all attributes are supposed to be influential. When we speak below of a representation of type I $\langle \trianglerighteq, S_i, V_i \rangle$ of a CDR, we mean one possible representation of type I of the CDR. As detailed in BP05a, BP07 the situation is different with CR. When all attributes are influential, they have a unique representation of type I. Similar remarks will hold for the representations of type II introduced below.

In the above definition, for each attribute i, S_i is interpreted as a non-strict preference relation on X_i. The asymmetric part P_i of S_i is the strict preference on X_i and V_i is the *veto* relation. Relation $\trianglerighteq$ is used in pairwise comparisons of alternatives, it compares coalitions of attributes in terms of their importance: if A and B denote subsets of attributes, $A \trianglerighteq B$ reads "the coalition of attributes A is at least as important as the coalition B". In the sequel, we shall use the notation $\triangleright$ (resp. $\triangleq$) to denote the asymmetric (resp. symmetric) part of $\trianglerighteq$. Consequently, $A \triangleright B$ (resp. $A \triangleq B$) reads "the coalition of attributes A is strictly more important (resp. equally important as) coalition B).

It is easy to see that the outranking relation of Electre I satisfies the above definition. In particular[5], the relation $\trianglerighteq$ is defined by:

$$A \trianglerighteq B \text{ if } \frac{\sum_{i \in A} w_i}{\sum_{j \in N} w_j} \geq s.$$

Rule (5) implies that the relation built in the Electre I procedure is reflexive.

Remark 3 (Outranking relations with attribute transitivity)
Due to (2) and (3), relations S_i and V_i in Electre I have additional properties, namely S_i is a semiorder, V_i is the asymmetric part of the semiorder U_i and S_i and U_i form a homogeneous chain of two semiorders. Concordance discordance relations with these additional properties have been defined and characterized in BP09b under the name *reflexive concordance-discordance relations with attribute transitivity* (R-CDR-AT). When required, the suffix "-AT" will be added to acronyms characterizing outranking relations, meaning that the corresponding relations also have the attribute transitivity property. •

Remark 4 (Concordance and non-discordance relations)
Condition (7) explicitly defines an outranking relation as a relation that satisfies two rules: a concordance rule $(S(x,y) \trianglerighteq S(y,x))$ and a non-discordance rule

[4]When a concordance discordance (resp. concordance relation) relation is irreflexive, reflexive or asymmetric we will use the acronyms I-CDR, R-CDR and A-CDR (resp. I-CR, R-CR and A-CR) when we want to emphasize this fact.

[5]See BP09b, p. 470, for more detail.

$(V(y,x) \neq \varnothing)$. Alternatively, an outranking relation $\mathcal{R}$ can be viewed as the intersection of two relations: a concordance relation $\mathcal{C}(\mathcal{R})$ and a non-discordance relation. The concordance relation $\mathcal{C}(\mathcal{R})$ is defined by $x\ \mathcal{C}(\mathcal{R})\ y$ if $S(x,y) \trianglerighteq S(y,x)$. The non-discordance relation $\mathcal{ND}(\mathcal{R})$ is defined by $x\ \mathcal{ND}(\mathcal{R})\ y$ if $V(y,x) = \varnothing$. Hence, we have $x\ \mathcal{R}\ y$ iff $[x\ \mathcal{C}(\mathcal{R})\ y$ and $x\ \mathcal{ND}(\mathcal{R})\ y]$.

Note that the concordance relation $\mathcal{C}(\mathcal{R})$ associated with a concordance-discordance relation in the sense of Definition 2 is itself a particular case of this definition, in which the veto relation V_i is assumed to be empty. Such relations were studied and characterized in (BP05a, BP07)[6]. •

3.3 Tactic

Another type of outranking relation has been introduced by Vansnick (1986). His Tactic method yields an asymmetric outranking relation interpreted as a strict preference. We briefly recall its definition. Let P_i be the asymmetric part of the semiorder S_i defined by (2) and let V_i be the asymmetric part of relation U_i defined by (3). Since U_i is complete, V_i is also the co-dual of U_i. An outranking relation $\mathcal{R}$ of the Tactic type is defined as follows:

$$x\ \mathcal{R}\ y \Leftrightarrow \sum_{i \in P(x,y)} w_i > \rho \sum_{j \in P(y,x)} w_j + \varepsilon \text{ and } V(y,x) = \varnothing, \tag{8}$$

where w_i is a weight assigned to attribute i, ρ is a multiplicative threshold with $\rho \geq 1$, ε is a nonnegative additive threshold, $P(y,x) = \{i \in N : x_i\ P_i\ y_i\}$ and $V(y,x) = \{i \in N : y_i\ V_i\ x_i\}$.

Such an outranking relation is clearly asymmetric by construction, hence irreflexive.

As with Electre I, Tactic inspires a general definition of outranking relations that we discuss in the next section.

3.4 An alternative definition of outranking relations

The alternative definition of a concordance-discordance relation that we give below is a variant of the one originally proposed in BP06, BP12, which was restricted to asymmetric preference relations. We drop this restriction and, in Lemma 6, we show that the unrestricted version of the definition is equivalent to Definition 2.

Definition 5
A binary relation $\mathcal{R}$ on $X = \prod_{i=1}^{n} X_i$ is a concordance-discordance relation *if there are:*

[6] This characterization was given for *reflexive* concordance relations (R-CR), but this restriction is inessential and our characterization is valid for general CR.

- *an* asymmetric *binary relation* P_i° *on each* X_i $(i = 1, 2, \dots, n)$,
- *an* asymmetric *binary relation* V_i° *on each* X_i $(i = 1, 2, \dots, n)$, *with* $V_i^\circ \subseteq P_i^\circ$,
- *a binary relation* $\trianglerighteq^\circ$ *between disjoint subsets of* N *that is* monotonic *w.r.t. inclusion, i.e., for all* $A, B, C, D \subseteq N$ *with* $A \cap B = \varnothing$ *and* $C \cap D = \varnothing$,

$$[A \trianglerighteq^\circ B, C \supseteq A, B \supseteq D] \Rightarrow C \trianglerighteq^\circ D, \tag{9}$$

such that, for all $x, y \in X$,

$$x \mathcal{R} y \Leftrightarrow [P^\circ(x, y) \trianglerighteq^\circ P^\circ(y, x) \text{ and } V^\circ(y, x) = \varnothing], \tag{10}$$

where $P^\circ(x, y) = \{i \in N : x_i P_i^\circ y_i\}$ *and* $V^\circ(y, x) = \{i \in N : y_i V_i^\circ x_i\}$.
We say that $\langle \trianglerighteq^\circ, P_i^\circ, V_i^\circ \rangle$ *is a* type II representation *of* $\mathcal{R}$.

It is readily checked that the outranking relation produced by TACTIC satisfies this definition. In TACTIC the relation $\trianglerighteq^\circ$ is asymmetric.

Lemma 6 (Equivalence of the definitions of CDR)
Definitions 2 and 5 are equivalent.

PROOF
Assume that $\mathcal{R}$ satisfies Definition 2 and that $\langle \trianglerighteq, S_i, V_i \rangle$ is a type I representation of $\mathcal{R}$. We construct a representation $\langle \trianglerighteq^\circ, P_i^\circ, V_i^\circ \rangle$ of type II satisfying (10) letting:

$$\begin{aligned} &P_i^\circ = P_i \text{ (the asymmetric part of } S_i), \forall i \in N \\ &V_i^\circ = V_i, \ \forall i \in N \end{aligned} \tag{11}$$

and $\trianglerighteq^\circ$ is defined as follows: for all $A, B \subseteq N$, with $A \cap B = \varnothing$,

$$A \trianglerighteq^\circ B \text{ if } (N \setminus B) \trianglerighteq (N \setminus A). \tag{12}$$

It is straightforward to check that $\trianglerighteq^\circ$ satisfies monotonicity condition (9) and that $\mathcal{R}$ satisfies condition (10).

Conversely, let $\mathcal{R}$ be a relation that satisfies (10) and $\langle \trianglerighteq^\circ, P_i^\circ, V_i^\circ \rangle$ a type II representation of $\mathcal{R}$. We build a type I representation $\langle \trianglerighteq, S_i, V_i \rangle$ of $\mathcal{R}$ by letting:

$$S_i = (P_i^\circ)^{cd} \text{ (the co-dual of } P_i^\circ), \forall i \in N \tag{13}$$

$$V_i = V_i^\circ, \ \forall i \in N \tag{14}$$

and $\trianglerighteq$ is defined as follows: for all $A, B \subseteq N$, with $A \cup B = N$,

$$A \trianglerighteq B \text{ if } (N \setminus B) \trianglerighteq^\circ (N \setminus A). \tag{15}$$

Again, it is easy to prove that $\trianglerighteq$ satisfies (6) and that $\mathcal{R}$ satisfies condition (7). Observe that because P_i°, it co-dual S_i is complete. □

The proof of the lemma has established a correspondence between representations of type I and type II of a CDR, which we state in the following definition.

Definition 7 (Conjugate representations)
Let $\mathcal{R}$ be a CDR and $\langle \trianglerighteq, S_i, V_i \rangle$ (resp. $\langle \trianglerighteq^\circ, P_i^\circ, V_i^\circ \rangle$) a representation of type I (resp. of type II) of $\mathcal{R}$. We say that these representations are conjugate if S_i and P_i° are linked by co-duality, i.e., $x_i\ S_i\ y_i$ if and only if $y_i\ P_i^\circ\ x_i$, $V_i = V_i^\circ$, and provided $\trianglerighteq$ and $\trianglerighteq^\circ$ are linked by (12) *or, equivalently, by* (15).

We state below three consequences of the equivalence of the two definitions of CDR.

1. It is immediate that an asymmetric outranking relation as yielded by the Tactic method described above satisfies the alternative definition of a CDR (Definition 5). By Lemma 6, it also satisfies Definition 2. Asymmetric concordance-discordance relations will be referred to by the acronym A-CDR.

2. If $\langle \trianglerighteq, S_i, V_i \rangle$ and $\langle \triangleright^\circ, P_i^\circ, V_i^\circ \rangle$) are dual representations of a CDR, $\mathcal{R}$, we see that condition (5) in Electre I could equivalently be formulated in terms of the strict preference P_i as
$$\frac{\sum_{i \in P(y,x)} w_i}{\sum_{j \in N} w_j} \leq 1 - s.$$

3. Let $\mathcal{R}$ be a CDR and let $\langle \trianglerighteq, S_i, V_i \rangle$, $\langle \trianglerighteq^\circ, P_i^\circ, V_i^\circ \rangle$ be conjugate representations of $\mathcal{R}$. The concordance part $\mathcal{C}(\mathcal{R})$ of the outranking relation $\mathcal{R}$ has been defined, using (7) and assuming $V_i = \varnothing$, by $x\ \mathcal{C}(\mathcal{R})\ y$ if $S(x,y) \trianglerighteq S(y,x)$. In the same spirit, we may use (10), assuming $V_i^\circ = \varnothing$, yielding $x\ \mathcal{C}(\mathcal{R})\ y$ if $P(x,y) \trianglerighteq P(y,x)$. Assuming that $\langle \trianglerighteq, S_i, V_i \rangle$ and $\langle \trianglerighteq^\circ, P_i^\circ, V_i^\circ \rangle$ are conjugate representations of $\mathcal{R}$, it is easy to see that the latter definition is equivalent with the initial one.

Summarizing, we may say that all crisp outranking relations, either those reflexive relations produced e.g., by the Electre I method, or the asymmetric ones produced e.g., by Tactic are CDR. Their distinctive structural features are mainly properties such as reflexivity vs irreflexivity, asymmetry or completeness. These properties of the outranking relations are reflected in their representations, more precisely, in corresponding properties of $\trianglerighteq$ or $\trianglerighteq^\circ$. The following proposition formally states some useful related results.

Proposition 8
Let $\mathcal{R}$ be a CDR and $\langle \trianglerighteq, S_i, V_i \rangle$ (resp. $\langle \trianglerighteq^\circ, P_i^\circ, V_i^\circ \rangle$) its representation of type I (resp. of type II). We have:

1. *$\mathcal{R}$ is either reflexive or irreflexive,*
   $\mathcal{R}$ is reflexive $\Leftrightarrow N \trianglerighteq N \Leftrightarrow \varnothing \trianglerighteq^\circ \varnothing$,
   otherwise, $\mathcal{R}$ is irreflexive,

2. *$\mathcal{R}$ is asymmetric $\Leftrightarrow \trianglerighteq$ is asymmetric $\Leftrightarrow \trianglerighteq^\circ$ is asymmetric,*

3. *$\mathcal{R}$ is complete*
   $\Leftrightarrow \quad (A \trianglerighteq B \text{ or } B \trianglerighteq A), \forall A, B \subseteq N, A \cup B = N$
   $\Leftrightarrow \quad (C \trianglerighteq^\circ D \text{ or } C \trianglerighteq^\circ D), \forall C, D \subseteq N, C \cap D = \varnothing.$

The straightforward proof is left to the reader. For Parts 2 and 3, the proof of the $\Rightarrow$ part uses the hypothesis that all attributes are influential (so that, for all $A, B \subseteq N$ such that $A \cup B = N$, there are $x, y \in X$ such that $S(x, y) = A$ and $S(y, x) = B$ and for all $C, D \subseteq N$ such that $C \cap D = \varnothing$, there are $z, w \in X$ such that $P^\circ(z, w) = C$ and $P^\circ(w, z) = D$).

These properties can immediately be applied to outranking relations built using Electre I or Tactic. If $\mathcal{R}$ arises from Electre I, it satisfies (5) so that $N \trianglerighteq N$. As a consequence of Proposition 8.1, $\mathcal{R}$ is reflexive. A relation $\mathcal{R}$ arising from Tactic satisfies (8). This implies $\varnothing \trianglerighteq^\circ \varnothing$ and $[C \trianglerighteq^\circ D \Rightarrow Not[D \trianglerighteq^\circ C]]$, for all $C, D \subseteq N$ with $C \cap D = \varnothing$. Hence, using Proposition 8.1 and 8.2, we see that $\mathcal{R}$ is irreflexive and asymmetric.

Remark 9 (The rôle of co-duality)
This section has shown that both reflexive and asymmetric outranking relations can be described in a common framework specified by either Definition 2 or, equivalently, Definition 5. In spite of this resemblance, their interpretations are rather contrasted since reflexive preferences are usually interpreted as "at least as good" relations while asymmetric preferences are interpreted as "better than" relations. With these interpretations, some sort of semantic relationship is intuitively expected between "at least as good" and "better than" relations. If we start with a reflexive preference relation $\mathcal{R}$, interpreted as an "at least as good" relation, the corresponding "better than" relation is the asymmetric part $\mathcal{R}^\alpha$ of $\mathcal{R}$. Conversely, starting with an asymmetric preference relation $\mathcal{R}$ (like in the Tactic method), how can we define the corresponding "at least as good" relation? At first glance, it is tempting to say that x is at least as good as y if y is not better than x, which amounts to define the "at least as good" relation as the co-dual of $\mathcal{R}$. Such a definition automatically yields a complete reflexive relation (in view of the properties of the co-dual operator that were recalled in Section 2). This is problematic since reflexive preference relations cannot always be assumed to be complete (see Deparis et al., 2012, for an experimental investigation of incomparability in preferences). In particular, in the context of outranking methods, pairs of alternatives may be incomparable, due for instance to veto effects (see Roy, 1996). Actually, an

asymmetric outranking relation can be, in general, the asymmetric part of several reflexive outranking relations. Determining a unique reflexive relation having a given asymmetric part requires additional information, namely the specification of the list of incomparable pairs of alternatives (referred to as the *incomparability relation* in Roy, 1996).

Although co-duality does not determine a correspondence between reflexive and asymmetric outranking relations, it plays a major rôle for understanding their relationship. Therefore, we devote the rest of Section 3 to investigate the effect of the the co-dual and asymmetric part operators on outranking relations. We first consider concordance relations (with empty veto relations) then we examine the case of concordance-discordance relations.

3.5 The co-dual and the asymmetric part of a concordance relation

We first define the co-dual of the relation comparing the coalitions of attributes in Definition 2 and state properties of such relations that will be useful in the sequel.

Definition 10
Let $\trianglerighteq$ be a relation between subsets of N having N for union. We call the co-dual of $\trianglerighteq$, the relation $\trianglerighteq^{cd}$ between subsets of N having N for union that is defined as follows: for all $A, B \subseteq N$, with $A \cup B = N$, we have $A \trianglerighteq^{cd} B \Leftrightarrow Not[B \trianglerighteq A]$.

Lemma 11
Let $\trianglerighteq$ and $\tilde{\trianglerighteq}$ be two relations between subsets of N having N for union and satisfying monotonicity condition (6).

1. *The intersection $\trianglerighteq \cap \tilde{\trianglerighteq}$ of these relations is a relation between subsets of N having N for union and satisfying (6).*
2. *The co-dual $\trianglerighteq^{cd}$ and the asymmetric part $\trianglerighteq^{\alpha}$ of $\trianglerighteq$ both satisfy condition (6).*

Proof. The proof is left to the reader.

Remark 12
Similar properties can be established for relations $\trianglerighteq^{\circ}$ that intervene in type II representations of concordance relations (CR). We emphasize that $\trianglerighteq^{\circ}$ is a relation between disjoint subsets of N and satisfies monotonicity condition (9). •

The co-dual of a CR is a CR and there is a correspondence between the type I representations of these relations.

Proposition 13
Let $\mathcal{R}$ be a CR that has a representation of type I, $\langle \trianglerighteq, S_i \rangle$. The co-dual $\mathcal{R}^{cd}$ of $\mathcal{R}$ is also a CR with a representation of type I that is $\langle \trianglerighteq^{cd}, S_i \rangle$.

Proof

We have $x \mathcal{R} y \Leftrightarrow [S(x,y) \trianglerighteq S(y,x)]$. The co-dual $\mathcal{R}^{cd}$ of $\mathcal{R}$ is such that $x \mathcal{R}^{cd} y \Leftrightarrow Not[y \mathcal{R} x]$. Hence, we have $x \mathcal{R}^{cd} y \Leftrightarrow Not[S(y,x) \trianglerighteq S(x,y)]$. The latter condition can be rewritten as $x \mathcal{R}^{cd} y \Leftrightarrow \left[S(x,y) \trianglerighteq^{cd} S(y,x)\right]$, where $\trianglerighteq^{cd}$ is the co-dual of $\trianglerighteq$. Using Lemma 11.2, we know that $\langle \trianglerighteq^{cd}, S_i \rangle$ is a type I representation of $\mathcal{R}^{cd}$. □

Remark 14

A quite similar correspondence holds for type II representations. If $\langle \trianglerighteq^{\circ}, P_i^{\circ} \rangle$ is a type II representation of $\mathcal{R}$, $\langle \trianglerighteq^{\circ\, cd}, P_i^{\circ} \rangle$ is a type II representation of $\mathcal{R}^{cd}$. Moreover, if the type I and II representations of $\mathcal{R}$ are dual of one another (Definition 7), the corresponding representations of $\mathcal{R}^{cd}$ are also dual of one another. In particular, if $\trianglerighteq$ and $\trianglerighteq^{\circ}$ are linked through relations (12) and (15) then their respective co-dual $\trianglerighteq^{cd}$ and $\trianglerighteq^{\circ cd}$ are linked through the same relations. •

Remark 15

The general properties of the co-dual operator, which were recalled in Section 2, apply to the particular case of concordance relations. In particular, the co-dual of a reflexive CR is an irreflexive CR and conversely. Also, the co-dual of an asymmetric CR is a complete CR and conversely. •

We now turn to considering the asymmetric part of a R-CR. Taking the asymmetric part of a R-CR yields a CR (that is of course asymmetric hence irreflexive).

Proposition 16

The asymmetric part of a R-CR, that has a type I representation $\langle \trianglerighteq, S_i \rangle$, is the A-CR that has a type I representation $\langle \trianglerighteq^{\alpha}, S_i \rangle$ with $\trianglerighteq^{\alpha}$, the asymmetric part of $\trianglerighteq$.

Proof

Let $\mathcal{R}$ be a R-CR. We have $x \mathcal{R} y \Leftrightarrow [S(x,y) \trianglerighteq S(y,x)]$. The asymmetric part $\mathcal{R}^{\alpha}$ of $\mathcal{R}$ is such that $x \mathcal{R}^{\alpha} y \Leftrightarrow x \mathcal{R} y$ and $Not[y \mathcal{R} x]$. Hence, we have $x \mathcal{R}^{\alpha} y \Leftrightarrow [S(x,y) \trianglerighteq S(y,x)$ and $Not[S(y,x) \trianglerighteq S(x,y)]]$. The latter condition can be rewritten as $x \mathcal{R}^{\alpha} y \Leftrightarrow [S(x,y) \trianglerighteq^{\alpha} S(y,x)]$, where $\trianglerighteq^{\alpha}$ is the asymmetric part of $\trianglerighteq$. Using Lemma 11.2, we know that $\langle \trianglerighteq^{\alpha}, S_i \rangle$ is a type I representation of $\mathcal{R}^{\alpha}$. □

Remark 17 (Type II representations)

A remark similar to Remark 14 can be formulated for type II representations of $\mathcal{R}$ and $\mathcal{R}^{\alpha}$. In particular, let $\langle \trianglerighteq^{\circ}, P_i^{\circ} \rangle$ be a type II representation of $\mathcal{R}$. The asymmetric part $\mathcal{R}^{\alpha}$ of $\mathcal{R}$ has a type II representation $\langle \trianglerighteq^{\circ\alpha}, P_i^{\circ} \rangle$, with $\trianglerighteq^{\circ\alpha}$, the asymmetric part of $\trianglerighteq^{\circ}$. •

Summarizing, if a R-CR is complete, its asymmetric part is an A-CR which is also its co-dual. On the other hand, for a given A-CR, there are several R-CR having it as their asymmetric part. One of them is a complete relation and its co-dual. Note also that all what we said for CR remains valid for CR-AT, i.e., CR with attribute transitivity (as defined in Remark 3).

3.6 Vetoes and bonuses

We now address the general case of outranking relations with veto. Considering concordance-discordance relations changes the picture. The correspondence between R-CR and A-CR described in the previous section no longer holds. In particular, the asymmetric part of a R-CDR is not, in general, an A-CDR. We investigate such issues below.

Let $\mathcal{R}$ be a R-CDR. For all $x, y \in X$, we have that $x \ \mathcal{R} \ y$ if $x \ \mathcal{C}(\mathcal{R}) \ y$ and $x \ \mathcal{ND}(\mathcal{R}) \ y$, where $\mathcal{C}(\mathcal{R})$ is the R-CR associated with $\mathcal{R}$ and $x \ \mathcal{ND}(\mathcal{R}) \ y$ if $Not[y_i \ V_i \ x_i]$, for all $i \in N$ (see Remark 4).

The asymmetric part $\mathcal{R}^\alpha$ of $\mathcal{R}$ obtains as follows:

$$\begin{aligned} x \ \mathcal{R}^\alpha \ y &\Leftrightarrow x \ \mathcal{R} \ y \text{ and } Not[y \ \mathcal{R} \ x] \\ &\Leftrightarrow \begin{cases} [x \ \mathcal{C}(\mathcal{R}) \ y \text{ and } x \ \mathcal{ND}(\mathcal{R}) \ y] \text{ and} \\ [Not[y \ \mathcal{C}(\mathcal{R}) \ x] \text{ or } Not[y \ \mathcal{ND}(\mathcal{R}) \ x]] \end{cases} \end{aligned}$$

It is easy to see that the above definition can equivalently be written as:

$$x \ \mathcal{R}^\alpha \ y \Leftrightarrow$$
$$\begin{cases} [x \ \mathcal{C}(\mathcal{R}) \ y \text{ and } Not[y \ \mathcal{C}(\mathcal{R}) \ x] \text{ and } x \ \mathcal{ND}(\mathcal{R}) \ y] \text{ or} \\ [x \ \mathcal{C}(\mathcal{R}) \ y \text{ and } y \ \mathcal{C}(\mathcal{R}) \ x \text{ and } x \ \mathcal{ND}(\mathcal{R}) \ y \text{ and } Not[y \ \mathcal{ND}(\mathcal{R}) \ x]] . \end{cases}$$

Hence we have $x \ \mathcal{R}^\alpha \ y$ if and only if one of the following two exclusive conditions is fulfilled:

1. (x, y) belongs to the asymmetric part of $\mathcal{C}(\mathcal{R})$ and $x \ \mathcal{ND}(\mathcal{R}) \ y$ or
2. (x, y) belongs to the symmetric part of $\mathcal{C}(\mathcal{R})$, $x \ \mathcal{ND}(\mathcal{R}) \ y$ and, for some $i \in N$, $x_i \ V_i \ y_i$

Case 1 corresponds to the definition of an A-CDR since, by Proposition 16, the asymmetric part of $\mathcal{C}(\mathcal{R})$, which is a R-CR, is an A-CR, a non-discordance condition is imposed on it.

Case 2 looks a bit more unexpected. There is no such condition in the definition of an A-CDR or in the TACTIC motivating example of an A-CDR. With Case 2, we have $x \ \mathcal{R}^\alpha \ y$ when (x, y) belongs to the *symmetric* part of $\mathcal{C}(\mathcal{R})$ and there is no veto of y against x ($\forall j,\ Not[y_j\ V_j\ x_j]$) but on some attribute i, we have $x_i\ V_i\ y_i$, which means that x_i is a much better performance than y_i on attribute i. The presence of a veto in favor of x against y can thus have a positive effect in breaking a tie in the concordance relation. We call such an effect a *bonus*.

In contrast with Proposition 16, the asymmetric part of an R-CDR is not, in general, an A-CDR, due to the possible occurrence of *bonus effects*.

Let us now examine the effect of the co-dual operator on concordance-discordance relations. Assume, for instance, that $\mathcal{R}$ is an A-CDR. By definition of an A-CDR, we have: $x\ \mathcal{R}\ y$ if $x\ \mathcal{C}(\mathcal{R})\ y$ and $x\ \mathcal{ND}(\mathcal{R})\ y$, where $\mathcal{C}(\mathcal{R})$ is the A-CR associated with $\mathcal{R}$. The co-dual $\mathcal{R}^{cd}$ of $\mathcal{R}$ is such that:

$$x\ \mathcal{R}^{cd}\ y \Leftrightarrow Not[y\ \mathcal{C}(\mathcal{R})\ x] \text{ or } Not[y\ \mathcal{ND}(\mathcal{R})\ x]. \tag{16}$$

The first condition in the righthand side of (16) states that (x, y) belongs to the co-dual of $\mathcal{C}(\mathcal{R})$. We know that the co-dual of this A-CR is a complete R-CR (Proposition 13). The second condition in the righthand side of (16), means again that there may be a *bonus* effect, i.e., that $x_i\ V_i\ y_i$ (for any $i \in N$) entails $x\ \mathcal{R}^{cd}\ y$. Condition (16) defining the co-dual of an A-CDR is very similar to the one defining a R-CDR except that veto plays a positive rôle here. In contrast with Proposition 13, the co-dual of a CDR is not, in general, a CDR, due to the possible occurrence of bonus effects. It is a concordance relation with bonus (CRB) as defined below. In this definition, the non-veto condition in Definition 2 is just transformed into a bonus condition.

Definition 18 (Concordance relation with bonus (CRB))
A binary relation $\mathcal{R}$ on $X = \prod_{i=1}^{n} X_i$ is a concordance relation with bonus *(CRB) if there are:*

1. *a* complete *binary relation S_i on each X_i ($i = 1, 2, \ldots, n$) (with asymmetric part P_i and symmetric part I_i),*

2. *an* asymmetric *binary relation V_i on each X_i ($i = 1, 2, \ldots, n$) such that $V_i \subseteq P_i$,*

3. *a binary relation $\trianglerighteq$ between subsets of N having N for union that is* monotonic *w.r.t. inclusion, i.e., for all $A, B, C, D \subseteq N$ with $A \cup B = N$ and $C \cup D = N$,*
$$[A \trianglerighteq B, C \supseteq A, B \supseteq D] \Rightarrow C \trianglerighteq D, \tag{17}$$

such that, for all $x, y \in X$,

$$x\ \mathcal{R}\ y \Leftrightarrow [S(x, y) \trianglerighteq S(y, x) \ \textit{or}\ V(x, y) \neq \varnothing], \tag{18}$$

where $S(x, y) = \{i \in N : x_i\ S_i\ y_i\}$ and $V(x, y) = \{i \in N : x_i\ V_i\ y_i\}$.

We say that $\langle \trianglerighteq, S_i, V_i \rangle$ is a representation *of $\mathcal{R}$ as a CRB.*

Note that a *concordance relation* (CR) is a CRB in which all the V_i relations are empty. As for concordance-discordance relations, we may distinguish reflexive CRB's (R-CRB) on the one hand and asymmetric CRB's (A-CRB) on the other hand. The alternative definition of a CDR established in Lemma 6 can also be transposed for CRB's without any difficulty and we omit the details.

It is easy to see that CRB's and CDR's are related through co-duality as stated in the following proposition.

Proposition 19
Relation $\mathcal{R}$ on X is a concordance-discordance relation if and only if its co-dual $\mathcal{R}^{cd}$ is a concordance relation with bonus and conversely.

Proof
Let $\mathcal{R}$ be a CDR having $\langle \trianglerighteq, S_i, V_i \rangle$ as a representation of type I. Using Proposition 13, we know that $\mathcal{R}^{cd}$ is a concordance relation that has $\langle \trianglerighteq^{cd}, S_i \rangle$ as a representation of type I as a CR. Since $\mathcal{R}^{cd}$ is defined, for all $x, y \in X$, by

$$\begin{aligned} x\, \mathcal{R}^{cd}\, y &\Leftrightarrow Not[S(y,x) \trianglerighteq S(x,y) \text{ and } V(y,x) = \varnothing] \\ &\Leftrightarrow Not[S(y,x) \trianglerighteq S(x,y)] \text{ or } [V(y,x) \neq \varnothing] \\ &\Leftrightarrow [S(x,y) \trianglerighteq^{cd} S(y,x)] \text{ or } [V(y,x) \neq \varnothing], \end{aligned}$$

we see that it is a CRB having a representation of type I, which is $\langle \trianglerighteq^{cd}, S_i, V_i \rangle$.

The converse is also true. Starting with $\mathcal{R}$, a CRB that has a type I representation $\langle \trianglerighteq^{cd}, S_i \rangle$, we apply the co-dual operator as follows:

$$\begin{aligned} x\, \mathcal{R}^{cd}\, y &\Leftrightarrow Not[S(y,x) \trianglerighteq S(x,y) \text{ or } V(y,x) \neq \varnothing] \\ &\Leftrightarrow Not[S(y,x) \trianglerighteq S(x,y)] \text{ and } [V(y,x) = \varnothing] \\ &\Leftrightarrow [S(x,y) \trianglerighteq^{cd} S(y,x)] \text{ and } [V(y,x) = \varnothing]. \end{aligned}$$

Relation $\mathcal{R}^{cd}$ is a CDR that admits the type I representation $\langle \trianglerighteq^{cd}, S_i, V_i \rangle$. □

Because the asymmetric part $\mathcal{R}^{\alpha}$ of a R-CDR, $\mathcal{R}$ may involve at the same time bonus and veto effects, simple examples show that it is neither an asymmetric CDR nor an asymmetric CRB. Such relations require an analysis that is more complex than the one for CDR or CRB. It is detailed in Bouyssou and Pirlot (2013).

3.7 Summary

Summarizing the above analysis of the relationship between non-strict and strict outranking relations, we draw the reader's attention to the following points.

1. As long as we are concerned with concordance relations, without considering vetoes, we see that there is no deep difference in nature between non-strict and strict concordance relations: R-CR and A-CR satisfy the same definition (Definition 2). They just differ by the additional property that they are respectively reflexive or asymmetric. Moreover, the asymmetric part of a R-CR is an A-CR. The co-dual of an A-CR is a complete R-CR. Knowing an A-CR, there is no way of distinguishing indifference from incomparability in view of reconstructing an hypothetic original R-CR of which the A-CR that we know would be the asymmetric part. From a practical point of view, this can be seen as an advantage of R-CR models over A-CR's. The preferential information encoded in a R-CR permits to distinguish incomparable pairs of alternatives from indifferent ones, while A-CR's do not allow for that.

2. Allowing for vetoes changes the picture. The asymmetric part of a R-CDR is not an A-CDR but a more complex object, in general consisting of two disjoint relations: on the one hand, the intersection of the asymmetric part of the associated concordance relation and the non discordance relation (as expected). On the other hand, a part of the indifference relation of the associated concordance relation determined as follows: if one of the two arcs linking a pair of alternatives in the indifference part of the concordance relation is broken due to a veto, while the other is not, then the remaining arc belongs to the asymmetric part of the R-CDR. In this case, the veto relation acts as a bonus.

In the rest of this paper, we take advantage of the just explored co-duality relationships between strict and and non-strict outranking relations, in order to unify and deepen the axiomatic analysis that we presented in several previous papers (BP02b, BP05a, BP06, BP07, BP09b).

4 Axiomatic analysis

We start by recalling some of earlier results on the characterization of CR and CDR. Then we study the effect of the co-dual operator on our axioms. We derive new axiomatic characterizations of classical outranking relations as well as we obtain characterizations of preference relations involving *bonuses* instead of vetoes. Our main goal is to offer a unified and comprehensive framework allowing clear understanding of the relationships between strict and non strict outranking relations.

4.1 Background

We briefly recall the axioms used in the characterization of reflexive CDR obtained in BP09b, under the assumption that all attributes are influential.

Definition 20 (Axioms $RC1$, $RC2$)
Let $\mathcal{R}$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. This relation is said to satisfy:

$$RC1_i \text{ if } \left.\begin{array}{c}(x_i, a_{-i}) \mathcal{R} (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \mathcal{R} (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(z_i, a_{-i}) \mathcal{R} (w_i, b_{-i}) \\ \text{or} \\ (x_i, c_{-i}) \mathcal{R} (y_i, d_{-i}),\end{array}\right.$$

$$RC2_i \text{ if } \left.\begin{array}{c}(x_i, a_{-i}) \mathcal{R} (y_i, b_{-i}) \\ \text{and} \\ (y_i, c_{-i}) \mathcal{R} (x_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(z_i, a_{-i}) \mathcal{R} (w_i, b_{-i}) \\ \text{or} \\ (w_i, c_{-i}) \mathcal{R} (z_i, d_{-i}),\end{array}\right.$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$. We say that $\mathcal{R}$ satisfies $RC1$ (resp. $RC2$) if it satisfies $RC1_i$ (resp. $RC2_i$) for all $i \in N$.

An interpretation of these axioms was provided in BP02a. Basically, axiom $RC1_i$ amounts to say that all preference differences (x_i, y_i) on X_i can be weakly ordered. Axiom $RC2_i$ establishes a link between opposite differences of preferences such as (x_i, y_i) and (y_i, x_i). Note that $RC2$ entails that $\mathcal{R}$ is an independent preference relation. Since they will be useful in the sequel, we recall the precise definition of the weak orders on preference differences induced on each attribute X_i as well as the main properties linking them to axioms $RC1$ and $RC2$.

Definition 21 (Relations $\succsim_i^*$, $\succsim_i^{**}$)
Let $\mathcal{R}$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. We define the binary relations $\succsim_i^*$ and $\succsim_i^{**}$ on X_i^2 letting, for all $x_i, y_i, z_i, w_i \in X_i$,

$$(x_i, y_i) \succsim_i^* (z_i, w_i) \Leftrightarrow$$
$$\forall a_{-i}, b_{-i} \in X_{-i}, [(z_i, a_{-i}) \mathcal{R} (w_i, b_{-i}) \Rightarrow (x_i, a_{-i}) \mathcal{R} (y_i, b_{-i})],$$

$$(x_i, y_i) \succsim_i^{**} (z_i, w_i) \Leftrightarrow [(x_i, y_i) \succsim_i^* (z_i, w_i) \text{ and } (w_i, z_i) \succsim_i^* (y_i, x_i)].$$

These relations allow to give a precise meaning to the comparison of preference differences on each attribute (see BP02a, for more detail). In the same paper, we have shown the following result.

Lemma 22 (Bouyssou and Pirlot, 2002a, Lemma 1)

1. *$RC1_i \Leftrightarrow$ [$\succsim_i^*$ is complete],*

2. *$RC2_i \Leftrightarrow$*
   [for all $x_i, y_i, z_i, w_i \in X_i$, $Not[(x_i, y_i) \succsim_i^* (z_i, w_i)] \Rightarrow (y_i, x_i) \succsim_i^* (w_i, z_i)$],

3. $[RC1_i$ *and* $RC2_i] \Leftrightarrow [\succsim_i^{**}$ *is complete*$]$.

Since $\succsim_i^*$ and $\succsim_i^{**}$ are transitive by definition, the above lemma states that $\succsim_i^*$ (resp. $\succsim_i^{**}$ is a weak order if and only if $RC1_i$ holds (resp. $RC1_i$ and $RC2_i$ hold).

A crucial feature of CR and CDR is that they induce relations $\succsim_i^*$ and $\succsim_i^{**}$ having a limited number of equivalence classes. This is the motivation for the following two conditions.

Definition 23 (Axioms $M1$, $M2$)
Let $\mathcal{R}$ be a binary relation on a set $X = \prod_{i=1}^n X_i$. This relation is said to satisfy:

$$M1_i \text{ if } \left.\begin{array}{c} (x_i, a_{-i}) \mathcal{R} (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \mathcal{R} (w_i, d_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \mathcal{R} (x_i, b_{-i}) \\ \text{or} \\ (w_i, a_{-i}) \mathcal{R} (z_i, b_{-i}) \\ \text{or} \\ (x_i, c_{-i}) \mathcal{R} (y_i, d_{-i}), \end{array}\right. \quad (19)$$

$$M2_i \text{ if } \left.\begin{array}{c} (x_i, a_{-i}) \mathcal{R} (y_i, b_{-i}) \\ \text{and} \\ (y_i, c_{-i}) \mathcal{R} (x_i, d_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \mathcal{R} (x_i, b_{-i}) \\ \text{or} \\ (z_i, a_{-i}) \mathcal{R} (w_i, b_{-i}) \\ \text{or} \\ (z_i, c_{-i}) \mathcal{R} (w_i, d_{-i}), \end{array}\right. \quad (20)$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$. We say that $M1$ (resp. $M2$) holds if $M1_i$ (resp. $M2_i$) holds for all $i \in N$.

The interpretation of $M1_i$ and $M2_i$, respectively conditional on $RC2_i$ and $RC1_i$, results from the following lemma.

Lemma 24

1. *If $\mathcal{R}$ satisfies $RC2_i$, then $\mathcal{R}$ satisfies $M1_i \Leftrightarrow$*
   [for all $x_i, y_i, z_i, w_i \in X_i$, $Not[(y_i, x_i) \succsim_i^* (x_i, y_i)] \Rightarrow (x_i, y_i) \succsim_i^* (z_i, w_i)$],

2. *If $\mathcal{R}$ satisfies $RC1_i$, then $\mathcal{R}$ satisfies $M2_i \Leftrightarrow$*
   [for all $x_i, y_i, z_i, w_i \in X_i$, $Not[(y_i, x_i) \succsim_i^* (x_i, y_i)] \Rightarrow (z_i, w_i) \succsim_i^* (y_i, x_i)$],

Proof
The proof of item 1 (resp. item 2) results from the combination of BP07, Lemma 11.1 and 11.3 (resp. 11.2 and 11.4) and BP05a, Lemma 16.1 (resp. 16.2). □

Let us call a *positive preference difference* (resp. *negative preference difference*) one that is at least (resp. at most) as large as the opposite preference difference. Under $RC2_i$, $M1_i$ says that a positive preference difference is at least as large as any other preference difference. In other terms, there is only one class of positive preference differences. Under $RC1_i$, $M2_i$ states the symmetric property for negative preference differences. For more detail on the interpretation of $M1$ and $M2$, see BP05a, BP07).

Remark 25 (Axioms *UC* and *LC*)
A simpler–and slightly stronger–version of axioms $M1_i$, $M2_i$ was used in our initial characterization of concordance relations in BP05a. These axioms, respectively labeled UC_i and LC_i, obtain by dropping the second in the three possible conclusions in the definitions of $M1_i$ and $M2_i$. The substitution of UC_i and LC_i by $M1_i$ and $M2_i$ in the characterization of reflexive CR was motivated by the need for independent sets of axioms. By Lemma 16.3 in BP05a, we established indeed that $RC2_i$, UC_i and LC_i imply $RC1_i$. The second in the three possible conclusions in the definitions of $M1_i$ and $M2_i$ has precisely the effect of guaranteeing the independence of the set of axioms $RC1_i, RC2_i, M1_i$ and $M2_i$, as is shown in the next theorem. •

Theorems 26 and 28 below are variants of Theorems 13 in BP07 and Theorem 19 in BP09b, respectively. The theorems stated below are slightly more general than their previous versions in two respects. First they are stated for general binary relations, instead of reflexive relations. The reflexivity property was actually playing no rôle in the proofs of the previous characterizations, which remain unchanged and are thus omitted. The second detail is that the independence of the axioms is now stated in the class of *complete* relations (instead of the class of reflexive ones). Most of the examples previously used to show the independence of the axioms were complete relations. For the sake of completeness, we recall these examples below and provide an additional one that is needed for proving Theorem 28.

Theorem 26
The binary relation $\mathcal{R}$ on $X = \prod_{i=1}^{n} X_i$ is a concordance relation (CR) iff it satisfies $RC1$, $RC2$, $M1$ and $M2$. These axioms are independent in the class of complete binary relations.

Proof
As we said before, the proof of Theorem 13 in BP07 remains valid for general binary relations and we omit it. The independence of the axioms in the class of complete relations results from the following examples (see Appendix):

Violated axiom	$RC1_i$	$RC2_i$	$M1_i$	$M2_i$
Example	78	76	77	82

□

We now introduce a weakened version of $M2_i$, axiom $M3_i$, which allows for vetoes, i.e., "large negative" preference differences forbidding that a pair of alternatives may belong to the global preference relation.

Definition 27
Let $\mathcal{R}$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. This relation is said to satisfy:

$$M3_i \text{ if } \left.\begin{array}{c} (x_i, a_{-i})\ \mathcal{R}\ (y_i, b_{-i}) \\ \text{and} \\ (y_i, c_{-i})\ \mathcal{R}\ (x_i, d_{-i}) \\ \text{and} \\ (z_i, e_{-i})\ \mathcal{R}\ (w_i, f_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i})\ \mathcal{R}\ (x_i, b_{-i}) \\ \text{or} \\ (z_i, a_{-i})\ \mathcal{R}\ (w_i, b_{-i}) \\ \text{or} \\ (z_i, c_{-i})\ \mathcal{R}\ (w_i, d_{-i}), \end{array}\right.$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i}, e_{-i}, f_{-i} \in X_{-i}$. We say that $\mathcal{R}$ satisfies $M3$ if it satisfies $M3_i$ for all $i \in N$.

We observe that $M3_i$ only differs from $M2_i$ by the adjunction of the third premise, implying that $M3_i$ is a weakening of $M2_i$. The interpretation of $M3_i$, under the hypothesis that $RC1_i$ holds, results from that of $M2_i$ as stated in Lemma 24.2. Assuming $RC1_i$ amounts to say that $\succsim_i^*$ is complete. Hence if the first two premises of $M3_i$ hold and neither the first nor the third conclusion do, then we have $(x_i, y_i) \succ_i^* (y_i, x_i) \succ_i^* (z_i, w_i)$. In these circumstances, the second conclusion cannot be true, since this would imply that $(z_i, w_i) \succ_i^* (y_i, x_i)$, a contradiction with $(y_i, x_i) \succ_i^* (z_i, w_i)$. Hence, none of the three conclusions holds and $M3_i$ can only be satisfied if it never happens that $(z_i, e_{-i})\ \mathcal{R}\ (w_i, f_{-i})$. This means that the pair (z_i, w_i) represents an unacceptable preference difference, leading to a veto. We have the following result.

Theorem 28
The binary relation $\mathcal{R}$ on $X = \prod_{i=1}^{n} X_i$ is a concordance-discordance relation (CDR) iff it satisfies $RC1$, $RC2$, $M1$ and $M3$. These axioms are independent in the class of complete binary relations.

Proof
As said before, the proof of Theorem 19 in BP09b remains valid for general binary relations and we omit it. In order to prove the independence of the axioms in the class of complete relations, we may invoke again those examples used in the proof of Theorem 26. It only remains to exhibit an example of a complete relation satisfying $RC1$, $RC2$, $M1$ and $M3_i$ on all attributes but one. Example 82 in appendix fulfills this requirement. □

4.2 Characterizations of CR via co-duality

From Proposition 13 we know that the co-dual of a concordance relation $\mathcal{R}$ is also a CR. Starting from the axioms above, it is not difficult to reformulate them in terms of the co-dual relation $\mathcal{R}^{cd}$. Let $\mathcal{R}$ be any relation on X. Consider for instance axiom $RC1_i$. Using contraposition, we obtain:
$\mathcal{R}$ satisfies $RC1_i$, i.e.,

$$\left.\begin{array}{c} Not[(x_i, c_{-i})\ \mathcal{R}\ (y_i, d_{-i})] \\ \text{and} \\ Not[(z_i, a_{-i})\ \mathcal{R}\ (w_i, b_{-i})] \end{array}\right\} \Rightarrow \left\{\begin{array}{c} Not[(x_i, a_{-i})\ \mathcal{R}\ (y_i, b_{-i})] \\ \text{or} \\ Not[(z_i, c_{-i})\ \mathcal{R}\ (w_i, d_{-i})] \end{array}\right.$$

if and only if $\mathcal{R}^{cd}$ satisfies:

$$\left.\begin{array}{c} (y_i, d_{-i})\ \mathcal{R}^{cd}\ (x_i, c_{-i}) \\ \text{and} \\ (w_i, b_{-i})\ \mathcal{R}^{cd}\ (z_i, a_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, b_{-i})\ \mathcal{R}^{cd}\ (x_i, a_{-i}) \\ \text{or} \\ (w_i, d_{-i})\ \mathcal{R}^{cd}\ (z_i, c_{-i}), \end{array}\right.$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$. Clearly, the above condition is axiom $RC1_i$ imposed on relation $\mathcal{R}^{cd}$. Hence, $\mathcal{R}$ satisfies $RC1_1$ iff its co-dual does. It can be similarly shown that it is also the case for $RC2_i$. We refer to this property saying that axioms $RC1$ and $RC2$ are *self co-dual.*

The picture is not exactly the same for $M1$ and $M2$. Let us recall axioms *Maj*1 and *Maj*2 that have been introduced for characterizing strict concordance relations in BP05b, BP06, Th. 2.

Definition 29 (Axioms *Maj*1, *Maj*2)
Let $\mathcal{R}$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. This relation is said to satisfy:

$$Maj1_i \text{ if } \left.\begin{array}{c} (x_i, a_{-i})\ \mathcal{R}\ (y_i, b_{-i}) \\ \textit{and} \\ (z_i, a_{-i})\ \mathcal{R}\ (w_i, b_{-i}) \\ \textit{and} \\ (z_i, c_{-i})\ \mathcal{R}\ (w_i, d_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i})\ \mathcal{R}\ (x_i, b_{-i}) \\ \textit{or} \\ (x_i, c_{-i})\ \mathcal{R}\ (y_i, d_{-i}), \end{array}\right. \tag{21}$$

$$Maj2_i \text{ if } \left.\begin{array}{c} (x_i, a_{-i})\ \mathcal{R}\ (y_i, b_{-i}) \\ \textit{and} \\ (w_i, a_{-i})\ \mathcal{R}\ (z_i, b_{-i}) \\ \textit{and} \\ (y_i, c_{-i})\ \mathcal{R}\ (x_i, d_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i})\ \mathcal{R}\ (x_i, b_{-i}) \\ \textit{or} \\ (z_i, c_{-i})\ \mathcal{R}\ (w_i, d_{-i}), \end{array}\right. \tag{22}$$

*for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$. We say that Maj*1 *(resp. Maj*2*) holds if $Maj1_i$ (resp. $Maj2_i$) holds for all $i \in N$.*

Contraposition of $M1_i$ yields: $\mathcal{R}$ satisfies $M1_i$ if and only if $\mathcal{R}^{cd}$ satisfies the following condition: for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$,

$$\left.\begin{array}{c} (x_i, b_{-i}) \; \mathcal{R}^{cd} \; (y_i, a_{-i}) \\ \text{and} \\ (z_i, b_{-i}) \; \mathcal{R}^{cd} \; (w_i, a_{-i}) \\ \text{and} \\ (y_i, d_{-i}) \; \mathcal{R}^{cd} \; (x_i, c_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, b_{-i}) \; \mathcal{R}^{cd} \; (x_i, a_{-i}) \\ \text{or} \\ (w_i, d_{-i}) \; \mathcal{R}^{cd} \; (z_i, c_{-i}). \end{array}\right. \tag{23}$$

It is readily seen that this condition is axiom $Maj2_i$ imposed on relation $\mathcal{R}^{cd}$. Indeed expressions (22) and (23) only differ by the positions of a_{-i} and b_{-i}, c_{-i} and d_{-i}, z_i and w_i, which have been interchanged, and by the substitution of $\mathcal{R}$ with $\mathcal{R}^{cd}$. Paraphrasing this result, we state that imposing $M1_i$ on $\mathcal{R}$ is tantamount to imposing $Maj2_i$ on its co-dual $\mathcal{R}^{cd}$ and conversely.

In a similar way, starting from condition $M2_i$ imposed on $\mathcal{R}$, we obtain the following equivalent condition imposed on $\mathcal{R}^{cd}$: for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$,

$$\left.\begin{array}{c} (x_i, b_{-i}) \; \mathcal{R}^{cd} \; (y_i, a_{-i}) \\ \text{and} \\ (w_i, b_{-i}) \; \mathcal{R}^{cd} \; (z_i, a_{-i}) \\ \text{and} \\ (w_i, d_{-i}) \; \mathcal{R}^{cd} \; (z_i, c_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, b_{-i}) \; \mathcal{R}^{cd} \; (x_i, a_{-i}) \\ \text{or} \\ (x_i, d_{-i}) \; \mathcal{R}^{cd} \; (y_i, c_{-i}). \end{array}\right. \tag{24}$$

We observe that the latter condition is axiom $Maj1_i$ imposed on relation $\mathcal{R}^{cd}$ (with the positions of a_{-i} and b_{-i}, c_{-i} and d_{-i}, z_i and w_i having been interchanged). Imposing $M2_i$ on $\mathcal{R}$ is equivalent to imposing $Maj1_i$ on its co-dual $\mathcal{R}^{cd}$ and conversely.

We collect our findings in the next lemma. Its proof results from the above observations.

Lemma 30
Let $\mathcal{R}$ be any relation on X and $\mathcal{R}^{cd}$ its co-dual. The following statements hold, for all $i \in N$:

1. *$\mathcal{R}$ satisfies $RC1_i$ iff $\mathcal{R}^{cd}$ satisfies $RC1_i$,*
2. *$\mathcal{R}$ satisfies $RC2_i$ iff $\mathcal{R}^{cd}$ satisfies $RC2_i$,*
3. *$\mathcal{R}$ satisfies $M1_i$ iff $\mathcal{R}^{cd}$ satisfies $Maj2_i$,*
4. *$\mathcal{R}$ satisfies $M2_i$ iff $\mathcal{R}^{cd}$ satisfies $Maj1_i$.*

Co-duality induces a simple correspondence between the relations comparing preference differences on each attribute, namely the relations $\succsim_i^*$ (resp. $\succsim_i^{**}$) associated with a relation $\mathcal{R}$ and its co-dual $\mathcal{R}^{cd}$. To avoid ambiguity, we write $\succsim_i^*(\mathcal{R})$, $\succsim_i^{**}(\mathcal{R})$ (resp. $\succsim_i^*(\mathcal{R}^{cd})$, $\succsim_i^{**}(\mathcal{R}^{cd})$) to denote the two relations comparing preference differences on attribute i associated with $\mathcal{R}$ (resp. $\mathcal{R}^{cd}$). Using Definition 21 and that of $\mathcal{R}^{cd}$, it is straightforward to establish the following result.

Lemma 31
Let $\mathcal{R}$ be any relation on X and $\mathcal{R}^{cd}$ its co-dual. For all $i \in N$, for all $x_i, y_i, z_i, w_i \in X_i$, we have:

1. $(x_i, y_i) \succsim_i^*(\mathcal{R}^{cd})\ (z_i, w_i)$ *iff* $(w_i, z_i) \succsim_i^*(\mathcal{R})\ (y_i, x_i)$,
2. $(x_i, y_i) \succsim_i^{**}(\mathcal{R}^{cd})\ (z_i, w_i)$ *iff* $(x_i, y_i) \succsim_i^{**}(\mathcal{R})\ (z_i, w_i)$.

Remark 32
In Remark 25, we pointed out that $M1_i$ (resp. $M2_i$) is a weakening of axiom UC_i (resp. LC_i) that was used in an earlier–non independent–characterization of concordance relations. It is easy to see that also axioms $Maj1_i$ and $Maj2_i$ are respectively weakened forms of UC_i and LC_i, obtained by imposing an additional clause (the second one) as a premise. We thus have that UC_i entails $Maj1_i$ and LC_i entails $Maj2_i$, a property that will be used below.

In the context of the present paper, UC and LC are at an advantage w.r.t. $M1$ and $M2$ or $Maj1$ and $Maj2$ since they form a pair of *cross co-dual* conditions. It is indeed easy to check that imposing UC_i on relation $\mathcal{R}$ is equivalent to imposing LC_i on its co-dual $\mathcal{R}^{cd}$ and conversely. •

Starting from the characterization of a reflexive concordance relation (Theorem 26) and using the results of Lemma 30, we easily obtain a "dual" characterization of irreflexive CR's. Actually, the characterization of reflexive CR's is also valid for irreflexive CR's and conversely. The following lemma will help us establishing characterizations that are valid for both reflexive and irreflexive concordance relations. Recall that a CR is either reflexive or irreflexive (Proposition 8.1).

Lemma 33
The following implications hold for all $i \in N$:

1. *$M1_i$ and $RC2_i$ entail $Maj1_i$,*
2. *$Maj1_i$ and $RC1_i$ entail $M1_i$,*
3. *$M2_i$ and $RC1_i$ entail $Maj2_i$,*
4. *$Maj2_i$ and $RC2_i$ entail $M2_i$.*

Under $RC1_i$ and $RC2_i$, we have:

1. $M1_i \Leftrightarrow Maj1_i$,
2. $M2_i \Leftrightarrow Maj2_i$.

Proof

1. Assume that $RC2_i$ and $M1_i$ hold for a relation $\mathcal{R}$. If we have $Not[(w_i, a_{-i})\ \mathcal{R}\ (z_i, b_{-i})]$ in (19), $M1_i$ entails the two remaining possible consequences in (21). On the contrary, if $(w_i, a_{-i})\ \mathcal{R}\ (z_i, b_{-i})$ holds true, since we have that $(z_i, c_{-i})\ \mathcal{R}\ (w_i, d_{-i})$, we may apply $RC2_i$ yielding $(y_i, a_{-i})\ \mathcal{R}\ (x_i, b_{-i})$ or $(x_i, c_{-i})\ \mathcal{R}\ (y_i, d_{-i})$, which are the desired consequences in order to establish that $Maj1_i$ holds.

2. Assume that $RC1_i$ and $Maj1_i$ hold. If $(z_i, a_{-i})\ \mathcal{R}\ (w_i, b_{-i})$ is true then $Maj1_i$ implies that the first or the third conclusion of $M1_i$ is true. Otherwise, we have $(x_i, a_{-i})\ \mathcal{R}\ (y_i, b_{-i})$ and $(z_i, c_{-i})\ \mathcal{R}\ (w_i, d_{-i})$. Applying $RC1_i$ we get either $(z_i, a_{-i})\ \mathcal{R}\ (w_i, b_{-i})$ or $(x_i, c_{-i})\ \mathcal{R}\ (y_i, d_{-i})$. We have assumed that the former does not hold, hence the latter is true, which establishes $M1_i$.

3. Assuming that $RC1_i$ and $M2_i$ hold, we show that $Maj2_i$ is satisfied. If the second consequence in (20) does not hold, i.e., if we have $Not[(z_i, a_{-i})\ \mathcal{R}\ (w_i, b_{-i})]$, then $M2_i$ entails one or the other consequence in $Maj2_i$. On the contrary, if $(z_i, a_{-i})\ \mathcal{R}\ (w_i, b_{-i})$ holds, considering the third premise of $Maj2_i$, i.e., $(y_i, c_{-i})\ \mathcal{R}\ (x_i, d_{-i})$ and using $RC1_i$, we get either consequence of $Maj2_i$.

4. Finally, assuming that $RC2_i$ and $Maj2_i$ hold, we derive $M2_i$. This is immediate whenever $(w_i, a_{-i})\ \mathcal{R}\ (z_i, b_{-i})$ is true since then $Maj2_i$ implies that the first or the third conclusion of $M2_i$ is true. In the opposite case, from $(x_i, a_{-i})\ \mathcal{R}\ (y_i, b_{-i})$ and $(y_i, c_{-i})\ \mathcal{R}\ (x_i, d_{-i})$, we obtain, using $RC2_i$, that $(w_i, a_{-i})\ \mathcal{R}\ (z_i, b_{-i})$ or $(z_i, c_{-i})\ \mathcal{R}\ (w_i, d_{-i})$. Since the former has been assumed to be false, the latter, which is the third conclusion of $M2_i$, is true, concluding the proof.

The equivalence of $M1_i$ and $Maj1_i$ under $RC1_i$ and $RC2_i$ results from the first two items. The equivalence of $M2_i$ and $Maj2_i$, under $RC1_i$ and $RC2_i$ is a consequence of items 3 and 4. □

We also have the following result.

Lemma 34
If relation $\mathcal{R}$ satisfies $RC2_i$, $M1_i$ and $Maj2_i$, then it satisfies $RC1_i$.

Proof

By Lemma 11.3 in BP07, we have that $RC2_i$ and $M1_i$ imply UC_i. By co-duality arguments, this implies that $RC2_i$ and $Maj2_i$ imply LC_i. By Lemma 8.3 in BP07, $RC2_i$, UC_i and LC_i imply $RC1_i$. □

Theorem 35 (Characterizations of CR)
Let $\mathcal{R}$ be a relation on X. The following statements are equivalent:

1. *$\mathcal{R}$ is a concordance relation (CR),*
2. *$\mathcal{R}$ satisfies $RC1$, $RC2$, $M1$ and $M2$,*
3. *$\mathcal{R}$ satisfies $RC1$, $RC2$, $Maj1$ and $Maj2$.*

The axioms used in each of the above characterizations are independent in the class of complete relations and in the class of asymmetric relations.

Proof

1. Let us first note that any relation $\mathcal{R}$ that satisfies $RC2$ is either reflexive or irreflexive. The relation is irreflexive if for all $x \in X$, we have $Not[x \mathcal{R} x]$. Assume there is some x such that $x \mathcal{R} x$ and consider any $z \in X$. From $(x_i, x_{-i}) \mathcal{R} (x_i, x_{-i})$ and $RC2_i$, we deduce that $(z_i, x_{-i}) \mathcal{R} (z_i, x_{-i})$. For $j \neq i$, using $RC2_j$, we can similarly show that $(z_i, z_j, x_{-ij}) \mathcal{R} (z_i, z_j, x_{-ij})$. Continuing in a similar way, we finally obtain $z \mathcal{R} z$.

2. By Proposition 8.1, we know that any concordance relation $\mathcal{R}$ is either reflexive or irreflexive. If $\mathcal{R}$ is reflexive, Theorem 26 establishes the first characterization. Since any irreflexive CR is the co-dual of a reflexive CR, Lemma 30 implies that the second characterization holds for irreflexive CR's.

3. Lemma 33 establishes that under conditions $RC1$ and $RC2$, $M1$ is equivalent to $Maj1$ and $M2$ is equivalent to $Maj2$. As a consequence, both characterizations are valid for reflexive CR's. Using Lemma 30, this implies that both characterizations are also valid for irreflexive CR's.

4. We know (Theorem 26) that axioms $RC1$, $RC2$, $M1$ and $M2$ (first characterization) are independent in the class of complete relations. The following examples (the same as for Theorem 26) show their independence. Each axiom is violated on a single attribute, referred to by i, and satisfied on all other attributes.

Violated axiom	$RC1_i$	$RC2_i$	$M1_i$	$M2_i$
Example	78	76	77	82

The following examples (see in appendix) prove that $RC1$, $RC2$, $Maj1$ and $Maj2$ are also independent in the class of complete relations:

Violated axiom	$RC1_i$	$RC2_i$	$Maj1_i$	$Maj2_i$
Example	79	76	77	82

Using co-duality, this implies that both sets of axioms are independent in the class of asymmetric relations. □

Remark 36
For showing the independence of $RC1$ in both characterizations, we need two different examples (we used Examples 78 and 79). It is indeed a consequence of Lemma 34 that there is no relation satisfying $RC2_i$, $M1_i$, $Maj1_i$, $M2_i$, $Maj2_i$ and $Not[RC1_i]$. •

Remark 37 (Earlier characterization of asymmetric CR)
For asymmetric relations, the properties in the third item in Theorem 35 have been previously shown to constitute a characterization of a concordance relation by a set of independent axioms (BP06, Theorem 2). •

Remark 38 (Other characterizations)
In view of Lemma 33, it is clear that

- $\mathcal{R}$ satisfies $RC1$, $RC2$, $Maj1$ and $M2$,
- $\mathcal{R}$ satisfies $RC1$, $RC2$, $M1$ and $Maj2$,

are two alternative characterizations of a CR. The examples used in the proof of Theorem 35 for showing that $RC1$, $RC2$, $M1$ and $M2$ are independent in the class of complete relations also show that $RC1$, $RC2$, $Maj1$ and $M2$ are independent in the same class. By co-duality, this implies that $RC1$, $RC2$, $Maj1$ and $M2$ are independent in the class of asymmetric relations. This means that these axioms constitute a third independent characterization of CR. In contrast, $RC1$, $RC2$, $M1$ and $Maj2$ do not form an independent family of axioms, be it in the class of complete or in the class of asymmetric relations, as implied by Lemma 34. We have no simple explanation for this asymmetry. We conjecture that it is linked to the fact that the respective rôles of $RC1$ and of $RC2$ are not symmetric in our analysis. •

Remark 39 (Axioms *UC* and *LC* and co-duality)
Since UC and LC are cross co-dual conditions (see Remark 32), the family of axioms $RC1, RC2$, UC and LC clearly offer a characterization of concordance relations within both reflexive or irreflexive relations (as well as within both complete or asymmetric relations). Unfortunately, these axioms are not independent since BP05a, Lemma 16 establishes that $RC2_i$, UC_i and LC_i imply $RC1_i$. Dropping $RC1$, however, yields an independent characterization of a CR since we know (BP05a, Theorem 18) that a binary relation is a CR iff it satisfies $RC2$, UC and LC . Moreover, the latter axioms are independent in the class of complete relations as attested by Examples 76, 77 and 82, in appendix. The dependence of $RC1, RC2$, UC and LC lead us to introduce axioms $M1$ and $M2$, which are discussed in BP07. Theorem 35 tells us that $Maj1$ and $Maj2$, introduced for characterizing asymmetric relations, can substitute $M1$ and $M2$, also in the case of complete relations, without hampering the independence of the axioms. •

4.3 A new independent self co-dual characterization of CR

Axioms $M1_i$ and $Maj1_i$ (resp. $M2_i$ and $Maj2_i$) admit a common weaker formulation, $MM1_i$ (resp. $MM2_i$), that simplifies the characterizations of CR and will be useful in the sequel.

Definition 40 (Axioms *MM1*, *MM2*)
Let $\mathcal{R}$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. This relation is said to satisfy:

$$MM1_i \text{ if } \left.\begin{array}{c}(x_i, a_{-i}) \mathcal{R} (y_i, b_{-i})\\ \text{and}\\ (z_i, a_{-i}) \mathcal{R} (w_i, b_{-i})\\ \text{and}\\ (z_i, c_{-i}) \mathcal{R} (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \mathcal{R} (x_i, b_{-i})\\ \text{or}\\ (w_i, a_{-i}) \mathcal{R} (z_i, b_{-i})\\ \text{or}\\ (x_i, c_{-i}) \mathcal{R} (y_i, d_{-i}),\end{array}\right.$$

$$MM2_i \text{ if } \left.\begin{array}{c}(x_i, a_{-i}) \mathcal{R} (y_i, b_{-i})\\ \text{and}\\ (w_i, a_{-i}) \mathcal{R} (z_i, b_{-i})\\ \text{and}\\ (y_i, c_{-i}) \mathcal{R} (x_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(y_i, a_{-i}) \mathcal{R} (x_i, b_{-i})\\ \text{or}\\ (z_i, a_{-i}) \mathcal{R} (w_i, b_{-i})\\ \text{or}\\ (z_i, c_{-i}) \mathcal{R} (w_i, d_{-i}),\end{array}\right.$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$. We say that $MM1$ (resp. $MM2$) holds if $MM1_i$ (resp. $MM2_i$) holds for all $i \in N$.

Note that $MM1_i$, without its second premise, is identical to $M1_i$. $MM1_i$, without its second conclusion, is $Maj1_i$. $MM1_i$, without both its second premise and its second conclusion, is UC_i. $MM1_i$ is clearly a weaker condition than both $M1_i$ and

$Maj1_i$. Similar observations can be made, linking $MM2_i, M2_i$ and $Maj2_i$. $MM2_i$ is a weakened variant of the two others. However, under $RC1_i$ and $RC2_i$, $MM1_i$ can be shown to be equivalent to $M1_i$, $Maj1_i$ and UC_i, as stated in the following lemma. A similar statement holds for $MM2_i$.

Lemma 41
The following implications hold for all $i \in N$:

1. *$MM1_i$ and $RC1_i$ entail $M1_i$,*
2. *$MM1_i$ and $RC2_i$ entail $Maj1_i$,*
3. *$MM2_i$ and $RC2_i$ entail $M2_i$,*
4. *$MM2_i$ and $RC1_i$ entail $Maj2_i$.*

Under $RC1_i$ and $RC2_i$, we have:

1. *$MM1_i \Leftrightarrow M1_i \Leftrightarrow Maj1_i \Leftrightarrow UC_i$,*
2. *$MM2_i \Leftrightarrow M2_i \Leftrightarrow Maj2_i \Leftrightarrow LC_i$.*

PROOF
The proofs are very similar to those used to establish Lemma 33. We prove the first implication, leaving the three others to the reader. Assume that $MM1_i$ and $RC1_i$ hold. We show that $M1_i$ must be true. Applying $RC1_i$ to the premises of $M1_i$ yields $(x_i, c_{-i}) \mathcal{R} (y_i, d_i)$ or $(z_i, a_{-i}) \mathcal{R} (w_i, b_{-i})$. If the former is true, then $M1_i$ is verified. Else, all three premises of $MM1_i$ are satisfied, which entails the disjunction of three conclusions that is common to $M1_i$ and $MM1_i$. The proofs of the two equivalences directly results from the four implications, Lemma 33 and Lemma 11.3 and 11.4 in BP07. The latter says that $RC2_i$ and $M1_i$ entail UC_i and that $RC1_i$ and $M2_i$ entail LC_i. By definition, UC_i entails $M1_i$ and LC_i entails $M2_i$. □

Axioms $MM1_i$ and $MM2_i$ are *cross co-dual*, as are UC_i and LC_i (Remark 32). More precisely, we have:

Lemma 42
Let $\mathcal{R}$ be a binary relation on X and $\mathcal{R}^{cd}$ its co-dual. The following hold for all $i \in N$:

1. *$\mathcal{R}$ satisfies $MM1_i$ iff $\mathcal{R}^{cd}$ satisfies $MM2_i$,*
2. *$\mathcal{R}$ satisfies $MM2_i$ iff $\mathcal{R}^{cd}$ satisfies $MM1_i$.*

Proof

The proof results immediately from contraposition and using the definition of the co-dual. □

Using Lemmas 41 and 42, it is easy to derive the following new characterization result.

Theorem 43

The relation $\mathcal{R}$ on X is a CR iff it satisfies $RC1, RC2, MM1$ and $MM2$. These axioms are independent both in the class of complete relations and in the class of asymmetric relations.

Proof

Using Lemma 41.1 and 41.3, we obtain that a relation $\mathcal{R}$ satisfying $RC1$, $RC2$, $MM1$ and $MM2$ also satisfies $M1$ and $M2$. Theorem 35 entails that $\mathcal{R}$ is a CR. The converse is also true by Theorem 35 and the fact that $M1$ implies $MM1$ and $M2$ implies $MM2$. The examples used to show the independence of the axioms in Theorem 35 can be used here. This is due to the fact, on the one hand, that $M1$ and $Maj1$ imply $MM1$, hence if $\mathcal{R}$ is an example of relation satisfying one of the former, it satisfies the latter. On the other hand, if $\mathcal{R}$ is an example that does not satisfy $M1_i$ or $Maj1_i$ (resp. $M2_i$ or $Maj2_i$), while satisfying $RC1$ and $RC2$, it cannot satisfy $MM1_i$ (resp. $MM2_i$), due to Lemma 41. □

4.4 Concordance relations with attribute transitivity

An additional property of CR, called *attribute transitivity*, was defined and studied in BP05a, BP07. Attribute transitivity amounts to assuming that the relations S_i in Definition 2 are *semiorders* as is the case in most ordinal aggregation methods. We have shown in the two above-mentioned papers that reflexive concordance relations with attribute transitivity (R-CR-AT) can be characterized by adding axioms $AC1, AC2$ and $AC3$, which are similar to $RC1$ and $RC2$ and were introduced and discussed in BP02a, BP04. We recall these axioms and examine how they behave w.r.t. co-duality. Reflexive CR with attribute transitivity have been characterized in BP05a, BP07. We use co-duality to derive characterizations of irreflexive CR with attribute transitivity.

Axioms $AC1$, $AC2$ and $AC3$ are recalled in the following definition.

Definition 44 (Axioms $AC1$, $AC2$, $AC3$)

Let $\mathcal{R}$ be a binary relation on a set $X = \prod_{i=1}^{n} X_i$. This relation is said to satisfy:

$$AC1_i \text{ if } \left.\begin{array}{c}(x_i, a_{-i}) \mathcal{R} (y_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i}) \mathcal{R} (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(z_i, a_{-i}) \mathcal{R} (y_i, b_{-i}) \\ \text{or} \\ (x_i, c_{-i}) \mathcal{R} (w_i, d_{-i}),\end{array}\right.$$

$$AC2_i \text{ if } \left.\begin{array}{c}(x_i, a_{-i}) \mathcal{R} (y_i, b_{-i}) \\ and \\ (z_i, c_{-i}) \mathcal{R} (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(x_i, a_{-i}) \mathcal{R} (w_i, b_{-i}) \\ or \\ (z_i, c_{-i}) \mathcal{R} (y_i, d_{-i})\end{array}\right.$$

$$AC3_i \text{ if } \left.\begin{array}{c}(x_i, a_{-i}) \mathcal{R} (y_i, b_{-i}) \\ and \\ (y_i, c_{-i}) \mathcal{R} (w_i, d_{-i})\end{array}\right\} \Rightarrow \left\{\begin{array}{c}(x_i, a_{-i}) \mathcal{R} (z_i, b_{-i}) \\ or \\ (z_i, c_{-i}) \mathcal{R} (w_i, d_{-i})\end{array}\right.$$

for all $x_i, y_i, z_i, w_i \in X_i$, *all* $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$.

We say that $\mathcal{R}$ *satisfies* $AC1$ *(resp.* $AC2$, $AC3$*) if it satisfies* $AC1_i$ *(resp.* $AC2_i$, $AC3_i$*) for all* $i \in N$.

An interpretation of these axioms was provided in BP04. Essentially, these axioms are related to the existence of linear arrangements of the elements (levels) of X_i. $AC1_i$ suggests that the elements of X_i can be linearly ordered relatively to "upward dominance": if x_i "upward dominates" z_i, then $(z_i, c_{-i}) \mathcal{R} (w_i, d_{-i})$ entails $(x_i, c_{-i}) \mathcal{R} (w_i, d_{-i})$. $AC2_i$ has a similar interpretation regarding "downward dominance". $AC3_i$ ensures that the upward and downward dominance orders are not incompatible. The following gives a precise definition of the upward and downward dominance relations.

Definition 45 (Relations $\succsim_i^+$, $\succsim_i^-$ and $\succsim_i^\pm$)
Let $\mathcal{R}$ *be a binary relation on a set* $X = \prod_{i=1}^n X_i$. *We define the binary relations* $\succsim_i^+$, $\succsim_i^-$ *and* $\succsim_i^\pm$ *on* X_i *letting, for all* $x_i, y_i \in X_i$,

$$x_i \succsim_i^+ y_i \Leftrightarrow \forall a_{-i} \in X_{-i}, b \in X, [(y_i, a_{-i}) \mathcal{R}\, b \Rightarrow (x_i, a_{-i}) \mathcal{R}\, b], \tag{25}$$

$$x_i \succsim_i^- y_i \Leftrightarrow \forall a \in X, b_{-i} \in X_{-i}, [a \mathcal{R} (x_i, b_{-i}) \Rightarrow a \mathcal{R} (y_i, b_{-i})], \tag{26}$$

$$x_i \succsim_i^\pm y_i \Leftrightarrow x_i \succsim_i^+ y_i \text{ and } x_i \succsim_i^- y_i. \tag{27}$$

By definition, $\succsim_i^+$, $\succsim_i^-$ and $\succsim_i^\pm$ are transitive relations. Axioms $AC1_i$, $AC2_i$ and $AC3_i$ ensure that they are complete, as restated in the next lemma.

Lemma 46 (Bouyssou and Pirlot (2004), Lemma 3.1–4)
Let $\mathcal{R}$ *be a binary relation on a set* $X = \prod_{i=1}^n X_i$. $\mathcal{R}$ *satisfies:*

1. $AC1_i \Leftrightarrow \succsim_i^+$ *is complete,*

2. $AC2_i \Leftrightarrow \succsim_i^-$ *is complete,*

3. $AC3_i \Leftrightarrow [Not[x_i \succsim_i^+ y_i] \Rightarrow y_i \succsim_i^- x_i] \Leftrightarrow [Not[x_i \succsim_i^- y_i] \Rightarrow y_i \succsim_i^+ x_i]$,

4. $[AC1_i$, $AC2_i$ *and* $AC3_i] \Leftrightarrow \succsim_i^\pm$ *is complete.*

As implied by Remark 3, a concordance relation with attribute transitivity is a CR having a representation $\langle \trianglerighteq, S_i \rangle$, with relations S_i that are semiorders. The following theorem characterizes CR with attribute transitivity. As for Theorems 26 and 28, it is a slight variation on Theorem 26 in BP07, which is formulated here for general binary relations. The independence of the axioms is not only valid for reflexive but also complete relations.

Theorem 47
The binary relation $\mathcal{R}$ on $X = \prod_{i=1}^{n} X_i$ is a concordance relation with attribute transitivity (CR-AT), i.e., has a representation $\langle \trianglerighteq, S_i \rangle$ in which all S_i are semiorders, iff it satisfies $RC1$, $RC2$, $AC1$, $AC3$, $M1$, $M2$. In the class of complete relations, these axioms are independent

PROOF
The proof of the characterization result in Theorem 26 in BP07 remains valid for general binary relations and it is omitted. The latter was formulated for reflexive relations but this hypothesis plays no rôle in the proof so that the result is valid for general relations. We recall the examples establishing the independence of the axioms in the proof of Lemma 19 in BP07:

Violated axiom	$RC1_i$	$RC2_i$	$AC1_i$	$AC3_i$	$M1_i$	$M2_i$
Example	78	76	80	81	77	82

All these relations are complete. Therefore the axioms are independent in the set of complete relations. □

Remark 48
Note that axiom $AC2$ does not appear in this characterization because it is not independent of the other axioms. Indeed, Lemma 27.1 in BP06 and Lemma 11, items 3 and 4 in BP07 imply that under $RC1$, $RC2$, $M1$ and $M2$, axioms $AC1$ and $AC2$ are equivalent. $AC2$ can thus substitute $AC1$ in the characterization of reflexive CR with attribute transitivity. •

We now examine how axioms $AC1$, $AC2$, $AC3$ can be transposed in terms of the co-dual relation $\mathcal{R}^{cd}$. Using contraposition, as we have done above with $RC1_i$, we can easily prove the following results.

Lemma 49
Let $\mathcal{R}$ be any relation on X and $\mathcal{R}^{cd}$ its co-dual. The following equivalences hold, for all $i \in N$:

1. *$\mathcal{R}$ satisfies $AC1_i$ iff $\mathcal{R}^{cd}$ satisfies $AC2_i$,*
2. *$\mathcal{R}$ satisfies $AC2_i$ iff $\mathcal{R}^{cd}$ satisfies $AC1_i$,*

3. $\mathcal{R}$ *satisfies* $AC3_i$ *iff* $\mathcal{R}^{cd}$ *satisfies* $AC3_i$.

In the theorem below we extend the characterizations obtained in Theorem 35 to CR with attribute transitivity (CR-AT). The next lemma will be used in the proof of the theorem.

Lemma 50
Let $\mathcal{R}$ *be a relation on* X *and* $\mathcal{R}^d$ *its dual relation. We have the following:*

1. $\mathcal{R}$ *satisfies* $RC1_i$ *(resp.* $RC2_i$, $AC3_i$, $M1_i$, $Maj1_i$, $M2_i$, $Maj2_i$, UC_i, LC_i*) for some* $i \in N$ *if and only if* $\mathcal{R}^d$ *satisfies the same property,*

2. $\mathcal{R}$ *satisfies* $AC1_i$ *(resp.* $AC2_i$*) for some* $i \in N$ *if and only if* $\mathcal{R}^d$ *satisfies* $AC2_i$ *(resp.* $AC1_i$*) for the same* i.

Proof
The proof consists in checking that each of the equivalences holds, starting from the properties definition. It is easy once it is noted that, for all $i \in N$ and all $x_i, y_i, z_i, w_i \in X_i$,

$$x_i \succsim_i^+(\mathcal{R}^d)\ y_i \Leftrightarrow y_i \succsim_i^-(\mathcal{R})\ x_i,$$
$$x_i \succsim_i^-(\mathcal{R}^d)\ y_i \Leftrightarrow y_i \succsim_i^+(\mathcal{R})\ x_i,$$
$$(x_i, y_i) \succsim_i^*(\mathcal{R}^d)\ (z_i, w_i) \Leftrightarrow (y_i, x_i) \succsim_i^*(\mathcal{R})\ (w_i, z_i),$$

where $\succsim_i^+(K)$ (resp. $\succsim_i^-(K)$, $\succsim_i^*(K)$) denotes the relation $\succsim_i^+$ (resp. $\succsim_i^-$, $\succsim_i^*$) using K as the base relation. □

A result similar to Lemma 31 can be established for the upward and downward dominance relations $\succsim_i^+, \succsim_i^-, \succsim_i^\pm$. As in this lemma, our notation makes explicit whether the upward and downward dominance relations refer to $\mathcal{R}$ or its co-dual $\mathcal{R}^{cd}$.

Lemma 51
Let $\mathcal{R}$ *be a binary relation on* X *and* $\mathcal{R}^{cd}$ *its co-dual. For all* $i \in N$*, for all* $x_i, y_i \in X_i$*, we have:*

1. $x_i \succsim_i^+(\mathcal{R}^{cd})\ y_i$ *iff* $x_i \succsim_i^-(\mathcal{R})\ y_i$,

2. $x_i \succsim_i^-(\mathcal{R}^{cd})\ y_i$ *iff* $x_i \succsim_i^+(\mathcal{R})\ y_i$,

3. $x_i \succsim_i^\pm(\mathcal{R}^{cd})\ y_i$ *iff* $x_i \succsim_i^\pm(\mathcal{R})\ y_i$.

Proof
The proof follows immediately from Definition 45 and from that of $\mathcal{R}^{cd}$. □

Theorem 52 (Characterization of CR-AT)
The relation $\mathcal{R}$ on X is a concordance relation with attribute transitivity (CR-AT) iff $\mathcal{R}$ satisfies $RC1$, $RC2$, $AC1$, $AC3$, $MM1$ and $MM2$. These axioms are independent in the class of complete relations and in the class of asymmetric relations. In this characterization, $AC1$ can be substituted by $AC2$ without any other change.

Proof
If $\mathcal{R}$ is a CR-AT, we know by Theorem 47 that it satisfies $RC1$, $RC2$, $AC1$, $AC3$, $M1$ and $M2$. $M1$ (resp. $M2$) implies $MM1$ (resp. $MM2$). Conversely, if $\mathcal{R}$ satisfies $RC1$, $RC2$, $AC1$, $AC3$, $MM1$ and $MM2$, it satisfies $M1$ and $M2$ (Lemma 41) hence it is a CR-AT.

By Remark 48, we may substitute $AC1$ by $AC2$ in the characterization since, under $RC1$, $RC2$, $M1$ and $M2$, axioms $AC1$ and $AC2$ are equivalent.

To prove the independence of the axioms in the set of complete relations, the examples used in the proof of Theorem 47 are also suitable here since these examples satisfy $M1_i$ whenever they satisfy $MM1_i$ and similarly for $M2_i$ and $MM2_i$. According with Lemma 50, t he duals of these examples show that substituting $AC1$ by $AC2$ in the characterization preserves the independence of the axioms. The co-duals of the same examples and of their duals are asymmetric relations showing the independence of the axioms characterizing CR-AT in the class of asymmetric relations. □

Corollary 53
The relation $\mathcal{R}$ on X is a concordance relation with attribute transitivity (CR-AT) iff $\mathcal{R}$ satisfies $RC1$, $RC2$, $AC1$, $AC3$, $Maj1$ and $Maj2$. These axioms are independent in the class of complete relations and in the class of asymmetric relations. In this characterization, $AC1$ can be substituted by $AC2$ without any other change.

Proof
Under $RC1$ and $RC2$, $MM1$ is equivalent to $Maj1$ and $MM2$ to $Maj2$ (Lemma 41). This new characterization hence results from Theorem 52. The independence of the axioms in the set of complete relations results from the following examples (in appendix):

Violated axiom	$RC1_i$	$RC2_i$	$AC1_i$	$AC3_i$	$Maj1_i$	$Maj2_i$
Example	$(79)^d$	76	80	81	77	82

Note that $(79)^d$ denotes the dual[7] of the relation described in Example 79. In view of Lemma 50, this relation does not satisfy $RC1_1$. It satisfies $AC1_1$ but not $AC2_1$. The possibility of substituting $AC1$ by $AC2$ and keep the independence of

[7] We really mean the *dual*, not the co-dual.

the axioms is shown as in Theorem 52 by taking the dual of the examples. The independence of the axioms in the set of asymmetric relations is established by taking the co-dual of the examples. □

4.5 Characterizations of a CDR

In the last section we have shown that two different "dual" characterizations of concordance relations (Theorem 47 and Corollary 53) can be obtained using co-duality. The picture is not the same for concordance-discordance relations, which are significantly more complex objects than CR. As discussed in Section 3.6, reflexes of automatic "co-dualization" must be abandoned since the co-dual of a CDR is not a CDR but a CRB (Proposition 19). Nonetheless, two characterizations of reflexive and irreflexive CDR can be obtained from previous results.

With Theorem 28, we have recalled a characterization of a reflexive concordance-discordance relation (R-CDR). It involves axiom $M1$ and a weakening of axiom $M2$, called $M3$ (see Definition 27). Examining the proof of this result in BP09b shows that the reflexivity of the relation plays no rôle, so that this characterization is valid both for reflexive and irreflexive CDR's.

A similar characterization of asymmetric CDR was given in BP06, using axiom $Maj1$ and a weakening of $Maj2$, that was called $Maj3$. This axiom was constructed as $M3$ from $M2$, by adding a premise to $Maj2$.

In the same spirit as we introduced, in the previous section, axiom $MM1$ (resp. $MM2$) generalizing both $M1$ and $Maj1$ (resp. $M2$ and $Maj2$), we now define the new axiom $MM3$ as follows.

Definition 54 (*MM*3 and *Maj*3)
A relation $\mathcal{R}$ on X satisfies

$$MM3_i \textit{ if } \left.\begin{array}{c} (x_i, a_{-i}) \mathcal{R} (y_i, b_{-i}) \\ \textit{and} \\ (w_i, a_{-i}) \mathcal{R} (z_i, b_{-i}) \\ \textit{and} \\ (y_i, c_{-i}) \mathcal{R} (x_i, d_{-i}) \\ \textit{and} \\ (z_i, e_{-i}) \mathcal{R} (w_i, f_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i}) \mathcal{R} (x_i, b_{-i}) \\ \textit{or} \\ (z_i, a_{-i}) \mathcal{R} (w_i, b_{-i}) \\ \textit{or} \\ (z_i, c_{-i}) \mathcal{R} (w_i, d_{-i}), \end{array}\right. \tag{28}$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i}, e_{-i}, f_{-i} \in X_{-i}$. $Maj3_i$ is the same condition as $MM3_i$ except that the second conclusion has been removed. We say that $\mathcal{R}$ satisfies $MM3$ (resp. $Maj3$) if it satisfies $MM3_i$ (resp. $Maj3_i$) for all $i \in N$.

Dropping the second premise in $MM3_i$ yields $M3_i$. Removing the second conclusion yields $Maj3_i$. Obviously, $M3_i$ (resp. $Maj3_i$) entails $MM3_i$. Under $RC1_i$ and

$RC2_i$, axioms $M3_i$, $Maj3_i$ and $MM3_i$ are equivalent as shown in the following lemma.

Lemma 55
The following implications hold:

1. *$MM3_i$ and $RC2_i$ entail $M3_i$,*
2. *$MM3_i$ and $RC1_i$ entail $Maj3_i$,*
3. *$M3_i$ and $RC1_i$ entail $Maj3_i$,*
4. *$Maj3_i$ and $RC2_i$ entail $M3_i$.*

Under $RC1_i$ and $RC2_i$, we have:

$$MM3_i \Leftrightarrow M3_i \Leftrightarrow Maj3_i.$$

PROOF
The proof, similar to that of Lemma 41, is left to the reader. □

Remark 56
Note that axioms $M3_i$ and $Maj3_i$ are not linked by co-duality. The co-dual counterpart of $MM3_i$ has not been met before. This is related with the fact that the co-dual of a CDR is not a CDR, in general, but a CRB, i.e., a concordance relation with bonus (Proposition 19). Such relations will be studied and characterized in Section 4.7. •

We are in position to produce a new characterization result, valid for any CDR, which is the following.

Theorem 57 (Characterization of CDR)
The relation $\mathcal{R}$ on X is a concordance-discordance relation (CDR) iff $\mathcal{R}$ satisfies $RC1$, $RC2$, $MM1$ and $MM3$. These axioms are independent in the set of complete relations and in the set of asymmetric relations.

PROOF
By Theorem 28, we know that a relation $\mathcal{R}$ that is a CDR satisfies $RC1$, $RC2$, $M1$ and $M2$. Since $M1$ implies $MM1$ and $M3$ implies $MM3$, $\mathcal{R}$ also satisfies $MM1$ and $MM3$. Conversely, if a relation $\mathcal{R}$ satisfies $RC1$ and $RC2$, then $MM1$ (resp. $MM3$) is equivalent to $M1$ (resp. $M3$) (by Lemmas 41 and 55). Hence, using Theorem 28, we have that $\mathcal{R}$ is a CDR.

In the class of complete relations, the following examples (in appendix) prove the independence of the axioms.

Violated axiom	$RC1_i$	$RC2_i$	$MM1_i$	$MM3_i$
Example	79	76	77	82

For proving the independence of the axioms in the class of asymmetric relations, we can still use co-duality arguments for obtaining part of the required examples but not all of them. Since Examples 79 and 76 satisfy $MM1$ and $MM2$, their co-dual also satisfy these two properties (Lemma 42), hence they satisfy $MM3$. In addition, Example $(79)^{cd}$ satisfies $RC2_i$ but not $RC1_i$ and conversely for Example $(76)^{cd}$. Example 82 satisfies $RC1$, $RC2$, $MM1$, $MM2_j$, for $j \neq 1$ but not $MM2_1$. Its co-dual satisfies $RC1, RC2, MM1_j$ for $j \neq 1$, $MM2$, hence $MM3$, but not $MM1_1$. It thus proves the independence of $MM1_1$ for asymmetric relations. For proving the independence of $MM3$, we need a new example. Example 83 is an asymmetric relation verifying $RC1, RC2, MM1$, but not $MM3_1$. To sum up, the following examples (in appendix) prove the independence of the axioms in the class of asymmetric relations:

Violated axiom	$RC1_i$	$RC2_i$	$MM1_i$	$MM3_i$
Example	$(79)^{cd}$	$(76)^{cd}$	$(82)^{cd}$	83

□

Corollary 58
The relation $\mathcal{R}$ on X is a concordance-discordance relation (CDR)

1. *iff $\mathcal{R}$ satisfies $RC1$, $RC2$, $Maj1$ and $Maj3$,*
2. *iff $\mathcal{R}$ satisfies $RC1$, $RC2$, $M1$ and $M3$.*

Both sets of axioms are independent in the class of complete relations and in the class of asymmetric relations.

Proof
Under $RC1$ and $RC2$, axiom $MM1$ (resp. $MM3$) is equivalent to $Maj1$ (resp. $Maj3$) by Lemmas 41 and 55. Using the same lemmas also entails that, under $RC1$ and $RC2$, axiom $M1$ (resp. $M3$) is equivalent to $MM1$ (resp. $MM3$). The new characterizations are thus a direct consequence of Theorem 57. The independence of the axioms for complete relations as well as for asymmetric relations is established by the same examples as in Theorem 57, except in one case. For complete relations, in order to prove the independence of $RC1$ from the other axioms in the second characterization, we need to invoke Example 78, which is a complete relations satisfying $RC2$, $M1$, $M3$ and $RC1_j$ for $j \neq 1$, but not $RC1_1$. □

4.6 CDR with attribute transitivity

A CDR with attribute transitivity (CDR-AT) is not just a CDR which admits a representation in which S_i are semiorders. A certain relationship between S_i and V_i must also be verified. CDR-AT have been studied in BP09b, Section 5.2. We first recall the definition of a CDR-AT and the characterization result obtained in BP09b.

Definition 59 (CDR with attribute transitivity)
A CDR with attribute transitivity (CDR-AT) is a CDR for which, for all $i \in N$:

- *S_i is a semiorder with asymmetric part P_i,*
- *V_i is the asymmetric part of a semiorder U_i with $U_i \supseteq S_i$ and, hence, $V_i \subseteq P_i$,*
- *(S_i, U_i) form a* homogeneous *chain of semiorders.*

The following is Theorem 29 in BP09b. The independence of the axioms is stated for reflexive relations.

Theorem 60
The relation $\mathcal{R}$ on X is a CDR-AT iff $\mathcal{R}$ satisfies $RC1$, $RC2$, $AC1$, $AC2$, $AC3$, $M1$ and $M3$. These axioms are independent in the class of reflexive relations.

The question of the independence–or not–of the axioms in the class of complete relations and in the class of asymmetric relations is more delicate for CDR-AT than for CR-AT (Theorem 52) or for CDR (Theorem 57). In view of examining the independence issue for CDR-AT in a simpler way, we relax axioms $M1$ and $M3$ into $MM1$ and $MM3$ respectively. In view of Lemmas 41 and 55, it is clear that axioms $RC1$, $RC2$, $AC1$, $AC2$, $AC3$, $MM1$ and $MM3$ yield another characterization of CDR-AT. In the class of complete relations, this set of axioms, although weaker than those used in Theorem 60, are not independent as attested by Proposition 73 in Appendix. Similarly, Proposition 75 in Appendix shows that, if $\mathcal{R}$ is a relation (that may not be complete or asymmetric) on X satisfying $RC2$, $AC1$, $AC2$, $AC3$, $Maj1$ and $Maj3$, then it also also satisfies $RC1$. These are other cases of asymmetry in our results for which we do not presently have a clear explanation.

Our next result is a new characterization theorem for CDR-AT, in the general case and in the case of complete and of asymmetric relations.

Theorem 61

1. *The relation $\mathcal{R}$ on X is a CDR-AT iff $\mathcal{R}$ satisfies $RC1$, $RC2$, $AC1$, $AC2$, $AC3$, $MM1$ and $MM3$. These axioms are independent in the class of reflexive relations.*

2. *If $\mathcal{R}$ is a complete relation on X, it is a CDR-AT iff $\mathcal{R}$ satisfies $RC2$, $AC1$, $AC2$, $AC3$, $MM1$ and $MM3$. These axioms are independent in the class of complete relations.*

3. *If $\mathcal{R}$ is an asymmetric relation on X, it is a CDR-AT iff $\mathcal{R}$ satisfies $RC1$, $RC2$, $AC1$, $AC2$, $AC3$, $MM1$ and $MM3$. These axioms are independent in the class of asymmetric relations.*

Proof

The usual argument, based on Lemmas 41 and 55 allows us to substitute $M1$ by $MM1$ and $M3$ by $MM3$ in the characterization of (general) CDR-AT provided in Theorem 60. The independence of the axioms is established by the same examples as in Theorem 60.

For complete CDR-AT, axiom $RC1$ can be dropped from their characterization, in view of Proposition 73. In the class of complete relations, axioms $RC2$, $AC1$, $AC2$, $AC3$, $MM1$ and $MM3$ are independent as attested by the following examples:

Violated axiom	$RC2_i$	$AC1_i$	$AC2_i$	$AC3_i$	$MM1_i$	$MM3_i$
Example	76	84	85	81	77	82

In the class of asymmetric CDR-AT, the axioms $RC1, RC2$, $AC1$, $AC2$, $AC3$, $MM1$ and $MM3$ are independent as shown by the examples below.

Violated axiom	$RC1$	$RC2_i$	$AC1_i$	$AC2_i$	$AC3_i$	$MM1_i$	$MM3_i$
Example	89	$(76)^{cd}$	87	88	$(81)^{cd}$	86	83

$(76)^{cd}$ designates the co-dual of the relation in Example 76 and similarly for $(81)^{cd}$. □

Remark 62

Substituting $MM1$ by $M1$ or by $Maj1$ and/or $MM3$ by $M3$ or by $Maj3$ in one of the characterizations in Theorem 61 leads to other characterizations of CDR-AT. The resulting sets of axioms remain independent in the class of complete relations but this is not always the case in the class of asymmetric relations, as we shall see.

1. The case of complete relations. The examples used in the proof of Theorem 61 for showing the independence of $RC2_i$, $AC1_i$, $AC2_i$ and $AC3_i$, namely Examples 76, 84, 85 and 81, all satisfy axioms $M1$, $Maj1$, $M3$, $Maj3$. Examples 77 (resp. 82) showing the independence of $MM1$ (resp. $MM3$) satisfies neither $M1$ nor $Maj1$ (resp. neither $M3$ nor $Maj3$).

2. The case of asymmetric relations. In view of Proposition 75 in Appendix, $RC1_i$ is implied by $RC2_i$, $AC1_i$, $AC2_i$, $AC3_i$, $Maj1_i$ and $Maj3_i$. It is also the case when $Maj3$ is substituted by $M3$ (since Lemma 55.4 tells us that $Maj3_i$ and $RC2_i$ entail $M3_i$). The following sets of axioms however are independent in the class of asymmetric relations:

   (a) $RC1$, $RC2$, $AC1$, $AC2$, $AC3$, $M1$ and $M3$,
   (b) $RC1$, $RC2$, $AC1$, $AC2$, $AC3$, $Maj1$ and $M3$.

   The independence of these axioms results from the same examples as those used in Theorem 61 for asymmetric relations. Indeed, Examples 89, $(76)^{cd}$, 87, 88 and $(81)^{cd}$ all satisfy $MM1$, $M1$, $Maj1$, $MM3$, $M3$ and $Maj3$. Example 86 violates not only $MM1$ but also $M1$ and $Maj1$. Example 83 violates not only $MM3$ but also $M3$ and $Maj3$. •

4.7 Concordance relations with bonus

We know that the co-dual of a CDR is a CRB, i.e., a concordance relation with bonus (Definition 18) by Proposition 19. Starting from the characterization of a CDR given in Theorem 57, we can easily derive a characterization of a CRB using contraposition and co-duality.

Lemmas 30 and 42 entail that the co-dual of a CDR is a relation that satisfies $RC1$, $RC2$, $MM2$ and an axiom that is obtained from $MM3$ by using contraposition and co-duality. We call the latter $DMM3$ and define it below.

Definition 63 (Axiom *DMM3*)
A relation $\mathcal{R}$ on X satisfies

$$DMM3_i \text{ if } \left.\begin{array}{c} (x_i, a_{-i})\ \mathcal{R}\ (y_i, b_{-i}) \\ \text{and} \\ (z_i, a_{-i})\ \mathcal{R}\ (w_i, b_{-i}) \\ \text{and} \\ (z_i, c_{-i})\ \mathcal{R}\ (w_i, d_{-i}) \end{array}\right\} \Rightarrow \left\{\begin{array}{c} (y_i, a_{-i})\ \mathcal{R}\ (x_i, b_{-i}) \\ \text{or} \\ (w_i, a_{-i})\ \mathcal{R}\ (z_i, b_{-i}) \\ \text{or} \\ (x_i, c_{-i})\ \mathcal{R}\ (y_i, d_{-i}) \\ \text{or} \\ (z_i, e_{-i})\ \mathcal{R}\ (w_i, f_{-i}), \end{array}\right. \tag{29}$$

for all $x_i, y_i, z_i, w_i \in X_i$ and all $a_{-i}, b_{-i}, c_{-i}, d_{-i}, e_{-i}, f_{-i} \in X_{-i}$. We say that $\mathcal{R}$ satisfies $DMM3$ if it satisfies $DMM3_i$ for all $i \in N$.

Note that dropping the second conclusion of $DMM3_i$ yields an axiom that is the "co-dual" of $M3_i$ and which we call $DM3_i$. In a similar way, dropping the second premise of $DMM3_i$ yields an axiom that is the "co-dual" of $Maj3_i$ and which we shall call $DMaj3_i$. We note these results in the following lemma.

Lemma 64
The relation $\mathcal{R}$ *on* X *satisfies* $MM3_i$ *(resp.* $M3_i, Maj3_i$*) iff its co-dual* $\mathcal{R}^{cd}$ *satisfies* $DMM3_i$ *(resp.* $DM3_i, DMaj3_i$*).*

Comparing $DMM3_i$ with $MM1_i$, we observe that the former only differ from the latter by an additional conclusion. We thus have the following.

Lemma 65
If the relation $\mathcal{R}$ *on* X *satisfies* $MM1_i$ *then it satisfies* $DMM3_i$*.*

As compared with $MM1_i$, $DMM3_i$ offers a fourth possible conclusion, which interprets, under $RC1_i$ and $RC2_i$, as the possible existence of a "preference difference" (z_i, w_i) on attribute i that is "so large" that we always have $(z_i, e_{-i})\ \mathcal{R}\ (w_i, f_{-i})$ whatever the levels e_{-i} and f_{-i} on the other attributes can be. Such a large difference of preference was called a *bonus* in Section 3.6. This interpretation is established in the next lemma.

Lemma 66
Let $\mathcal{R}$ *be a binary relation on* X*. If* $\mathcal{R}$ *satisfies* $RC1_i$*,* $RC2_i$ *and* $DMM3_i$*, then, for all* $x_i, y_i, z_i, w_i, r_i, s_i \in X_i$*, if* $(z_i, w_i) \succ_i^* (x_i, y_i) \succ_i^* (y_i, x_i)$*, we then have:*

1. $(z_i, w_i) \succsim_i^* (r_i, s_i)$*,*
2. $(z_i, e_{-i})\ \mathcal{R}\ (w_i, f_{-i})$*, for all* $e_{-i}, f_{-i} \in X_{-i}$*.*

Proof
If $(z_i, w_i) \succ_i^* (x_i, y_i) \succ_i^* (y_i, x_i)$, there are $a_{-i}, b_{-i}, c_{-i}, d_{-i} \in X_{-i}$, such that (i) $(x_i, a_{-i})\ \mathcal{R}\ (y_i, b_{-i})$, (ii) $Not[(y_i, a_{-i})\ \mathcal{R}\ (x_i, b_{-i})]$, (iii) $(z_i, c_{-i})\ \mathcal{R}\ (w_i, d_{-i})$ and (iv) $Not[(x_i, c_{-i})\ \mathcal{R}\ (y_i, d_{-i})]$. Applying $RC1_i$ to (i) and (iii), and taking (iv) into account yields (v) $(z_i, a_{-i})\ \mathcal{R}\ (w_i, b_{-i})$. Since (i), (iii) and (v) match the premises of $DMM3_i$, we get one of the four possible conclusions. The first and the third one are in contradiction with (ii) and (iv). Due to $RC2_i$ and Lemma 22.2, we obtain that $(y_i, x_i) \succsim_i^* (w_i, z_i)$. From this and (ii) we deduce that the third conclusion is not true. The only remaining possibility is thus the fourth conclusion of $DMM3_i$, which establishes the second part of the lemma and implies the first part. □

Starting from Theorem 57 and considering a relation $\mathcal{R}$ that is the co-dual of a CDR, we obtain directly the following characterization of a CRB.

Theorem 67 (Characterization of CRB)
The relation $\mathcal{R}$ *on* X *is a* CRB *iff it satisfies* $RC1, RC2, MM2$ *and* $DMM3$*. These axioms are independent both in the class of complete and in the class of asymmetric relations.*

Proof

This result is a direct consequence of two facts:

- by definition, the co-dual of a CRB is a CDR and conversely,
- $\mathcal{R}$ satisfies $RC1$, $RC2$, $MM2$ and $DMM3$ iff its co-dual $\mathcal{R}^{cd}$ satisfies $RC1$, $RC2$, $MM1$ and $MM3$ (Lemmas 30, 42 and 64).

Examples showing the independence of the axioms are obtained by taking the co-dual of those used in the proof of Theorem 57 to show the independence of the axioms characterizing a CDR. □

Corollary 68

The relation $\mathcal{R}$ on X is a CRB

1. *iff it satisfies $RC1, RC2, M2$ and $DM3$,*
2. *iff it satisfies $RC1, RC2, Maj2$ and $DMaj3$.*

These two families of axioms are independent both in the class of complete and in the class of asymmetric relations.

Proof

These characterizations, as well as the independence of the axioms, result from Theorem 28 and Corollary 58 respectively, by the same argument as we used to prove Theorem 67 starting from Theorem 57. □

4.8 CRB with attribute transitivity

The co-dual of a CDR-AT is a CRB with attribute transitivity (CRB-AT), i.e., a CRB which satisfies $AC1$, $AC2$ and $AC3$ since the first two axioms are cross co-dual and the latter is self co-dual (Lemma 49). In view of Proposition 19, and its proof, the co-dual of a CDR, $\mathcal{R}$, having a type I representation $\langle \trianglerighteq, S_i, V_i \rangle$, is a CRB having a type I representation which is $\langle \trianglerighteq^{cd}, S_i, V_i \rangle$, with the same relations S_i, V_i as for $\mathcal{R}$. If it happens that $\mathcal{R}$ is a CDR-AT, S_i, V_i form an homogeneous chain of semiorders as defined in Section 2. These properties are thus inherited by the co-dual of $\mathcal{R}$, which prompts the following definition of a CRB-AT.

Definition 69 (CRB with attribute transitivity)

A CRB with attribute transitivity (CRB-AT) is a CRB for which, for all $i \in N$:

- *S_i is a semiorder with asymmetric part P_i,*
- *V_i is the asymmetric part of a semiorder U_i with $U_i \supseteq S_i$ and, hence, $V_i \subseteq P_i$,*
- *(S_i, U_i) form an* homogeneous *chain of semiorders.*

We obtain a characterization of a CRB-AT from that of a CDR-AT, by co-duality arguments. This yields the following theorem, which is similar to Theorem 61

Theorem 70

1. *The relation $\mathcal{R}$ on X is a CRB-AT iff $\mathcal{R}$ satisfies $RC1$, $RC2$, $AC1$, $AC2$, $AC3$, $MM2$ and $DMM3$. These axioms are independent in the class of irreflexive relations.*

2. *If $\mathcal{R}$ is an asymmetric relation on X, it is a CDR-AT iff $\mathcal{R}$ satisfies $RC2$, $AC1$, $AC2$, $AC3$, $MM2$ and $DMM3$. These axioms are independent in the class of asymmetric relations.*

3. *If $\mathcal{R}$ is a complete relation on X, it is a CDR-AT iff $\mathcal{R}$ satisfies $RC1$, $RC2$, $AC1$, $AC2$, $AC3$, $MM2$ and $DMM3$. These axioms are independent in the class of complete relations.*

Proof

The proof of this theorem obtains from that of Theorem 61 by co-duality arguments. In particular, the co-dual of the examples used to prove the independence of the axioms in the three cases considered in Theorem 61 can be used here in the co-dual case. We emphasize that co-duality transforms complete relations into asymmetric ones and conversely. □

Remark 71

The independence result in the class of irreflexive relations (Part 1 of Theorem 70) is not semantically attractive in a preference modelling context, since strict preference relations are not only irreflexive but also asymmetric. The relevant result for strict preference relations is contained in Part 2 of Theorem 70. For non-strict preference relations, which are just supposed to be reflexive, Part 3 is the relevant result, since independence in the class of complete relations entails independence in the larger class of reflexive relations. •

Remark 72

Remark 62 can be transposed by co-duality to yield alternative characterizations of CRB-AT. In particular, for asymmetric CRB-AT, independent characterizations are obtained by substituting $MM2$ by $M2$ or by $Maj2$ and/or $DMM3$ by $DM3$ or by $DMaj3$ in the characterization of asymmetric CDR-AT given in the previous theorem. For complete CRB-AT, the following sets of axioms constitute independent characterizations:

- $RC1$, $RC2$, $AC1$, $AC2$, $AC3$, $Maj2$ and $DM3$,
- $RC1$, $RC2$, $AC1$, $AC2$, $AC3$, $M2$ and $DM3$. •

5 Conclusion

From the present research and a series of previous papers investigating outranking relations, we draw the following lessons.

1. It is possible to analyze concordance relations and concordance-discordance relations, both reflexive (such as Electre) and asymmetric (such as Tactic), in the framework and with the classical tools of conjoint measurement,
2. This research has illustrated the interest of an axiomatic analysis by showing
   (a) that new models (namely, concordance relations with bonus) can be defined and characterized just by using such a simple transformation as co-duality,
   (b) that new characterizations of known models can be obtained using such a transformation,
   (c) that axiomatic analysis allows to present a corpus of models (reflexive and asymmetric outranking relations) in a unified framework and to better understand their inter-relations,

Note also that the aim of characterizing methods is not just better understanding: the axioms used in the characterizations are testable in practice, they are expressed in a language, that of preferences, which allows them to be refuted experimentally.

Co-duality has played an important rôle in our analysis. While the co-dual of a concordance relation is a concordance relation, it is no longer the case as soon as vetoes come into play.

A noticeable product of our investigation using co-duality is the observation that the asymmetric part of a reflexive concordance-discordance relation is not a concordance-discordance relation since it involves both veto and bonus effects. Knowing the properties of such relations (i.e., the asymmetric part of a reflexive concordance discordance relation) is of importance since they are used in some multi-criteria sorting methods, namely the *optimistic* version of the Electre Tri method (Roy and Bouyssou, 1993, p.391). The pessimistic version of this method is well-understood (characterized in Słowiński et al. (2002), Bouyssou and Marchant (2007a,b)) and methods for learning its parameters on the basis of assignment examples were developed since 1998 (see e.g., Mousseau and Słowiński (1998), Mousseau et al. (2006), Leroy et al. (2011)). It is not the case with the optimistic version. No axiomatic characterization is known. A method for learning its parameters was recently proposed (Zheng et al., 2011). The recent interest for this method in applications (Metchebon Takougang et al., 2014) motivates further investigation. For lack of place, an axiomatic characterization of the asymmetric part of a concordance-discordance relation was not included in the present paper. We leave it for another publication.

In closing, two brief remarks on possible additional developments arising from the present analysis.

With the idea of *bonus*, some light was shed on what could be called an optimistic counterpart of the notion of veto. The notion of bonus could make sense in practical situations. Indeed, taking for granted that the usual outranking concept is relevant for modeling preferences in certain cases, the notion of bonus naturally comes into play in the asymmetric part of the traditional non-strict outranking relations as it became apparent in our analysis. Alternative outranking models could thus consider the possibility of bonuses instead of vetoes.

Another interesting issue is related to recent work by Bisdorff (2010, 2013). This author adopts a logicist and argumentative viewpoint in his interpretation of outranking. This is in line with the usual presentation of the outranking concept according to which alternative x outranks alternative y if there are enough reasons for asserting that x is at least as good as y while there is no reason that strongly opposes this assertion (Roy, 1991). Bisdorff starts with the same observation that we made in Remark 9: for preferences that are not complete relations, their co-dual is not their asymmetric part, hence the interpretation of the co-dual as the "better than" relation corresponding to the preference viewed as an "at least as good" relation is impaired. In order to restore this relationship viewed as essential in the framework of an argumentative interpretation of outranking relations, R. Bisdorff uses a bipolar representation of concordance and discordance relations (on a $[-1, 1]$ scale, with 0 playing the special rôle of coding contradictory information). He proposes an adapted definition of an outranking relation, which restores the identity of the co-dual and the asymmetric part of the relation.

The latter remarks show that new and interesting models of preference can be developed in the spirit of the classical outranking relations by combining ingredients such as concordance, vetoes and bonuses, in a way that preserves intuitively appealing properties. The usefulness of such models for representing actual preferences in practical applications has yet to be investigated.

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Appendices

A Propositions 73 and 75

Proposition 73
If $\mathcal{R}$ is a complete binary relation on X satisfying $RC2$, $AC1$, $AC2$, $AC3$, $MM1$ and $MM3$, then $\mathcal{R}$ satisfies $RC1$.

For proving this proposition, we need the following lemma.

Lemma 74
Let $\mathcal{R}$ be a binary relation on X satisfying $RC2_i$, $AC1_i$, $AC2_i$, $AC3_i$ and $Maj1_i$, on some attribute i. Consider four levels $x, y, z, w \in X_i$ such that the pairs (x, y) and (z, w) are not comparable w.r.t. the relation $\succsim_i^*$, which we denote by $(x, y) \bowtie (z, w))$. The relative positions of these pairs and the opposite pairs are as follows:

1. $[(y, x) \sim_i^* (w, z)] \succ_i^* [(x, y) \bowtie (z, w)]$,
2. *furthermore, one of the following configurations holds true:*
   (a) $[(y, z) \sim_i^* (y, x) \sim_i^* (w, z)] \succ_i^* [(x, y) \bowtie (z, w)] \succ_i^* (z, y)$
   (b) $[(w, x) \sim_i^* (y, x) \sim_i^* (w, z)] \succ_i^* [(x, y) \bowtie (z, w)] \succ_i^* (x, w)$.

   In the above, the notation $[(x, y) \bowtie (z, w)]$ means that the incomparable pairs $[(x, y)$ and (z, w) have the same relationships with the other pairs listed.

Proof (of Lemma 74)
1. Let $x, y, z, w \in X_i$ be such that the pairs (x, y) and (z, w) are incomparable w.r.t. relation $\succsim_i^*$, i.e., we have:

$$Not[(x, y) \succsim_i^* (z, w)] \text{ and } Not[(z, w) \succsim_i^* (x, y)]. \tag{30}$$

In view of Definition 21, this means that there are $a, b, c, d \in X_{-i}$ such that:

$$\begin{array}{ll} (x, c) \mathcal{R} (y, d), & Not[(z, c) \mathcal{R} (w, d)], \\ (z, a) \mathcal{R} (w, b), & Not[(x, a) \mathcal{R} (y, b)], \end{array} \tag{31}$$

in other words, $\mathcal{R}$ does not satisfy $RC1_i$.

Using $RC2_i$ and Lemma 22.2 imply that we have $(y, x) \succsim_i^* (w, z)$ and $(w, z) \succsim_i^* (y, x)$, yielding:

$$(y, x) \sim_i^* (w, z) \tag{32}$$

The same axiom and lemma entail that (x, y) and (y, x) are comparable w.r.t. $\succsim_i^*$, i.e., we must have $(x, y) \succsim_i^* (y, x)$ or $(y, x) \succsim_i^* (x, y)$. The former is incompatible with $Maj1_i$ as we shall see. Note that $\mathcal{R}$ satisfies $Maj1_i$ by Lemma 33.1.

2. We show that assuming $(x,y) \succsim_i^* (y,x)$ leads to a contradiction. From $(x,y) \succsim_i^* (y,x)$, we first derive the following consequences:

1. $(z,w) \succ_i^* (w,z)$. Assuming $Not[(z,w) \succsim_i^* (w,z)]$ implies, by Lemma 22.2, that $(w,z) \succsim_i^* (z,w)$. Hence we would have: $(x,y) \succsim_i^* (y,x) \sim_i^* (w,z) \succsim_i^* (z,w)$. Using the transitivity of $\succsim_i^*$ leads to $(x,y) \succsim_i^* (z,w)$, a contradiction. The same contradiction can be derived if we suppose $(z,w) \sim_i^* (w,z)$.

2. $(x,y) \succ_i^* (y,x)$. Else, from $(x,y) \sim_i^* (y,x)$ we would derive $(z,w) \succ_i^* (w,z) \sim_i^* (y,x) \sim_i^* (x,y)$, from which we deduce $(z,w) \succ_i^* (x,y)$, a contradiction.

3. $(x,y) \succ_i^* (w,z)$. Assuming $Not[(x,y) \succsim_i^* (w,z)]$ implies, by Lemma 22.2, that $(y,x) \succsim_i^* (z,w)$. Hence we would have: $(w,z) \sim_i^* (y,x) \succsim_i^* (z,w)$. Using the transitivity of $\succsim_i^*$ leads to $(w,z) \succsim_i^* (z,w)$, a contradiction. The same contradiction can be derived if we suppose $(x,y) \sim_i^* (w,z)$.

4. $(z,w) \succ_i^* (y,x)$ is established in a similar way as the previous item.

We thus have the following situation: (x,y) and (z,w) are incomparable differences w.r.t. $\succsim_i^*$, both are strictly preferred to (y,x) and (w,z), which are indifferent pairs.

We now use $AC1_i$, $AC2_i$ and $AC3_i$. The main consequence of these axioms is that the relations $\succsim_i^+, \succsim_i^-$ and $\succsim_i^\pm$ are complete (Lemma 46). Moreover, we have, for all $s,t,u,v \in X_i$:

$$s \succsim_i^+ t \Rightarrow (s,u) \succsim_i^* (t,u) \tag{33}$$

$$s \succsim_i^- t \Rightarrow (v,t) \succsim_i^* (v,s) \tag{34}$$

(direct consequence of $AC1_i, AC2_i$ and the definitions of $\succsim_i^+, \succsim_i^-$ and $\succsim_i^*$).

Consider the pairs (x,y) and (z,w). We claim that there are $u,v \in X_i$ such that $(u,v) \succ_i^* (x,y)$ and $(u,v) \succ_i^* (z,w)$. Furthermore, (u,v) is either (x,w) or (z,y). Observe first that we cannot have:

1. $x \succsim_i^\pm z$ and $w \succsim_i^\pm y$. Else, using (33) and (34), we would have $(x,y) \succsim_i^* (z,y) \succsim_i^* (z,w)$, a contradiction with the fact that (x,y) and (z,w) are incomparable,

2. $z \succsim_i^\pm x$ and $y \succsim_i^\pm w$. Else, using (33) and (34), we would have $(z,w) \succsim_i^* (x,w) \succsim_i^* (x,y)$, a contradiction with the fact that (x,y) and (z,w) are incomparable.

Since $\succsim_i^\pm$ is complete, we thus have either $[x \succsim_i^\pm z$ and $y \succsim_i^\pm w]$ or $[z \succsim_i^\pm x$ and $w \succsim_i^\pm y]$. Consider the former case. Using (33) and (34) yields $(x,w) \succsim_i^* (z,w)$ and $(x,w) \succsim_i^* (x,y)$. We can have neither $(x,w) \sim_i^* (z,w)$ nor $(x,w) \sim_i^* (x,y)$, because this would imply that (x, y) and (z, w) are comparable. Our claim is thus proved with $(u, v) = (x, w)$. If the situation was such that $z \succsim_i^{\pm} x$ and $w \succsim_i^{\pm} y$, then we would have that $(z, y) \succsim_i^* (z, w)$ and $(z, y) \succsim_i^* (x, y)$. The rôle of (u, v) would be played by (z, y). Our claim is proved.

Assume first that $(u, v) = (x, w)$. From $(x, c) \mathcal{R} (y, d)$ in (31) and $(x, w) \succsim_i^* (x, y)$, we derive $(x, c) \mathcal{R} (w, d)$. Similarly, $(z, a) \mathcal{R} (w, b)$ and $(x, w) \succsim_i^* (z, w)$ entail $(x, a) \mathcal{R} (w, b)$. This allows to derive a contradiction with $Maj1_i$. Indeed, we have $(x, c) \mathcal{R} (w, d)$, $(x, a) \mathcal{R} (w, b)$ and $(z, a) \mathcal{R} (w, b)$. Using $Maj1_i$ yields either $(w, a) \mathcal{R} (z, b)$ or $(z, c) \mathcal{R} (w, d)$. None of this conclusions holds true. The latter is false by hypothesis (see (31)) and the former cannot be true since $(x, y) \succ_i^* (w, z)$ and $Not[(x, a) \mathcal{R} (y, b)]$. The case in which we assume $(u, v) = (z, y)$ yields a similar contradiction. As a conclusion, we have established that $(y, x) \succ_i^* (x, y)$.

3. We draw the consequences of the fact that $(y, x) \succ_i^* (x, y)$, by adapting the ideas that we used in Part 2 of the present proof, under the opposite hypothesis.

The fact that $(y, x) \succsim_i^* (x, y)$ entails the following:

1. $(w, z) \succ_i^* (z, w)$. Assuming $Not[(w, z) \succsim_i^* (z, w)]$ implies, by Lemma 22.2, that $(z, w) \succsim_i^* (w, z)$. Hence we would have: $(z, w) \succsim_i^* (w, z) \sim_i^* (y, x) \succsim_i^* (x, y)$. Using the transitivity of $\succsim_i^*$ leads to $(z, w) \succsim_i^* (x, y)$, a contradiction. The same contradiction arises if we suppose $(z, w) \sim_i^* (w, z)$.

2. $(y, x) \succ_i^* (x, y)$. Else, from $(y, x) \sim_i^* (x, y)$ we would derive $(x, y) \sim_i^* (y, x) \sim_i^* (w, z) \succ_i^* (z, w)$, from which we deduce $(x, y) \succ_i^* (z, w)$, a contradiction.

3. $(y, x) \succ_i^* (z, w)$. Assuming $Not[(y, x) \succsim_i^* (z, w)]$ implies, by Lemma 22.2, that $(x, y) \succsim_i^* (w, z)$. Hence we would have: $(x, y) \succsim_i^* (w, z) \sim_i^* (y, x)$. Using the transitivity of $\succsim_i^*$ leads to $(x, y) \succsim_i^* (y, x)$, a contradiction. The same contradiction can be derived if we suppose $(y, x) \sim_i^* (z, w)$.

4. $(w, z) \succ_i^* (x, y)$ is established in a similar way as the previous item.

We thus have the following situation: (y, x) and (w, z) are incomparable differences w.r.t. $\succsim_i^*$. Both are strictly preferred to (x, y) and (z, w), which are indifferent pairs.

Using $AC1_i$, $AC2_i$ and $AC3_i$, we derive exactly the same consequences as in Part 2, i.e., we have either $[x \succsim_i^{\pm} z$ and $y \succsim_i^{\pm} w]$ or $[z \succsim_i^{\pm} x$ and $w \succsim_i^{\pm} y]$.

If $[x \succsim_i^{\pm} z$ and $y \succsim_i^{\pm} w]$, we conclude that $(x, y) \succsim_i^* (z, y)$ and $(z, w) \succsim_i^* (z, y)$. We can have neither $(x, y) \sim_i^* (z, y)$ nor $(z, w) \sim_i^* (z, y)$, because this would imply that (x, y) and (z, w) are comparable. Since we have $Not[(z, y) \succsim_i^* (x, y)]$, we deduce that $(y, z) \succsim_i^* (y, x)$, using Lemma 22.2. Having $(y, z) \succ_i^* (y, x)$ is impossible since this would contradict $Maj1_i$. Indeed, assume that there are $e, f \in$

X_{-i} such that $(y,e)\ \mathcal{R}\ (z,f)$ and $Not[(y,e)\ \mathcal{R}\ (x,f)]$. Since $(y,x) \succ_i^* (z,w)$ and $(z,a)\ \mathcal{R}\ (w,b)$, we get $(y,a)\ \mathcal{R}\ (x,b)$. From $(y,z) \succsim_i^* (y,x)$ and $(y,a)\ \mathcal{R}\ (x,b)$, we derive $(y,a)\ \mathcal{R}\ (z,b)$. By (31), we also have $Not[(x,a)\ \mathcal{R}\ (y,b)]$. The following configuration is not compatible with $Maj1_i$: $(y,a)\ \mathcal{R}\ (x,b)$, $(y,a)\ \mathcal{R}\ (z,b)$, $(y,e)\ \mathcal{R}\ (z,f)$, $Not[(x,a)\ \mathcal{R}\ (y,b)]$, $Not[(y,e)\ \mathcal{R}\ (x,f)]$. We have thus established that $(y,z) \sim_i^* (y,x)$. Starting from $Not[(z,y) \succsim_i^* (z,w)]$, one proves similarly that $(y,z) \sim_i^* (w,z)$ and we finally have that $(y,z) \sim_i^* (w,z) \sim_i^* (y,x)$.

In the case in which $[z \succsim_i^{\pm} x$ and $w \succsim_i^{\pm} y]$, one proves in an analogous way that $(x,y) \succ_i^* (x,w)$, $(z,w) \succ_i^* (x,w)$ and $(w,x) \sim_i^* (y,x) \sim_i^* (w,z)$.

This concludes the proof of Lemma 74. □

Proof (of Proposition 73)
Since $\mathcal{R}$ satisfies $MM1$ and $RC2$, it satisfies also $Maj1$ (Lemma 41.2). Let us assume that $\mathcal{R}$ does not verify $RC1_i$ on some attribute i. We shall derive a contradiction from this assumption. If $RC1_i$ is not verified by $\mathcal{R}$, there exist four levels $x,y,z,w \in X_i$ such that (x,y) and (z,w) are incomparable w.r.t. relation $\succsim_i^*$, or, in other words, there are $a,b,c,d \in X_{-i}$ such that:

$$\begin{array}{ll} (x,c)\ \mathcal{R}\ (y,d) & Not[(z,c)\ \mathcal{R}\ (w,d)] \\ (z,a)\ \mathcal{R}\ (w,b) & Not[(x,a)\ \mathcal{R}\ (y,b)] \end{array} \tag{35}$$

Hence $\mathcal{R}$ is in the conditions of application of Lemma 74. We shall assume that the configuration described in conclusion 2.(a) of the lemma holds true, i.e., we have:

$$[(y,z) \sim_i^* (y,x) \sim_i^* (w,z)] \succ_i^* [(x,y) \bowtie (z,w)] \succ_i^* (z,y). \tag{36}$$

Note that case 2.(b) can be dealt with similarly. We leave it to the reader.

Since $\mathcal{R}$ satisfies $MM3$ and $RC2$, it satisfies $M3$ (Lemma 55.1). In the configuration described by (36), $M3_i$ implies that the pair (z,y) is a veto. Indeed, assume that there are $e,f \in X_{-i}$ such that $(z,e)\ \mathcal{R}\ (y,f)$. We have $(x,y) \succ_i^* (z,y)$, which means there are $g,h \in X_{-i}$ such that $(x,g)\ \mathcal{R}\ (y,h)$ and $Not[(z,g)\ \mathcal{R}\ (y,h)]$. It holds true that $(y,a)\ \mathcal{R}\ (x,b)$ since, by (35), $(z,a)\ \mathcal{R}\ (w,b)$ and, by (36), $(y,x) \succ_i^* (z,w)$. Finally, we have $Not[(x,a)\ \mathcal{R}\ (y,b)]$ by (35) and $Not[(z,a)\ \mathcal{R}\ (y,b)]$ since $(x,y) \succ_i^* (z,y)$. Gathering the relevant preferences, i.e., $(y,a)\ \mathcal{R}\ (x,b)$, $(x,g)\ \mathcal{R}\ (y,h)$, $(z,e)\ \mathcal{R}\ (y,f)$, $Not[(x,a)\ \mathcal{R}\ (y,b)]$, $Not[(z,a)\ \mathcal{R}\ (y,b)]$ and $Not[(z,g)\ \mathcal{R}\ (y,h)]$, yields a contradiction with $M3_i$. We thus have shown that for all $e,f \in X_{-i}$, we have

$$Not[(z,e)\ \mathcal{R}\ (y,f)] \tag{37}$$

The fact that $\mathcal{R}$ is complete enters into play in the following way. Since $\mathcal{R}$ is complete, (37) entails that for all $e,f \in X_{-i}$, we have $(y,e)\ \mathcal{R}\ (z,f)$. Since (36) tells us that $(y,z) \sim_i^* (w,z) \sim_i^* (y,x)$, we also have, for all $e,f \in X_{-i}$,

$(w,e)\ \mathcal{R}\ (z,f)$ and $(y,e)\ \mathcal{R}\ (x,f)$. In other words, $(y,z),(w,z)$ and (y,x) are bonuses as defined in Section 3.6.

The relation $\mathcal{R}$ induces not only a relation $\succsim_i^*$ comparing pairs of levels on X_i, but also a similar relation $\succsim_{-i}^*$ on the pairs of elements of X_{-i}. For $e,f,g,h \in X_{-i}$, we have $(e,f) \succsim_{-i}^* (g,h)$ iff, for all $u,v \in X_i$, $[(u,g)\ \mathcal{R}\ (v,h)] \Rightarrow [(u,e)\ \mathcal{R}\ (v,f)]$. The assumption (35) also means that the pairs $(a,b),(c,d) \in X_{-i} \times X_{-i}$ are not comparable w.r.t. $\succsim_{-i}^*$. This relation is transitive by definition and complete iff $RC1_i$ holds.

If $AC1, AC2$ and $AC3$ hold, we claim that there are $g,h \in X_{-i}$ with $(a,b) \succsim_{-i}^* (g,h)$ and $(c,d) \succsim_{-i}^* (g,h)$. $AC1, AC2$ and $AC3$ imply that $\succsim_j^{\pm}$ is a complete weak order for all $j \in N$. We define g (resp.h) by specifying its level g_j (resp. h_j) for each $j \neq i$ as follows: for all $j \neq i$,

$$g_j = \min\{a_j, c_j\} = \begin{cases} a_j & \text{if } c_j \succsim_j^{\pm} a_j \\ c_j & \text{if } a_j \succsim_j^{\pm} c_j \end{cases} \tag{38}$$

$$h_j = \max\{b_j, d_j\} = \begin{cases} b_j & \text{if } b_j \succsim_j^{\pm} d_j \\ d_j & \text{if } d_j \succsim_j^{\pm} b_j \end{cases} \tag{39}$$

Starting from the trivial $(a,b) \succsim_{-i}^* (a,b)$ and applying repeatedly (33) and (34), using g and h, we obtain $(a,b) \succsim_{-i}^* (g,h)$. One proves similarly that $(c,d) \succsim_{-i}^* (g,h)$.

We finish the proof by showing that the above induces a contradiction with $M3_i$. We have that $Not[(x,a)\ \mathcal{R}\ (y,b)]$ entails $Not[(x,g)\ \mathcal{R}\ (y,h)]$ and $Not[(z,c)\ \mathcal{R}\ (w,d)]$ entails $Not[(z,g)\ \mathcal{R}\ (w,h)]$ (since a difference on X_{-i} is substituted by a smaller one w.r.t. $\succsim_{-i}^*$). Since (y,x) is a bonus, we have in particular $(y,g)\ \mathcal{R}\ (x,h)$. By (35), we have $(x,c)\ \mathcal{R}\ (y,d)$ and $(z,a)\ \mathcal{R}\ (w,b)$. Gathering the relevant preferences, i.e., $(y,g)\ \mathcal{R}\ (x,h)$, $(x,c)\ \mathcal{R}\ (y,d)$, $(z,a)\ \mathcal{R}\ (w,b)$, $Not[(x,g)\ \mathcal{R}\ (y,h)]$, $Not[(z,g)\ \mathcal{R}\ (w,h)]$ and $Not[(z,c)\ \mathcal{R}\ (w,d)]$, yields a contradiction with $M3_i$. □

The proposition below is another result, besides Proposition 73, showing that $RC1$ has relationships with the other axioms even though the considered relations here are neither complete nor asymmetric.

Proposition 75
If $\mathcal{R}$ is a relation on X satisfying $RC2_i$, $AC1_i$, $AC2_i$, $AC3_i$, $Maj1_i$ and $Maj3_i$, for some $i \in N$, then $\mathcal{R}$ satisfies $RC1_i$.

Proof
Let us assume that $\mathcal{R}$ does not verify $RC1_i$ on some attribute i, i.e., there exist $x,y,z,w \in X_i$ and $a,b,c,d \in X_{-i}$ such that:

$$\begin{array}{ll} (x,c)\ \mathcal{R}\ (y,d) & Not[(z,c)\ \mathcal{R}\ (w,d)] \\ (z,a)\ \mathcal{R}\ (w,b) & Not[(x,a)\ \mathcal{R}\ (y,b)]. \end{array} \tag{40}$$

In other words, the pairs (x, y) and (z, w) are incomparable w.r.t. relation $\succsim_i^*$. Therefore, $\mathcal{R}$ is in the conditions of application of Lemma 74 and we have $[(y, x) \sim_i^* (w, z)] \succ_i^* [(x, y), (z, w)]$.

The latter is not compatible with $Maj3_i$ as we shall see. Since $(y, x) \succ_i^* (z, w)$ and using $(z, a)\ \mathcal{R}\ (w, b)$ in (40), we obtain that $(y, a)\ \mathcal{R}\ (x, b)$. From $(w, z) \sim_i^* (y, x)$ and $(y, a)\ \mathcal{R}\ (x, b)$, we derive $(w, a)\ \mathcal{R}\ (z, b)$. We also directly use the four clauses in (40). The following facts contradict $Maj3_i$: $(y, a)\ \mathcal{R}\ (x, b)$, $(w, a)\ \mathcal{R}\ (z, b)$, $(x, c)\ \mathcal{R}\ (y, d)$, $(z, a)\ \mathcal{R}\ (w, b)$, $Not[(x, a)\ \mathcal{R}\ (y, b)]$ and $Not[(z, c)\ \mathcal{R}\ (w, d)]$. □

B Examples

The examples below have been checked in order to determine whether they satisfy the following axioms:

$$RC1,\ RC2,\ AC1,\ AC2,\ AC3,\ UC,\ LC,\ M1,\ M2,\ Maj1,\ Maj2,$$
$$MM1,\ MM2,\ M3,\ Maj3,\ MM3,\ DMM3.$$

Those among these axioms that are not satisfied are mentioned below next to the example label. All axioms from the previous list that are not explicitly mentioned are proved to be satisfied. By default, the examples are complete relations. Relations that are asymmetric are explicitly labeled as such, as well as relations that are neither complete nor asymmetric.

Example 76 (*Not*[$RC2_i$])
This is example 25 in BP07. Let $N = \{1, 2\}$ and $X = \{x, y\} \times \{a, b\}$. Let $\mathcal{R}$ on X be identical to X^2 except that, $Not[(y, a)\ \mathcal{R}\ (x, a)]$ and $Not[(y, b)\ \mathcal{R}\ (x, a)]$. This relation is complete.

It is easy to check that we have:

- $(x, y), (x, x), (y, y) \succ_1^* (y, x)$ and
- $[(a, b), (b, b)] \succ_2^* [(a, a), (b, a)]$.

Using Lemma 22, it is easy to see that $RC1$ and $RC2_1$ hold but that $RC2_2$ is violated. Using Lemma 8.1 and 8.2 in BP07 it is clear that UC and LC hold so that the same is true for $M1$ and $M2$. As a consequence of Remark 32, we have that $\mathcal{R}$ satisfies $Maj1$ and $Maj2$. Since $M3$ (resp. $Maj3$) is entailed by $M2$ (resp. $Maj2$), $\mathcal{R}$ also satisfies $M3$ (resp. $Maj3$). Since $\mathcal{R}$ satisfies $M1$ (resp. $M2$) it satisfies its relaxed version $MM1$ (resp. $MM2$). As $\mathcal{R}$ satisfies $M2$ it fulfills $M3$ and $MM3$. As $\mathcal{R}$ satisfies $M1$ it fulfills $DMM3$.

Finally, using Lemma 15 in BP07, it is routine to check that we have :

- $x \succ_1^{\pm} y$,

- $a \succ_2^{\pm} b$.

Hence $AC1$, $AC2$ and $AC3$ hold. ◇

Example 77 ($\boldsymbol{Not[UC_i, M1_i, Maj1_i, MM1_i]}$)
This is Example 33 in BP05a. Also used in the proof of Part 5 of BP07, Lemma 11 and as Example 23 in BP07.

Let $X = \{a, b\} \times \{x, y, z\}$ and $\mathcal{R}$ on X be identical to the linear order:

$$(a, x)\ \mathcal{R}\ (a, y)\ \mathcal{R}\ (a, z)\ \mathcal{R}\ (b, x)\ \mathcal{R}\ (b, y)\ \mathcal{R}\ (b, z),$$

except that (a, z) and (b, x) are indifferent: $(a, z)\ \mathcal{R}\ (b, x)$ and $(b, x)\ \mathcal{R}\ (a, z)$ both hold true.

This is a complete relation.

We have, abusing notation,

- $(a, b) \succ_1^* [(a, a), (b, b)] \succ_1^* (b, a)$ and
- $(x, z) \succ_2^* [(x, x), (y, y), (z, z), (x, y), (y, z)] \succ_2^* [(y, x), (z, x), (z, y)]$,
- $a \succ_1^{\pm} b$ and $x \succ_2^{\pm} y \succ_2^{\pm} z$.

Using Lemma 22, it is easy to check that $\mathcal{R}$ satisfies $RC1$, $RC2$, $AC1$, $AC2$, $AC3$.

It is clear that UC_1, LC_1 and LC_2 hold. UC_2 is violated since we have $(x, y) \succ_2^* (y, x)$ and $Not[(x, y) \succsim_2^* (x, z)]$.

Parts 1 and 2 of Lemma 11 in BP07, show that conditions $M1_1$ and $M2$ hold. By Part 3 of Lemma 11 in BP07, $M1_2$ cannot hold. Using Lemma 33 shows that $Maj1_1$ and $Maj2$ hold while $Maj1_2$ does not. Using Lemma 41 shows that $MM1_1$ and $MM2$ hold while $MM1_2$ does not. Since $\mathcal{R}$ satisfies $M2$ (resp. $Maj2$, $MM2$), this implies that $M3$ (resp. $Maj3$, $MM3$) also holds. Since $M1_1$ holds, $DMM3_1$ holds too. We show that $DMM3_2$ also holds. Assume the contrary. Taking $RC1_2$ into account, this implies that there are $x_2, y_2, z_2, w_2 \in X_1$ such that $(z_2, w_2) \succ_2^* (x_2, y_2) \succ_2^* (y_2, x_2)$. Hence (z_2, w_2) can only be (x, z). The fourth conclusion of $DMM3_2$ is always true since $(u, x)\ \mathcal{R}\ (v, z)$ for all $u, v \in X_1 = \{a, b\}$. ◇

Example 78 ($\boldsymbol{Not[RC1_i, AC2_i, LC_i, Maj2_i, Maj3_1]}$)
This is Example 12 in BP07. Also used in Example 24 in the same paper.

Let $N = \{1, 2, 3\}$ and $X = \{x, y, z, w\} \times \{a, b\} \times \{p, q\}$. Let $\mathcal{R}$ on X be identical to X^2 except that, for all $\alpha_1, \beta_1 \in X_1$, all $\alpha_2, \beta_2 \in X_2$ and all $\alpha_3, \beta_3 \in X_3$ the following pairs are *missing*:

$$Not[(x, a, \alpha_3)\ \mathcal{R}\ (y, b, \beta_3)], \quad Not[(z, \alpha_2, p)\ \mathcal{R}\ (w, \beta_2, q)],$$
$$Not[(x, \alpha_2, p)\ \mathcal{R}\ (w, \beta_2, q)], \quad Not[(\alpha_1, a, p)\ \mathcal{R}\ (\beta_1, b, q)],$$

	xap	xaq	xbp	xbq	yap	yaq	ybp	ybq	zap	zaq	zbp	zbq	wap	waq	wbp	wbq
xap	–	–	–	×	–	–	×	×	–	–	–	×	–	×	–	×
xaq	–	–	–	–	–	–	×	×	–	–	–	–	–	–	–	–
xbp	–	–	–	–	–	–	–	–	–	–	–	–	–	×	–	×
xbq	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
yap	–	–	–	×	–	–	–	×	–	–	–	×	–	–	–	×
yaq	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
ybp	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
ybq	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
zap	–	–	–	×	–	–	–	×	–	–	–	×	–	×	–	×
zaq	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
zbp	–	–	–	–	–	–	–	–	–	–	–	–	–	×	–	×
zbq	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
wap	–	–	–	×	–	–	–	×	–	–	–	×	–	–	–	×
waq	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
wbp	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
wbq	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–

Table 1: Relation $\mathcal{R}$ in Example 78: the missing pairs are marked by a cross.

There is a total of 25 such pairs that are marked by a cross in Table 1.

It is not difficult to check that $\mathcal{R}$ is complete.

For $i \in \{2, 3\}$, it is easy to check that we have:

$$[(b,a),(a,a),(b,b)] \succ_2^* (a,b),$$
$$[(q,p),(p,p),(q,q)] \succ_3^* (p,q),$$
$$b \succ_2^{\pm} a, q \succ_3^{\pm} p,$$

which shows,

- using Parts 1 and 2 of Lemma 22, that $RC1_2$, $RC1_3$, $RC2_2$ and $RC2_3$ hold,
- using Lemma 46, that $AC1_2$, $AC1_3$, $AC2_2$, $AC2_3$, $AC3_2$ and $AC3_3$ hold.

Using Parts 1 and 2 of Lemma 8 in BP07, it is easy to check that LC_2, LC_3, UC_2 and UC_3 hold. Hence, using Parts 3 and 4 of Lemma 11 in BP07, we know that $M1_2$, $M1_3$, $M2_2$ and $M2_3$ hold. Using Lemma 33, we have also $Maj1_2$, $Maj1_3$, $Maj2_2$ and $Maj2_3$.

On attribute 1, it is easy to check that we have:

$$(c_1, d_1) \succ_1^* (x, y) \text{ and}$$
$$(c_1, d_1) \succ_1^* [(x, w), (z, w)],$$

for all $(c_1, d_1) \in \Gamma = \{(x,x),(x,z), (y,x),(y,y),(y,z),(y,w), (z,x), (z,y), (z,z), (w,x), (w,y), (w,z), (w,w)\}$. The pairs (x,w) and (z,w) are linked by $\sim_1^*$. The pairs (x,y) and (x,w) are not comparable in terms of $\succsim_1^*$ since $(x,a,p)\ \mathcal{R}\ (y,a,q)$ and $Not[(x,a,p)\ \mathcal{R}\ (w,a,q)]$, while $(x,a,p)\ \mathcal{R}\ (w,b,p)$ and $Not[(x,a,p)\ \mathcal{R}\ (y,b,p)]$. Similarly, the pairs (x,y) and (z,w) are not comparable in terms of $\succsim_1^*$. This shows, using Part 1 of Lemma 22, that $RC1_1$ is violated.

Using Part 2 of Lemma 22, it is easy to see that $RC2_1$ holds. Using Part 1 of Lemma 8 in BP07, shows that UC_1 holds. Hence, using Part 3 of Lemma 11 in BP07, we know that $M1_1$ holds.

In view of Part 6 of Lemma 16 in BP05a, LC_1 does not hold (since this lemma tells us that $RC2_1, UC_1$ and LC_1 entail $RC1_1$. We now check that $M2_1$ holds. The two premises of $M2_1$ are that $(a_1, a_{-1})\ \mathcal{R}\ (b_1, b_{-1})$ and $(b_1, c_{-1})\ \mathcal{R}\ (a_1, d_{-1})$. The three possible conclusions of $M2_1$ are that $(b_1, a_{-1})\ \mathcal{R}\ (a_1, b_{-1})$ or $(c_1, a_{-1})\ \mathcal{R}\ (d_1, b_{-1})$ or $(c_1, c_{-1})\ \mathcal{R}\ (d_1, d_{-1})$.

Suppose first that $(b_1, a_1) \in \Gamma$. In this case, we have $(b_1, a_1) \succsim_1^* (a_1, b_1)$, so that $(a_1, a_{-1})\ \mathcal{R}\ (b_1, b_{-1})$ implies $(b_1, a_{-1})\ \mathcal{R}\ (a_1, b_{-1})$. Hence, the first conclusion of $M2_1$ holds.

Suppose now that $(b_1, a_1) = (x, y)$.

If (c_1, d_1) is distinct from (x, w) and (z, w), we have $(c_1, d_1) \succsim_1^* (x, y)$, so that $(b_1, c_{-1})\ \mathcal{R}\ (a_1, d_{-1})$ implies $(c_1, c_{-1})\ \mathcal{R}\ (d_1, d_{-1})$ and the third conclusion of $M2_1$ holds.

If $(c_1, d_1) = (x, w)$, it is easy to check that there are no $a_{-1}, b_{-1} \in X_{-i}$ such that $(y, a_{-1})\ \mathcal{R}\ (x, b_{-1})$, $Not[(x, a_{-1})\ \mathcal{R}\ (y, b_{-1})]$ and $Not[(x, a_{-1})\ \mathcal{R}\ (w, b_{-1})]$, so that no violation of $M2_1$ is possible in this case. Since $(x, w) \sim_1^* (z, w)$, the same is true if $(c_1, d_1) = (z, w)$.

This shows that $M2_1$ cannot be violated if $(b_1, a_1) = (x, y)$. A similar reasoning shows that $M2_1$ cannot be violated if $(b_1, a_1) = (x, w)$ or if $(b_1, a_1) = (z, w)$. Hence, $M2_1$ holds and so does $M3_1$.

Using Remark 32, we know that $\mathcal{R}$ satisfies $Maj1_1$ since UC_i entails $Maj1_i$.

Since $\mathcal{R}$ satisfies $RC2_1$, $M1_1$ but not $RC1_1$, it cannot satisfy $Maj2_1$, as a consequence of Lemma 34. Since $\mathcal{R}$ satisfies $M1$ and $M2$, it also satisfies $MM1$, $DMM3$, $MM2$, $M3$ and $MM3$.

$Maj3_2$ (resp. $Maj3_3$) holds because $Maj2_2$ (resp. $Maj2_3$) holds but $Maj3_1$ is violated as shown by the following configuration (which also confirms that $Maj2_1$ is violated): $(y, a, p)\ \mathcal{R}\ (x, b, p)$, $(w, a, p)\ \mathcal{R}\ (z, a, p)$, $(w, a, p)\ \mathcal{R}\ (z, a, q)$, $(z, a, p)\ \mathcal{R}\ (w, a, p)$, $Not[(x, a, p)\ \mathcal{R}\ (y, b, p)]$, $Not[(z, a, p)\ \mathcal{R}\ (w, a, q)]$.

On attribute 1, it is easy to check that we have:

$$\{y, w\} \succ_1^+ z \succ_1^+ x.$$

Hence $AC1_1$ holds. Since (x, w) and (x, y) are not comparable w.r.t. $\succsim_1^*$, y and w are not comparable w.r.t. $\succ_1^-$, hence $AC2_1$ is violated. It is easy to check, using Lemma 15 in BP07, that $AC3_1$ is satisfied. $\diamond$

Remark The co-dual of Example 78 is an asymmetric relation that satisfies all axioms but $RC1, AC1, UC$ and $M1$.

Example 79 ($\boldsymbol{Not[RC1_i, AC1_i, UC_i, M1_i, DMaj3_i]}$)
The co-dual of the relation $\mathcal{R}$ described in Table 2 is a complete relation satisfying all properties except for $RC1$, $AC2$, $M1$ and $DMaj3$. Note that this relation verifies $DM3$ and $DMM3$.

Let $X = \{x, y, z\} \times \{a, b\} \times \{p, q\}$ and $\mathcal{R}$ consist of the set of pairs listed in Table 2. We have to show that $\mathcal{R}$ satisfies all properties but $RC1$, $AC1$,

	xap	xaq	xbp	xbq	yap	yaq	ybp	ybq	zap	zaq	zbp	zbq
xap	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$
xaq	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$
xbp	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$
xbq	–	–	–	–	–	–	–	–	–	–	–	
yap	–	–	$\mathcal{R}$	$\mathcal{R}$	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$
yaq	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$
ybp	–	–	–	–	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$
ybq	–	–	–	–	–	–	–	–	–	–	–	
zap	–	$\mathcal{R}$	–	$\mathcal{R}$	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$
zaq	–	–	–	–	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$
zbp	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$
zbq	–	–	–	–	–	–	–	–	–	–	–	

Table 2: Relation $\mathcal{R}$ in Example 79.

LC, $Maj2$, $Maj3$. It is easy to check that $\mathcal{R}$ is asymmetric. As for the comparison of preference differences on each attribute, we have for all $(\alpha, \beta) \in \Gamma = \{(x, x), (y, y), (z, z), (x, y), (x, z), (y, z), (z, y)\}$,

- $[(\alpha, \beta)] \succ_1^* (y, x)$ and $[(\alpha, \beta)] \succ_1^* (z, x)$, while (y, x) and (z, x) are incomparable in terms of $\succsim_1^*$,
- $(a, b) \succ_2^* [(a, a), (b, b)] \succ_2^* (b, a)$,
- $(p, q) \succ_3^* [(p, p), (q, q)] \succ_3^* (q, p)$.

The upward and downward dominance relations on attributes 2 and 3 are as follows:

- $a \succ_2^{\pm} b$,
- $p \succ_3^{\pm} q$.

On attribute 1, we have:

- $x \succ_1^+ y$, $x \succ_i^+ z$,
- y and z are not comparable w.r.t. $\succsim_i^+$ since, on the one hand, $zap \; \mathcal{R} \; xaq$ and $Not[yap \; \mathcal{R} \; xaq]$, and on the other hand, $yap \; \mathcal{R} \; xbp$ and $Not[zap \; \mathcal{R} \; xbp]$,
- $x \succ_1^- y \succ_i^- z$.

For $j \in \{2,3\}$, $RC1_j$, $RC2_j$, $AC1_j$, $AC2_j$, $AC3_j$, UC_j and LC_j are clearly satisfied, implying $M1_j$ and $M2_j$ (see Remark 25) as well as $Maj1_j$ and $Maj2_j$ (see Remark 32), $MM1_j$ and $MM2_j$. On attribute 1 it is easy to check that $RC2_1$, $AC2_1$ and $AC3_1$ are verified while $RC1_1$ and $AC1_1$ are violated. Using Lemma 8(1) in BP07, we observe that UC_1 is satisfied, implying $M1_1$, $Maj1_1$ and $MM1_1$. LC_1 does not hold but $M2_1$ does as we shall see.

Using the notation in condition (20), for establishing $M2_1$ we only have to consider the cases in which $(y_1, x) = (y, x)$ or $(y_1, x) = (z, x)$ (otherwise $(y_1, x) \succsim_1^* (x, y_1)$ and consequently the first conclusion of (20) is satisfied). Assume that $(y_1, x) = (y, x)$ and that the first conclusion is not satisfied. This means that either $a_{-1} = ap$ and $b_{-1} = aq$ or $a_{-1} = bp$ and $b_{-1} = bq$. We now distinguish two cases regarding (z_i, w_i):

1. if $(z_i, w_i) \neq (z, x)$, the third conclusion is always satisfied because of the second premise and the fact that $(z_i, w_i) \succsim_1^* (y, x)$,

2. if $(z_i, w_i) = (z, x)$ and $a_{-1} = ap$ and $b_{-1} = aq$ or $a_{-1} = bp$ and $b_{-1} = bq$, the second conclusion is satisfied because we have $zap \mathrel{\mathcal{R}} xaq$ and $zbp \mathrel{\mathcal{R}} xbq$.

The case in which $(y_1, x) = (z, x)$ is dealt with similarly. Consequently, $\mathcal{R}$ satisfies $M2_1$, $MM2_1$, $M3_1$ and $MM3_1$.

For establishing that $Maj2_1$ does not hold, we consider the case in which $(y_1, x) = (y, x)$ and use the notation of (22) in Definition 29. In the previous analysis we only need to reconsider the case in which the second conclusion of $M2_i$ was used, i.e., when $(z_i, w_i) = (z, x)$ and $a_{-1} = ap$ and $b_{-1} = aq$ or $a_{-1} = bp$ and $b_{-1} = bq$. We have $xap \mathrel{\mathcal{R}} yaq$, $Not[yap \mathrel{\mathcal{R}} xaq]$, $xap \mathrel{\mathcal{R}} zaq$, $yap \mathrel{\mathcal{R}} xbp$ and $Not[zap \mathrel{\mathcal{R}} xbp]$, which means that $Maj2_1$ does not hold. Since (z, x) is no veto (as we have, e.g., $zap \mathrel{\mathcal{R}} xaq$), the latter also shows that $\mathcal{R}$ does not satisfy $Maj3_1$. $\diamond$

Example 80 ($Not[AC1_i, AC2_i]$)
This is Example 36 in BP05a. Also used in Example 21 in BP07.

Let $X = \{a, b, c, d\} \times \{x, y\}$. We build $\mathcal{R}$ as the CR in which:

- $a \; I_1 \; b$, $a \; P_1 \; c$, $a \; I_1 \; d$, $b \; I_1 \; c$, $b \; P_1 \; d$, $c \; I_1 \; d$,

- $x \; P_2 \; y$,

- $\{1,2\} \rhd \varnothing$, $\{1,2\} \triangleq \{2\}$, $\{1,2\} \triangleq \{1\}$, $\{2\} \triangleq \{1\}$.

Therefore, $\mathcal{R}$ links any two elements of X except that we have: $(a, x) \mathrel{\mathcal{R}} (c, y)$ but $Not[(c, y) \mathrel{\mathcal{R}} (a, x)]$ and $(b, x) \mathrel{\mathcal{R}} (d, y)$ but $Not[(d, y) \mathrel{\mathcal{R}} (b, x)]$. Hence $\mathcal{R}$ is a complete relation. Since it is a CR, it satisfies $RC1$, $RC2$, UC, LC, $M1$, $M2$, $Maj1$, $Maj2$, $MM1$, $MM2$, $M3$, $Maj3$, $MM3$ and $DMM3$.

It is easy to see that $AC3$ and $AC1_2$ as well as $AC2_2$ hold. $AC1_1$ is violated since $(d,y)\ \mathcal{R}\ (a,x)$ and $(c,y)\ \mathcal{R}\ (b,x)$ but neither $(c,y)\ \mathcal{R}\ (a,x)$ nor $(d,y)\ \mathcal{R}\ (b,x)$. $AC2_1$ is also violated (Part 1 of Lemma 27 in BP05a). ◇

Example 81 ($\boldsymbol{Not[AC3_i]}$)
This is Example 35 in BP05a. Also used as Example 20 in BP07.

Let $X = \{a,b,c,d\} \times \{x,y\}$. We build the CR in which:

- $a\ P_1\ b$, $a\ I_1\ c$, $a\ P_1\ d$, $b\ I_1\ c$, $b\ P_1\ d$, $c\ I_1\ d$,
- $x\ P_2\ y$,
- $\{1,2\} \rhd \varnothing$, $\{1,2\} \triangleq \{2\}$, $\{1,2\} \triangleq \{1\}$, $\{2\} \triangleq \{1\}$.

Therefore, $\mathcal{R}$ links any two elements of X except that we have: $(a,x)\ \mathcal{R}\ (b,y)$ but $Not[(b,y)\ \mathcal{R}\ (a,x)]$, $(b,x)\ \mathcal{R}\ (d,y)$ but $Not[(d,y)\ \mathcal{R}\ (b,x)]$ and $(a,x)\ \mathcal{R}\ (d,y)$ but $Not[(d,y)\ \mathcal{R}\ (a,x)]$. Hence $\mathcal{R}$ is a complete relation. Since it is a CR, it satisfies $RC1$, $RC2$, UC, LC, $M1$, $M2$, $Maj1$, $Maj2$, $MM1$, $MM2$, $M3$, $Maj3$, $MM3$ and $DMM3$.

It is easy to see that $AC1$ holds and, hence, $AC2$ (by Part 1 of Lemma 27 in BP05a). One verifies that $AC3_2$ holds. $AC3_1$ is violated since $(c,y)\ \mathcal{R}\ (a,x)$, $(d,y)\ \mathcal{R}\ (c,x)$ but neither $(b,y)\ \mathcal{R}\ (a,x)$ nor $(d,y)\ \mathcal{R}\ (b,x)$. ◇

Remark. The co-dual of this relation is an asymmetric relation that satisfies all axioms of a CR-AT except $AC3_1$. In particular, it satisfies $M3$ and $Maj3$ since it satisfies $M2$ and $Maj2$.

Example 82 ($\boldsymbol{Not[LC_i, M2_i, Maj2_i, MM2_i, M3_i, Maj3_i, MM3_i]}$)
This is Example 38 in BP09a. It is used in Remark 16 in BP09b (but erroneously referred to as Example 39 in BP09a).

Let $X = X_1 \times X_2 \times X_3$ with $X_1 = \{x,y,z\}$, $X_2 = \{a,b\}$ and $X_3 = \{p,q\}$. Let us consider the relation $\mathcal{R}$ such that:

$$x\ \mathcal{R}\ y \Leftrightarrow \sum_{i=1}^{3} p_i(x_i, y_i) \geq 0,$$

the functions p_i being such that:

$$p_1(x,y) = p_1(x,z) = p_1(y,z) = p_1(x,x) = p_1(y,y) = p_1(z,z) = 4,$$
$$p_1(y,x) = p_1(z,y) = -1, p_1(z,x) = -4,$$
$$p_2(a,b) = 2, p_2(a,a) = p_2(b,b) = 0, p_2(b,a) = -2,$$
$$p_3(a,b) = 2, p_3(p,p) = p_3(q,q) = 0, p_3(q,p) = -2.$$

This is a complete relation. Indeed if $\sum_{i=1}^{3} p_i(x_i, y_i) < 0$, then $p_1(x, y_1) < 4$. This implies that $p_1(y_1, x) = 4$, hence $\sum_{i=1}^{3} p_i(y_i, x_i) \geq 0$.

It is easily checked that we have (with (α, α) standing for (x, x), (y, y) and (z, z)):

$$[(x, y) \sim_1^* (x, z) \sim_1^* (y, z) \sim_1^* (\alpha, \alpha)] \succ_1^* [(y, x) \sim_1^* (z, y)] \succ_1^* (z, x),$$
$$x \succ_1^{\pm} y \succ_1^{\pm} z,$$
$$(a, b) \succ_2^* [(a, a) \sim_2^* (b, b)] \succ_2^* (b, a),$$
$$a \succ_2^{\pm} b,$$
$$(p, q) \succ_3^* [(p, p) \sim_2^* (q, q)] \succ_2^* (q, p),$$
$$p \succ_3^{\pm} q.$$

This shows that $RC1$, $RC2$, $AC1$, $AC2$ and $AC3$ are satisfied. Using Parts 1 of Lemma 8 and Lemma 11 in BP07

shows that UC and $M1$ hold. Similarly, using Parts 2 of Lemma 8 and Lemma 11 in BP07

shows that $\mathcal{R}$ satisfies UC_2, UC_3, $M2_2$ and $M2_3$, which implies that $M3_2$ and $M3_3$ hold. Condition $M3_1$ is violated since $(x, b, q) \mathcal{R} (y, a, p)$, $(y, a, q) \mathcal{R} (x, b, q)$ and $(z, a, p) \mathcal{R} (x, b, q)$

while $Not[(y, b, q) \mathcal{R} (x, a, p)]$, $Not[(z, b, q) \mathcal{R} (x, a, p)]$ and $Not[(z, a, q) \mathcal{R} (x, b, q)]$.

Hence $M2_1$ is violated too. Lemma 11, Part 2, in BP07 implies that LC_1 is also violated. Using Lemmas 33, 41 and 55, we obtain that $\mathcal{R}$ satisfies $Maj1$, $MM1$, $Maj2_2$, $Maj2_3$, $MM2_2$, $MM2_3$, $Maj3_2$, $Maj3_3$, $MM3_2$, $MM3_3$ but neither $Maj2_1$ nor $Maj3_1$. In view of Lemmas 55, $MM3_1$ is also violated as well as $MM2_1$. Since $\mathcal{R}$ satisfies $M1$, it fulfills $DMM3$. ◇

Example 83 (Asymmetric, $Not[LC_i, M2_i, M3_i, Maj2_i, Maj3_i, MM2_i, MM3_i]$) This is Example 5 in BP06.

Let $X = \{x, y, z\} \times \{a, b\} \times \{p, q\}$ and $\mathcal{R}$ on X be be as described in Table 3:

It is easy to check that $\mathcal{R}$ is asymmetric. It is not difficult to see that we have, abusing notation,

- $[(x, y), (x, z), (y, z)] \succ_1^* [(x, x), (y, y), (z, z), (y, x), (z, y)] \succ_1^* (z, x)$,
- $(a, b) \succ_2^* [(a, a), (b, b)] \succ_2^* (b, a)$, and
- $(p, q) \succ_3^* [(p, p), (q, q)] \succ_3^* (q, p)$.

This shows that $RC1$, $RC2$ and $Maj1$ hold. It is easy to see that $Maj2_2$ and $Maj2_3$ hold so that $Maj3_2$ and $Maj3_3$ are satisfied. Condition $Maj3_1$ is violated since

	xap	xaq	xbp	xbq	yap	yaq	ybp	ybq	zap	zaq	zbp	zbq
xap	–	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$
xaq	–	–	–	$\mathcal{R}$	–	$\mathcal{R}$	–	$\mathcal{R}$	–	$\mathcal{R}$	–	$\mathcal{R}$
xbp	–	–	–	–	–	–	$\mathcal{R}$	$\mathcal{R}$	–	–	$\mathcal{R}$	$\mathcal{R}$
xbq	–	–	–	–	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$
yap	–	–	$\mathcal{R}$	$\mathcal{R}$	–	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$
yaq	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$	–	$\mathcal{R}$	–	$\mathcal{R}$
ybp	–	–	–	–	–	–	–	–	–	–	$\mathcal{R}$	$\mathcal{R}$
ybq	–	–	–	–	–	–	–	–	–	–	–	$\mathcal{R}$
zap	–	–	–	$\mathcal{R}$	–	–	$\mathcal{R}$	$\mathcal{R}$	–	–	$\mathcal{R}$	$\mathcal{R}$
zaq	–	–	–	–	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$
zbp	–	–	–	–	–	–	–	–	–	–	–	–
zbq	–	–	–	–	–	–	–	–	–	–	–	–

Table 3: Relation $\mathcal{R}$ in Example 83.

$(x, a, p)\ \mathcal{R}\ (y, a, p)$, $(x, a, p)\ \mathcal{R}\ (z, a, p)$, $(y, a, p)\ \mathcal{R}\ (x, b, p)$ and $(z, a, p)\ \mathcal{R}\ (x, b, q)$ but neither $(y, a, p)\ \mathcal{R}\ (x, a, p)$ nor $(z, a, p)\ \mathcal{R}\ (x, b, p)$.

Since $RC1$ and $RC2$ hold, Lemma 33 implies that $\mathcal{R}$ satisfies $M1$, $M2_2$, $M2_3$ but not $M2_1$. $M3_2$ and $M3_3$ hold while $M3_1$ is violated (Lemma 55). By Lemma 11 in BP07, $\mathcal{R}$ satisfies UC, LC_2, LC_3 but not LC_1. By Lemmas 41 and 55, we know that $MM2_1$ and $MM3_1$ are violated. $MM2_j$ and $MM3_j$ are satisfied for $j = 2, 3$. $\mathcal{R}$ satisfies $MM1$ so that it also satisfies $DMM3$.

From relations $\succsim_i^*$ described above, we infer the following:

$$x \succ_1^{\pm} y \succ_1^{\pm} z,$$
$$a \succ_2^{\pm} b,$$
$$p \succ_3^{\pm} q,$$

which implies that $\mathcal{R}$ satisfies $AC1$, $AC2$ and $AC3$. ◇

Example 84 ($Not[AC1_i, M2_i, Maj2_i, MM2_i, LC_i]$)
This is Example 36 in BP09a.

Let $X = X_1 \times X_2$ with $X_1 = \{x, y, z, w\}$ and $X_2 = \{a, b\}$. We build a CDR on X with:

- $z\ P_1\ x$, $z\ P_1\ y$, $z\ P_1\ w$, $x\ P_1\ w$, $x\ I_1\ y$, $y\ I_1\ w$,
- the relation V_1 is empty except that $z\ V_1\ y$,
- $b\ P_2\ a$,
- the relation V_2 is empty,
- $\{1, 2\} \rhd \varnothing$, $\{1, 2\} \triangleq \{2\}$, $\{1, 2\} \triangleq \{1\}$ and $\{1\} \triangleq \{2\}$.

By construction, $\mathcal{R}$ is a CDR. Hence, it satisfies $RC1$, $RC2$, $M1$, $Maj1$, $MM1$, $M3$, $Maj3$ and $MM3$ (Theorem 57 and Lemmas 41 and 55). It satisfies $M2_2$,

$Maj2_2$, $MM2_2$, but not $M2_1$ (due to the veto on X_1), not $Maj2_1$ (by Lemma 33) and not $MM2_1$ (by Lemma 41). Since $\mathcal{R}$ satisfies $M1$ it also fulfills $DMM3$. Using Lemma 11 in BP07, we get that UC and LC_2 are satisfied but LC_1 is violated.

The relation $\mathcal{R}$ contains all pairs in $X \times X$ except the following ones:

- $Not[(x, b) \ \mathcal{R} \ (z, a)]$, $Not[(y, b) \ \mathcal{R} \ (z, a)]$, $Not[(w, b) \ \mathcal{R} \ (z, a)]$, $Not[(w, b) \ \mathcal{R} \ (x, a)]$, due to the fact that $Not[\varnothing \trianglerighteq \{1, 2\}]$, and
- $Not[(y, a) \ \mathcal{R} \ (z, a)]$, $Not[(y, b) \ \mathcal{R} \ (z, b)]$, $Not[(y, b) \ \mathcal{R} \ (z, a)]$, $Not[(y, a) \ \mathcal{R} \ (z, b)]$, due to the fact that $z \ V_1 \ y$.

One pair is common to these two series of four pairs, so that $\mathcal{R}$ is equal to $X \times X$ minus the seven distinct pairs in the lists above. It is a complete relation.

On X_2, it is easy to check that we have $b \succ_2^{\pm} a$, so that $AC1_2$, $AC2_2$ and $AC3_2$ hold.

On X_1, it is easy to check that $\succsim_1^-$ is complete. We indeed have that:

$$z \succ_1^- x \succ_1^- [y \sim_1^- w].$$

The relation $\succsim_1^+$ is not complete. We have $z \succ_1^+ x$, $x \succ_1^+ y$ and $x \succ_1^+ w$ but neither $y \succsim_1^+ w$ nor $w \succsim_1^+ y$ since $(y, b) \ \mathcal{R} \ (x, a)$ but $Not[(w, b) \ \mathcal{R} \ (x, a)]$ and $(w, a) \ \mathcal{R} \ (z, a)$ but $Not[(y, a) \ \mathcal{R} \ (z, a)]$. This shows that $AC1_1$ is violated. Condition $AC3_1$ holds since $\succsim_1^+$ and $\succsim_1^-$ are not incompatible. ◇

Example 85 ($Not[AC2_i, M2_i, Maj2_i, MM2_i, LC_i]$)
This is Example 35 in BP09a. It is a slight variation on Example 84 obtained by reversing all relations S_i and V_i.

Let $X = X_1 \times X_2$ with $X_1 = \{x, y, z, w\}$ and $X_2 = \{a, b\}$. We build a CDR on X with:

- $w \ P_1 \ z$, $x \ P_1 \ z$, $y \ P_1 \ z$, $w \ P_1 \ x$, $y \ I_1 \ w$, $y \ I_1 \ x$ (and all I_1 loops),
- V_1 is empty except that $y \ V_1 \ z$,
- $a \ P_2 \ b$ (and all I_2 loops) and the relation V_2 is empty,
- $\{1, 2\} \rhd \varnothing$, $\{1, 2\} \triangleq \{2\}$, $\{1, 2\} \triangleq \{1\}$ and $\{1\} \triangleq \{2\}$.

Observe that S_1 is a semiorder (the weak order it induces ranks the elements of X_1 in the following order: w, y, x, z). The relation V_1 is a strict semiorder that is included in P_1. But (S_1, U_1) is not an homogeneous chain of semiorders on X_1 since the weak order induced by U_1 ranks y before w, while the weak order induced by S_1 does the opposite.

By construction, $\mathcal{R}$ is a CDR. Hence, it satisfies $RC1$, $RC2$, $M1$, $Maj1$, $MM1$, $M3$, $Maj3$ and $MM3$ (Theorem57 and Lemmas 41 and 55). It satisfies $M2_2$,

$Maj2_2$, $MM2_2$, but not $M2_1$ (due to the veto on X_1), not $Maj2_1$ (by Lemma 33) and not $MM2_1$ (by Lemma 41). Since $\mathcal{R}$ satisfies $M1$ it also fulfills $DMM3$. Using Lemma 11 in BP07, we get that UC and LC_2 are satisfied but LC_1 is violated.

The relation $\mathcal{R}$ contains all pairs in $X \times X$ except the following ones:

- $Not[(z,b)\ \mathcal{R}\ (w,a)]$, $Not[(z,b)\ \mathcal{R}\ (x,a)]$, $Not[(z,b)\ \mathcal{R}\ (y,a)]$, $Not[(x,b)\ \mathcal{R}\ (w,a)]$, due to the fact that $Not[\varnothing \trianglerighteq \{1,2\}]$, and
- $Not[(z,a)\ \mathcal{R}\ (y,a)]$, $Not[(z,a)\ \mathcal{R}\ (y,b)]$, $Not[(z,b)\ \mathcal{R}\ (y,a)]$, $Not[(z,b)\ \mathcal{R}\ (y,b)]$, due to the fact that $y\ V_1\ z$.

One pair is common to these two series of four pairs, so that $\mathcal{R}$ is equal to $X \times X$ minus the seven distinct pairs in the lists above. It is clear that $\mathcal{R}$ is complete.

On X_2, it is easy to check that we have $a \succ_2^{\pm} b$, so that $AC1_2$, $AC2_2$ and $AC3_2$ hold.

On X_1, it is easy to check that $\succsim_1^+$ is complete. We indeed have that:

$$[y \sim_1^+ w] \succ_1^+ x \succ_1^+ z.$$

The relation $\succsim_1^-$ is not complete. We have $w \succ_1^- x$, $y \succ_1^- x$ and $x \succ_1^- z$ but neither $y \succsim_1^- w$ nor $w \succsim_1^- y$ since $(z,a)\ \mathcal{R}\ (w,a)$ but $Not[(z,a)\ \mathcal{R}\ (y,a)]$ and $(x,b)\ \mathcal{R}\ (y,a)$ but $Not[(x,b)\ \mathcal{R}\ (w,a)]$. This shows that $AC2_1$ is violated. Condition $AC3_1$ holds since $\succsim_1^+$ and $\succsim_1^-$ are not incompatible. $\diamond$

Example 86 (Asymmetric, $Not[UC_i, M1_i, Maj1_i, MM1_i]$)
This is Example 3 in BP06.

Let $X = \{a,b\} \times \{x,y,z\}$ and $\mathcal{R}$ on X be identical to the strict linear order (abusing notation in an obvious way):

$$(a,x)\ \mathcal{R}\ (b,x)\ \mathcal{R}\ (a,y)\ \mathcal{R}\ (b,y)\ \mathcal{R}\ (a,z)\ \mathcal{R}\ (b,z),$$

except that we have also $(a,y)\ \mathcal{R}\ (b,x)$.

It is easy to see that $\mathcal{R}$ is asymmetric. We have, abusing notation:

- $(a,b) \succ_1^* [(a,a),(b,b)] \succ_1^* (b,a)$, and
- $[(x,z),(y,z)] \succ_2^* (x,y) \succ_2^* [(x,x),(y,y),(z,z)] \succ_2^* [(y,x),(z,x),(z,y)]$.

Using Lemma 22, it is easy to check that $\mathcal{R}$ satisfies $RC1$ and $RC2$.

It is clear that UC_1, LC_1 and LC_2 hold. This implies that $M2$, $Maj2$, $MM2$, $M3$, $Maj3$, $MM3$ hold as well as $M1_1$, $Maj1_1$ and $MM1_1$.

$Maj1_2$ is violated since $(a,x)\ \mathcal{R}\ (a,y)$, $(a,x)\ \mathcal{R}\ (a,z)$, $(b,x)\ \mathcal{R}\ (a,z)$ but neither $(a,y)\ \mathcal{R}\ (a,x)$ nor $(b,x)\ \mathcal{R}\ (a,y)$. As a consequence, UC_1 is also violated. Since $RC1$ and $RC2$ hold, Lemmas 33 and 41 imply that $M1_2$ and $MM1_2$ are violated.

Since $M1_1$ holds, $DMM3_1$ holds too. We show that $DMM3_2$ also holds. Assume the contrary. Taking $RC1_2$ into account, this implies that there are $a, b, z_2, w_2 \in X_2$ such that $(z_2, w_2) \succ_2^* (a, b) \succ_2^* (b, a)$. Hence (z_2, w_2) can only be (x, z) or (y, z). The fourth conclusion of $DMM3_2$ is always true since $(u, x)\ \mathcal{R}\ (v, z)$ and $(u, y)\ \mathcal{R}\ (v, z)$ for all $u, v \in X_1 = \{a, b\}$.

Using Lemma 46, we have:

- $a \succsim_1^{\pm} b$ and
- $x \succsim_2^{\pm} y \succsim_3^{\pm} z$.

Hence $AC1$, $AC2$ and $AC3$ hold. ◇

Example 87 (Asymmetric, $\boldsymbol{Not[AC1_i, M2_i, Maj2_i, MM2_i, LC_i]}$)
This example is an asymmetric variant of Example 84.

Let $X = X_1 \times X_2$ with $X_1 = \{x, y, z, w\}$ and $X_2 = \{a, b\}$. We build a CDR on X with:

- $z\ P_1^\circ\ x$, $z\ P_1^\circ\ y$, $z\ P_1^\circ\ w$, $x\ P_1^\circ\ w$, $y\ P_1^\circ\ w$,
- the relation V_1° is empty except that $z\ V_1^\circ\ y$,
- $b\ P_2^\circ\ a$,
- the relation V_2° is empty,
- $\{1\} \rhd^\circ \varnothing$, $\{2\} \rhd^\circ \{1\}$, $\{1, 2\} \rhd^\circ \varnothing$.

By construction, $\mathcal{R}$ is an asymmetric CDR. Hence, it satisfies $RC1$, $RC2$, $M1$, $Maj1$, $MM1$, $M3$, $Maj3$, $MM3$ (Theorem 57 and Lemmas 41 and 55). It satisfies $M2_2$ but not $M2_1$ (due to the veto on X_1), not $Maj2_1$ (by Lemma 41) and not $MM2_1$ (by Lemma 41). Using Lemma 11 in BP07, we get that UC and LC_2 are satisfied but LC_1 is violated[8].

Since $\mathcal{R}$ satisfies $M1$ it verifies $DMM3$.

The relation $\mathcal{R}$ contains the following pairs in $X \times X$:

- $(x, a)\ \mathcal{R}\ (w, a)$, $(x, b)\ \mathcal{R}\ (x, a)$, $(x, b)\ \mathcal{R}\ (y, a)$, $(x, b)\ \mathcal{R}\ (z, a)$, $(x, b)\ \mathcal{R}\ (w, a)$,
- $(y, a)\ \mathcal{R}\ (w, a)$, $(y, b)\ \mathcal{R}\ (x, a)$, $(y, b)\ \mathcal{R}\ (y, a)$, $(y, b)\ \mathcal{R}\ (w, a)$ (but, due to $z\ V_1^\circ\ y$, $Not[(y, b)\ \mathcal{R}\ (z, a)]$),
- $(z, a)\ \mathcal{R}\ (x, a)$, $(z, a)\ \mathcal{R}\ (y, a)$, $(z, a)\ \mathcal{R}\ (w, a)$, $(z, b)\ \mathcal{R}\ (x, a)$, $(z, a)\ \mathcal{R}\ (x, b)$, $(z, b)\ \mathcal{R}\ (y, a)$, $(z, b)\ \mathcal{R}\ (y, b)$, $(z, b)\ \mathcal{R}\ (z, a)$, $(z, b)\ \mathcal{R}\ (w, a)$, $(z, b)\ \mathcal{R}\ (w, b)$,

[8] This lemma was stated for reflexive relations but its proof does not depend on the reflexivity hypothesis. It is also valid for irreflexive, and a fortiori asymmetric, relations.

- $(w,b)\ \mathcal{R}\ (x,a)$, $(w,b)\ \mathcal{R}\ (y,a)$, $(w,b)\ \mathcal{R}\ (z,a)$, $(w,b)\ \mathcal{R}\ (w,a)$.

On X_2, it is easy to check that we have $b \succ_2^{\pm} a$, so that $AC1_2$, $AC2_2$ and $AC3_2$ hold.

On X_1, it is easy to check that $\succsim_1^-$ is complete. We indeed have that:

$$z \succ_1^- x \succ_1^- y \succ_1^- w.$$

The relation $\succsim_1^+$ is not complete. We have $z \succ_1^+ x$, $x \succ_1^+ y$ and $x \succ_1^+ w$ but neither $y \succsim_1^+ w$ nor $w \succsim_1^+ y$ since $(y,a)\ \mathcal{R}\ (w,a)$ but $Not[(w,a)\ \mathcal{R}\ (w,a)]$ and $(w,b)\ \mathcal{R}\ (z,a)$ but $Not[(y,b)\ \mathcal{R}\ (z,a)]$. This shows that $AC1_1$ is violated. Condition $AC3_1$ holds since $\succsim_1^+$ and $\succsim_1^-$ are not incompatible.

Each of P_1° and V_1° is the asymmetric part of some semiorder but these semiorders do not form an homogeneous chain of semiorders (the weak order induced by P_1° imposes that w is placed in the last position while that induced by V_1° imposes the last position to y). ◇

Example 88 (Asymmetric, $Not[AC2_i, M2_i, Maj2_i, LC_i]$)
This example is an asymmetric variant of Example 85

Let $X = X_1 \times X_2$ with $X_1 = \{x, y, z, w\}$ and $X_2 = \{a, b\}$. We build a CDR on X with:

- $w\ P_1^\circ\ y$, $w\ P_1^\circ\ z$, $x\ P_1^\circ\ z$, $y\ P_1^\circ\ z$,
- the relation V_1° is empty except that $y\ V_1^\circ\ z$,
- $a\ P_2^\circ\ b$,
- the relation V_2° is empty,
- $\{1\} \rhd^\circ \varnothing$, $\{2\} \rhd^\circ \{1\}$.

By construction, $\mathcal{R}$ is an asymmetric CDR. Hence, it satisfies $RC1$, $RC2$, $M1$, $Maj1$, $MM1$, $M3$, $Maj3$, $MM3$ (Theorem 57 and Lemmas 41 and 55). It satisfies $M2_2$ but not $M2_1$ (due to the veto on X_1), not $Maj2_1$ (by Lemma 41) and not $MM2_1$ (by Lemma 41). Using Lemma 11 in BP07, we get that UC and LC_2 are satisfied but LC_1 is violated[9].

Since $\mathcal{R}$ satisfies $M1$ it verifies $DMM3$.

The relation $\mathcal{R}$ contains the following pairs in $X \times X$:

- $(x,a)\ \mathcal{R}\ (x,b)$, $(x,a)\ \mathcal{R}\ (y,b)$, $(x,a)\ \mathcal{R}\ (z,a)$, $(x,a)\ \mathcal{R}\ (z,b)$, $(x,a)\ \mathcal{R}\ (w,b)$, $(x,b)\ \mathcal{R}\ (z,b)$,

[9] This lemma was stated for reflexive relations but its proof does not depend on the reflexivity hypothesis. It is also valid for irreflexive, and a fortiori asymmetric, relations.

- $(y,a)\ \mathcal{R}\ (x,b)$, $(y,a)\ \mathcal{R}\ (y,b)$, $(y,a)\ \mathcal{R}\ (z,a)$, $(y,a)\ \mathcal{R}\ (z,b)$, $(y,a)\ \mathcal{R}\ (w,b)$, $(y,b)\ \mathcal{R}\ (z,b)$,
- $(z,a)\ \mathcal{R}\ (x,b)$, $(z,a)\ \mathcal{R}\ (z,b)$, $(z,a)\ \mathcal{R}\ (w,b)$ (but $Not[(z,a)\ \mathcal{R}\ (y,b)]$, due to $y\ V_1^\circ\ z$),
- $(w,a)\ \mathcal{R}\ (x,b)$, $(w,a)\ \mathcal{R}\ (y,a)$, $(w,a)\ \mathcal{R}\ (y,b)$, $(w,a)\ \mathcal{R}\ (z,a)$, $(w,a)\ \mathcal{R}\ (z,b)$, $(w,a)\ \mathcal{R}\ (w,b)$, $(w,b)\ \mathcal{R}\ (y,b)$, $(w,b)\ \mathcal{R}\ (z,b)$.

On X_2, it is easy to check that we have $a \succ_2^\pm b$, so that $AC1_2$, $AC2_2$ and $AC3_2$ hold.

On X_1, it is easy to check that $\succsim_1^+$ is complete. We indeed have that:

$$w \succ_1^\pm [x \sim_1^+ y] \succ_1^+ z.$$

The relation $\succsim_1^-$ is not complete. We have $w \succ_1^- x$, $w \succ_1^- y$, $x \succ_1^- z$ and $y \succ_1^- z$ but neither $x \succsim_1^- y$ nor $y \succsim_1^- x$ since $(z,a)\ \mathcal{R}\ (x,b)$ but $Not[(z,a)\ \mathcal{R}\ (y,b)]$ and $(w,a)\ \mathcal{R}\ (y,a)$ but $Not[(w,a)\ \mathcal{R}\ (x,a)]$. This shows that $AC2_1$ is violated. Condition $AC3_1$ holds since $\succsim_1^+$ and $\succsim_1^-$ are not incompatible.

Each of P_1° and V_1° is the asymmetric part of some semiorder but these semiorders do not form an homogeneous chain of semiorders (the weak order induced by P_1° imposes that w is placed in the first position while that induced by V_1° imposes the first position to y). $\diamond$

Example 89 (Asymmetric $Not[RC1_i, M2_i, Maj2_i, MM2_i, Maj3_i]$)
This example was not published before.

Let $X = \{x,y,z,w\} \times \{a,b\} \times \{p,q\}$ and $\mathcal{R}$ consist of the set of pairs listed in Table 4.

	xap	xaq	xbp	xbq	yap	yaq	ybp	ybq	zap	zaq	zbp	zbq	wap	waq	wbp	wbq
xap	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	–	$\mathcal{R}$	–	$\mathcal{R}$	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$
xaq	–	–	–	$\mathcal{R}$	–	–	–	–	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$
xbp	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$
xbq	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
yap	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$
yaq	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$
ybp	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$
ybq	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
zap	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	–	–	–	–	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	–	–	$\mathcal{R}$	$\mathcal{R}$
zaq	–	–	–	$\mathcal{R}$	–	–	–	–	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$
zbp	–	–	–	$\mathcal{R}$	–	–	–	–	–	–	–	$\mathcal{R}$	–	–	–	–
zbq	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–
wap	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$	–	$\mathcal{R}$	$\mathcal{R}$	$\mathcal{R}$
waq	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$
wbp	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$	–	–	–	$\mathcal{R}$
wbq	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–	–

Table 4: Relation $\mathcal{R}$ in Example 89.

It is easy to see that $\mathcal{R}$ is an asymmetric relation.

As for the comparison of preference differences on each attribute, we have, for all $(\alpha,\beta) \in \Gamma = \{(x,x),(y,y),\ (z,z),(w,w),\ (x,z),(x,w),\ (y,x),(y,z),$ (y,w), (z,x),$(w,x),(w,y),(w,z)]\}$,

- $(\alpha,\beta) \succ_1^* (x,y) \succ_1^* (z,y)$ and $(\alpha,\beta) \succ_1^* (z,w) \succ_1^* (z,y)$, while (x,y) and (z,w) are incomparable in terms of $\succsim_1^*$,
- $(a,b) \succ_2^* [(a,a),(b,b)] \succ_3^* (b,a)$,
- $(p,q) \succ_3^* [(p,p),(q,q)] \succ_3^* (q,p)$.

The upward and downward dominance relations are as follows:

- $y \succ_1^{\pm} w \succ_1^{\pm} x \succ_1^{\pm} z$,
- $a \succ_2^{\pm} b$,
- $p \succ_3^{\pm} q$.

$RC1_1$ does not hold since the pairs (x,y) and (z,w) are not comparable w.r.t. $\succsim_1^*$, but $RC1_2$ and $RC1_3$ hold true. For $j \in \{1,2,3\}$, $RC2_j$, $AC1_j$, $AC2_j$, $AC3_j$ are clearly satisfied. For $j=2$ and $j=3$, using Lemma 24, we see that $\mathcal{R}$ fulfills $M1_j$ and $M2_j$ hence it satisfies UC_j and LC_j (by Lemma 11 in BP07), $Maj1_j$ and $Maj2_j$ (by Lemma33), UC_j and LC_j (by Lemma 11 in BP07), $M3_j$, $Maj3_j$, $MM1_j$, $MM2_j$ and $MM3_j$ (since each of the latter is implied by one of the previously established properties of $\mathcal{R}$).

$\mathcal{R}$ satisfies $M1_1$. Assume to the contrary that there are $s,t,u,v \in X_1$ and $S,T,U,V \in X_{-1}$ such that: (1) $(s,S)\ \mathcal{R}\ (t,T)$, (2) $(u,U)\ \mathcal{R}\ (v,V)$, (3) $Not[(t,S)\ \mathcal{R}\ (s,T)]$, (4) $Not[(s,U)\ \mathcal{R}\ (t,V)]$, (5) $Not[(v,S)\ \mathcal{R}\ (u,T)]$. Using (1), (3) and Lemma 22.2, we deduce that (s,t) can only be one of the pairs $(y,x),(y,z)$ or (w,z). In all three cases, (2) and (4) cannot both hold true since $(s,t) \succsim_1^* (u,v)$, for all $u,v \in X_1$, a contradiction.

$\mathcal{R}$ satisfies $Maj1_1$. Assume to the contrary that there are $s,t,u,v \in X_1$ and $S,T,U,V \in X_{-1}$ such that: (1) $(s,S)\ \mathcal{R}\ (t,T)$, (2) $(u,S)\ \mathcal{R}\ (v,T)$, (3) $(u,U)\ \mathcal{R}\ (v,V)$, (4) $Not[(t,S)\ \mathcal{R}\ (s,T)]$, (5) $Not[(s,U)\ \mathcal{R}\ (t,V)]$. Using (1), (4) and Lemma 22.2, we deduce that (s,t) can only be one of the pairs $(y,x),(y,z)$ or (w,z). In all three cases, (3) and (5) cannot both hold true since $(s,t) \succsim_1^* (u,v)$, a contradiction.

$\mathcal{R}$ satisfies $M3_1$. Assume to the contrary that there are $s,t,u,v \in X_1$ and $S,T,U,V,Q,R \in X_{-1}$ such that: (1) $(s,S)\ \mathcal{R}\ (t,T)$, (2) $(t,U)\ \mathcal{R}\ (s,V)$, (3) $(u,Q)\ \mathcal{R}\ (v,R)$, (4) $Not[(t,S)\ \mathcal{R}\ (s,T)]$, (5) $Not[(u,S)\ \mathcal{R}\ (v,T)]$, (6) $Not[(u,U)\ \mathcal{R}\ (v,V)]$. Using (1), (4) and Lemma 22.2, we deduce that (s,t) can only be one of the pairs

$(y, x), (y, z)$ or (w, z). If $(s, t) = (y, z)$, (2) never holds true. In case $(s, t) = (y, x)$, (1) and (4) imply $[S = ap$ and $T = bp]$ or $[S = aq$ and $T = bq]$. Contradicting (5), we have $(u, S)\ \mathcal{R}\ (v, T)$ for all $u, v \in X_1$ except for $(u, v) = (z, y)$, for which (3) does not hold. The case in which $(s, t) = (w, z)$ is dealt with similarly. As a conclusion, $M3_1$ holds for $\mathcal{R}$.

$\mathcal{R}$ violates $M2_1$. $M2_1$ does not hold if we can find $s, t, u, v \in X_1$ and $S, T, U, V \in X_{-1}$ such that: (1) $(s, S)\ \mathcal{R}\ (t, T)$, (2) $(t, U)\ \mathcal{R}\ (s, V)$, (3) $Not[(t, S)\ \mathcal{R}\ (s, T)]$, (4) $Not[(u, S)\ \mathcal{R}\ (v, T)]$, (5) $Not[(u, U)\ \mathcal{R}\ (v, V)]$. These 5 conditions can be simultaneously fulfilled by setting: $s = y, t = x, u = z, v = y$ and $S = ap, T = bp, U = ap, V = aq$. Since $MM2_i$ and $RC2_i$ entail $M2_i$ (Lemma 41.3), $\mathcal{R}$ violates $MM2_1$.

$\mathcal{R}$ violates $Maj3_1$. $Maj3_1$ does not hold if we can find $s, t, u, v \in X_1$ and $S, T, U, V, Q, R \in X_{-1}$ such that: (1) $(s, S)\ \mathcal{R}\ (t, T)$, (2) $(v, S)\ \mathcal{R}\ (u, T)$, (3) $(t, U)\ \mathcal{R}\ (s, V)$, (4) $(u, Q)\ \mathcal{R}\ (v, R)$, (5) $Not[(t, S)\ \mathcal{R}\ (s, T)]$, (6) $Not[(u, U)\ \mathcal{R}\ (v, V)]$. These 6 conditions can be simultaneously fulfilled by setting: $s = y, t = x, u = z, v = w$ and $S = ap, T = bp, U = ap, V = aq, Q = ap, R = bq$. Since $Maj2_i$ entails $Maj3_i$, $\mathcal{R}$ also violates $Maj2_1$.

Since $\mathcal{R}$ satisfies $M1_1$ (resp. $M3_1$) it satisfies $MM1_1$ (resp. $MM3_1$).

Since $\mathcal{R}$ satisfies $MM1_1$ it satisfies $DMM3_1$ (by Lemma 65). $\diamond$