Laboratoire d'Analyse et Modélisation de Systèmes pour
1 'Aide à la Décision
UMR 7243

CAHIER DU LAMSADE
349
Mars 2014

Optimal stock-out risk for a component mounted on several assembly lines in the case of emergency supplies

Vincent Giard*, Mustapha Sali**

*Professeur émérite à PSL- Université Paris-Dauphine – LAMSADE – vincent.giard@dauphine.fr

**Maître de conférences à PSL- Université Paris-Dauphine – DRM – mustapha.sali@dauphine.fr

www.cnrs.fr

Abstract: *This article examines the optimal stock-out risk for a component used by alternative modules mounted on several assembly lines. The studied context is a supply chain dedicated to the mass production of highly diversified products, which is common in the automotive industry. The material requirement planning (MRP) approach is adapted to monitor this chain; however, the distance between the production units leads to a mix between production to stock and production to order for the component of interest. To prevent stock-out propagation along the downstream part of the supply chain, the requirement for an emergency supply is triggered prior to the latter's occurrence. The definition of the optimal safety stock and its associated optimal stock-out risk are based on a mono-period model that considers the cost of a safety stock and the costs incurred by the emergency supply (transportation and production). The analytical solutions dependent on these costs are illustrated*.

Keywords: Stock-out risk, Emergency supplies, Safety stock, Supply Chain, Customized mass production

1. Introduction

In this article, we define the optimal stock-out risk for an order-up-to-level supply policy. We examine the particular context of the mass production of highly diversified products in which component requirements are supplied for the use of alternative and optional modules on final assembly lines; the products' overall production is deemed stable and predictable.

Supply chains (SCs) dedicated to the mass production of highly diversified products are characterized by a certain geographic dispersion of production facilities that is well-known in the automotive industry. In this context, the production is driven by several assembly lines that are geographically remote. Diversity is mainly ensured by alternative modules (AM, e.g., engines and gearboxes) mounted on several workstations in a final assembly line. Each workstation is dedicated to a different set of alternative modules (AMS), of which one AM must be mounted on the finished product that passes through this workstation. An alternative module can be used by many assembly lines and belongs to several alternative sets of modules, each set being specific to an assembly line. Because a fictitious module can be introduced as an AM, optional modules (e.g., sunroof) are considered particular AM that belongs to an AMS, with this fictitious AM. Production levels of final assembly lines are stable in the short term or their evolutions are known.

With a known daily production over several weeks for each line, the demand of systematically mounted components and of the components that they use is certain. In the absence of uncertainty with respect to quality, lead-times and production, the management of this component type is not difficult and is beyond the scope of our study.

The supply and production management of alternative modules − and the components they use − is more complex. We consider the classic scenario in which orders are delivered with similar periodicity. This operating mode is that of the MRP when the lot-for-lot policy is used. In this approach, a specified quantity (to supply or to produce) of a given reference of the bill of materials (BOM) is calculated periodically to ensure compliance with the requirements of the master production schedule (MPS), which derives the production of all productive units of the SC.

Giard and Sali (2012) and Sali (2012) proposed an adaptation of the MRP approach to control the production of components for mass customization in upstream SCs. In that context, the requirements of the MPS, which is used for pulling the production of components and alternative modules along the SC, are specified at the BOM level corresponding to the alternative modules known as the Planning BOM (PBOM). Over the frozen horizon, these requirements are uncertain and can be represented by random variables that are used to determine, for each alternative module and component, an order-up-to-level that integrates a safety stock. In these two studies, the chosen stock-out risk is not issued from an economic trade-off, and no rule specifies its level. This risk was arbitrarily fixed to a low level, assuming that it is preferable to prevent, as much as possible, a stockout, given its potential consequences for the downstream part of the supply chain. In real life, an emergency supply, which generates extra costs compared with the classical supply, is used to avoid shortages. Thus, it is relevant to consider the problem of the optimal stock-out risk calculation issued from a tradeoff between a shortage cost (emergency supply) and a holding cost (safety stock).

This study addresses the economic analysis that should be used to determine the optimal stock-out risk when an emergency supply is triggered systematically to prevent the propagation of a stock-out along the downstream part of a SC. We describe how the problem of the emergency supply can be characterized in the studied context. In the third section, we model the problem and provide the resulting analytical solutions before illustrating them through examples. A list of used parameters and variables appears in appendix 1.

2. Problem positioning

In mass customization, product diversity is too high, necessitating that the MPS be defined at the BOM level of alternative modules, which are limited in number. The requirements of systematically used components are certain when the total periodic production is known in advance. Thus, these components are beyond the scope of our study.

The requirements of alternative modules for periods covered by the frozen horizon H_F^l of an assembly line l are known. The frozen horizon delimits what can be produced to order in the upstream part of the SC. The remoteness of the production units in global SCs and the heterogeneity of the frozen horizons associated with the assembly lines prompt an adaption of the MRP approach that can mix make-to-order (MTO) and make-to-stock (MTS) productions. Such an adaptation of the MRP is proposed by Giard and Sali (2012); we summarize their analytical results in this paper (§ 2.1). In this proposal, for each component and alternative module, the authors define a reorder-up-to-level, used to address demand uncertainty, using an arbitrarily stock-out risk. The determination of the stock-out risk may result from an economic trade-off between the cost of triggering emergency supplies and the cost of holding safety stock. The data used for this trade-off are detailed (§ 2.2) to facilitate, in section 3, a general modeling of the problem.

2.1. Procurements in a revisited MRP by mixing MTO and MTS

We refer to the results obtained in (Giard and Sali, 2012) and generalized in (Sali, 2012).

The application in cascade of the BOM explosion leads to find a_{ik} units of component i, which belongs to level n of the BOM included in an alternative module k

belonging to the set $\mathscr{E}_i^l$. $\mathscr{E}_i^l$ is the set of exclusive alternative modules that require component i to be used in the assembly line l.

Moreover, in the MPS, the application in cascade of the lead-time offset generates a lag λ_{ik} between the period t of the production launch of reference unit i and the period $t+\lambda_{ik}$ of the requirements of the module k in the MPS. Thus, the gross requirements (GR_{it}) of reference i (level n of the BOM) at period t is linked to the requirements $MPS_{kt'}^l$ of module k (level 1 of the BOM) mounted on the assembly line l at period $t > t'$. This link is distinct from the classical link that binds the gross requirements of component i with the planned orders of the references of level $n-1$ of the BOM that use this component.

When the demand is certain, the stocks are useless and, if the lot-for-lot rule is followed, GR_{it} equals the net requirements (NR_{it}) and the planned order ($PO_{i,t-L_i}$), where L_i is the lead-time of component i. The assumption of the lot-for-lot rule in this paper is justified by the generalization of the lean management and by the low reliability of forecasts for distant periods. These values are related to the MPS requirements of the final assembly lines by equation (1).

$$PO_{it} = NR_{i,t+L_i} = GR_{i,t+L_i} = \sum_l \sum_{k \in \mathscr{E}_i^l} a_{ik} \times MPS_{k,t+\lambda_{ik}}^l \qquad (1)$$

Beyond the frozen horizon H_F^l of assembly line l, demand is uncertain and characterized only through its structure as recorded in the PBOMs. In this case, the coefficient c_k^l of a PBOM represents the average usage rate of the alternative module k among the set of alternative modules assembled on a workstation of the line l and is considered as the probability of assembling k on a product passing through this workstation.

The requirements of the MPS of assembly line l for the alternative module[1] k in the period $t' > H_F^l$ becomes a random variable $X_{k,t'}^l$. This variable follows a binomial distribution in which the number of events corresponds to the number of finished units of product that are assembled on line l during a review period T, and the probability of the event occurrence is the coefficient c_k^l of the PBOM.

$$GR_{i,t+L_i} = \sum_l \Big[\sum_{\mathscr{E}_i^l \mid \lambda_{ik} < H_F^l} a_{ik} \cdot MPS_{k,t+\lambda_{ik}}^l + \sum_{\mathscr{E}_i^l \mid \lambda_{ik} \geq H_F^l} a_{ik} \cdot X_{k,t+\lambda_{ik}}^l \Big] \qquad (2)$$

---

[1] If component i is required by several alternative modules on the workstation with the same coefficient a_{ik} and for the same period, a fictitious module k', which regroups that subset of alternative modules, is required. The coefficient of planning BOM for this fictitious module is the sum of the coefficients of modules included in this subset. This module allows our approach to be generalized by considering the commonality of components used by several alternative modules in the same assembly line.

This generalization is essential to plan the production of remote assembly lines dedicated to the mass production of diversified products with an MRP approach. The planned order PO_{it} calculated at the beginning of period t and delivered at the beginning of period $t+L_i$ equals the sum of a specific requirement for that period $t+L_i$ generated by the part of the MPS within the frozen horizon ($\sum_l \sum_{k\in\mathcal{K}_i^l} a_{ik}\times MPS_{k,t+\lambda_{ik}}^l$) and the difference between an order-up-to level $R_{i,t+L_i}$ and the projected available inventory during decision-making (cf. equation (3)). The projected available inventory equals the one-hand balance OHB_{it}, which is the physical stock at the beginning of period t, increased by the planning orders that will be delivered before the end of the lead-time period $\sum_{h=0}^{h=L_i-1} PO_{i,t-L_i+h}$ and decreased by the requirements $\sum_{h=0}^{h=L_i-1} \sum_{\mathcal{K}_i^l|\lambda_{ik}<H_G^l+h} a_{ik}\cdot MPS_{k,t+\lambda_{ik}-h}^l$ to deliver before the end of this period.

$$PO_{it}=\sum_l \sum_{k\in\mathcal{K}_i^l} a_{ik}\times MPS_{k,t+\lambda_{ik}}^l + R_{i,t+L_i} - [OHB_{it} + \sum_{h=0}^{h=L_i-1} PO_{i,t-L_i+h} - \sum_{h=0}^{h=L_i-1} \sum_{\mathcal{K}_i^l|\lambda_{ik}<H_G^l+h} a_{ik}\cdot MPS_{k,t+\lambda_{ik}-h}^l] \tag{3}$$

The order-up-to-level $R_{i,t+L_i}$ is the fractile associated with a predefined stock-out risk for the random variable $Y_{i,t+L_i}$ that corresponds to a weighted sum of random variables following a binomial distribution (cf. equation (4)).

$$Y_{i,t+L_i}=\sum_l \sum_{h=0}^{h=L_i-1} \sum_{\mathcal{K}_i^l|\lambda_{ik}\geq H_F^l-h} a_{ik}\cdot X_{k,t+\lambda_{ik}+h}^l \tag{4}$$

In the steady state, as characterized by the stability of the PBOMs, the $Y_{i,t+L_i}$ variable becomes Y_i, and $R_{i,t+L_i}$ is replaced becomes R_i.

$$Y_i=\sum_l \sum_{h=0}^{h=L_i-1} \sum_{\mathcal{K}_i^l|\lambda_{ik}\geq H_F^l-h} a_{ik}\cdot X_k^l \tag{5}$$

Subsequently, we conduct our investigation under steady state conditions to simplify the formulation. In all cases, this random variable, which serves as a reference to determine the order-up-to-level, is a weighted sum of binomial random variables whose distribution function is straightforward to determine using the Monte Carlo method. Expression (6) offers a generic formulation[2] of the random variable Y_i.

$$Y_i \to \sum_j w_j \times \mathscr{B}(n_j, p_j) \tag{6}$$

When the conditions of approximation by a normal distribution are satisfied for each binomial distribution, the random variable Y_i can be approximated by a normal distribution.

$$Y_i \to \mathscr{N}\left(\mu_i = \sum_j w_j \times n_j \times p_j, \sigma_i = \sqrt{\sum_j \left[w_j \times n_j \times p_j \times (1-p_j)\right]}\right) \tag{7}$$

In the case of mass production, this approximation is generally possible for describing the MPS stochastic requirement of an alternative module beyond the frozen horizon, thereby allowing a normal approximation for the stochastic requirement of component i pulled by the stochastic requirements of alternative modules.

2.2. Costs to consider for the determination of an optimal stock-out risk in the context of emergency supplies

In the studied context, a stock-out at any plant of the SC triggers an emergency procedure that prevents production stoppages. The emergency procedure assumes that the supplier can mobilize additional resources to promptly produce the missing units and that the lead-time can be shortened to deliver the missing quantities more rapidly than a normal delivery. Mobilizing an emergency procedure at a given level of the SC prevents stock-out propagation downstream in the SC.

Emergency supplies of the missing units can be analyzed in the context of an order-up-to-level policy that is characterized by an order-up-to-level R_i designed to cope with random demand according to a stock-out probability α_i. This policy generates two types of costs: costs directly incurred by the emergency supply to avoid stoppages and costs incurred by the unused units when the order is delivered that result from using safety stock.

- Emergency supply costs: an emergency supply may or may not generate a fixed cost c_{F_i} that is independent of the number of missing units. This cost may correspond to the payment of a special transport (charter a plane, for example) and/or the launch of exceptional production (set-up cost). It is possible to introduce a capacity W, expressed as the maximum number of components i to carry in an emergency transport, to yield the possibility of requesting additional emergency transports to satisfy an abnormally important shortage; the same fixed cost is assumed to be supported, although this hypothesis may be changed easily. An emergency supply

---

[2] The notations used in this formulation have no physical significance. They are used to obtain a generic mathematical expression for Y_i.

can also generate an additional variable cost c_{V_i} per missing unit. This cost can be the unit transportation cost of a logistics provider that specializes in rapid transit and/or an increase in the direct variable production cost of a missing unit (because of overtime, for example).

- Holding costs: if no stock-out occurs at the end of the review period and prior to the receipt of a new delivery, the residual stock generates a holding cost. Each component unit i held during the review period T generates a periodic holding cost p_i, which is the multiplication of an annual unitary holding cost π_i by the duration T in years.

The amount of these charges depends on the order-up-to level R_i. The minimization of the global cost of the procurement policy independently for each component i defines the optimal order-up-to level R_i^* associated with the optimal stock-out probability $\alpha_i^* = \mathrm{P}(X > R_i^*)$. The optimal stock-out risk is not necessarily identical for all of the components.

This type of inventory problem can be considered a variant of the newsvendor problem, which introduces, in addition to the "traditional" shortage cost, a lump-sum cost independent of the importance of the shortage in the case of a stock-out. This problem was approached by Wagner (1975) with a restrictive formulation (no proportional shortage cost). Hill et al. (1989) are interested in the management of spare parts for equipment that reach the end of their life cycle with a mono-period model that includes only a lump-sum cost.

The multi-period supplying policies "*S, s*", based on an order-up-to level *S* and an order point *s*, incorporate an ordering cost in addition to the holding and stock-out costs in the economic function to minimize. Such models are often based on a stochastic dynamic programming to define optimal policies in the steady state (Naddor, 1966) or within a given horizon. In the context of these models, Bel and Hamidi-Noori (1982) use an approximate formulation to resolve the periodic problem of the supply of foreign currencies in a banking agency. Aneja and Noori (1987) propose an "*S, s*" supply model by considering a single product multi-period inventory problem for which the penalty cost is twofold: a lump-sum portion independent of the shortage size and a portion which is linearly proportional to the shortage size. This approach was generalized by Benkherouf and Sethi (2010).

The problem studied here is characterized by a periodic decision, independent from those conducted previously. Stochastic dynamic programming is inappropriate for modeling that issue. Our bibliographic investigations did not enable us to find models addressing emergency supply close to the type of emergency supply that concerns us. The available formulations differ in particular respects from the formulation suggested in this article.

3. Determination and implementation of the optimal emergency supply policy

After reviewing the analytical formulation of the problem and highlighting the relationship that characterizes the optimal policy in different cost contexts (§ 3.1), we introduce the condition of dominance of an emergency policy based on fixed shortage cost versus an emergency policy based on proportional shortage cost (§ 3.2). This issue is encountered when decision makers must choose between these alternatives. We end this section with numerical examples that applies the found analytical solution (§ 3.3).

3.1. Emergency supply model and optimal solution

The cost function to minimize $\mathrm{C}(R_i)$, defined over the review period T, is the sum of a mathematical expectation of the holding cost $\mathrm{CH}(R_i)$ and a mathematical expectation of a stock-out cost $\mathrm{CS}(R_i)$. We use a discrete formulation of the problem (equations 9 to 11) followed by a continuous formulation.

$$\mathrm{C}(R_i) = \mathrm{CH}(R_i) + \mathrm{CS}(R_i) \tag{8}$$

The first term $\mathrm{CH}(R_i)$ is the product of the periodic holding cost of one unit of component i that is held during one review period and the mathematical expectation of the remaining stock at the end of this period. The remaining stock level depends on the order-up-to level R_i and the random demand Y_i of component i.

$$\mathrm{CH}(R_i) = p_i \cdot \sum_{y_i=R_i+1}^{\infty} (R_i - y_i) \cdot \mathrm{P}(Y_i = y_i) \tag{9}$$

The second term $\mathrm{CS}(R_i)$ in equation (8) depends on the order-up-to level R_i and the random demand Y_i. It involves the fixed and variable costs identified previously. The formulation $\mathrm{CS}(R_i)$ is given by equation (10).

$$\mathrm{CS}(R_i) = c_{V_i} \cdot \sum_{y_i=R_i+1}^{\infty} (y_i - R_i) \cdot \mathrm{P}(Y_i = y_i) + c_{F_i} \cdot \sum_{u=0}^{\infty} \mathrm{P}(Y_i \geq R_i + 1 + u \cdot W) \tag{10}$$

- The first part of this cost, i.e., $c_{V_i} \cdot \sum_{y_i=R_i+1}^{\infty} (y_i - R_i) \cdot \mathrm{P}(Y_i = y_i)$, corresponds to the mathematical expectation of the additional variable costs generated by the expected stock-out amount.
- The second part of this cost, i.e., $c_{F_i} \cdot \sum_{u=0}^{\infty} \mathrm{P}(Y_i \geq R_i + 1 + u \cdot W)$, is the mathematical expectation of a fixed cost. The capacity constraint of the emergency transport makes this cost dependent on the number of missing units. If the unit transportation capacity W is important (compared with the average demand), then the probability $\mathrm{P}(Y_i \geq R_i + W + 1)$ of the requirement for a second emergency transport can be neglected. If W is low, and the probability $\mathrm{P}(Y_i \geq R_i + 1)$ of using an emergency transport is low, then the probability $\mathrm{P}(Y_i \geq R_i + W + 1)$ is lower and may be neglected. Thereafter, we will privilege the case in which the possibility of requiring more than one emergency transport at the end of any review period is highly improbable; then, the second part of this shortage cost becomes $c_{F_i} \cdot \mathrm{P}(Y_i \geq R_i + 1)$.

The cost function to minimize is thus given by equation (11).

$$\mathrm{C}(R_i) = p_i \cdot \sum_{y_i=R_i+1}^{\infty} (R_i - y_i) \cdot \mathrm{P}(Y_i = y_i) + c_{V_i} \cdot \sum_{y_i=R_i+1}^{\infty} (y_i - R_i) \cdot \mathrm{P}(Y_i = y_i) + c_{F_i} \cdot \sum_{u=0}^{\infty} \mathrm{P}(Y_i \geq R_i + 1 + u \cdot W) \tag{11}$$

In the continuous case, (9) to (11) becomes (12) to (14).

$$\mathrm{CH}(R_i) = p_i \int_0^{R_i} (R_i - y_i) f(y_i) dy_i \tag{12}$$

$$\mathrm{CS}(R_i) = c_{V_i} \int_{R_i}^{\infty} (y_i - R_i) f(y_i) dy_i + c_{F_i} \cdot \sum_{u=0}^{\infty} \mathrm{P}(Y_i \geq R_i + 1 + u \cdot W) \tag{13}$$

$$\mathrm{C}(R_i) = p_i \int_0^{R_i} (R_i - y_i) f(y_i) dy_i + c_{V_i} \int_{R_i}^{\infty} (y_i - R_i) f(y_i) dy_i + c_{F_i} \cdot \sum_{u=0}^{\infty} \mathrm{P}(Y_i \geq R_i + 1 + u \cdot W) \tag{14}$$

We seek to determine the stock-out risk α_i^* associated with the order-up-to level R_i^* that minimizes the expected total cost $\mathrm{C}(R_i)$.

In the discrete case, the two cost functions are monotone (increasing for $\mathrm{CP}(R_i)$ and decreasing for $\mathrm{CS}(R_i)$), with R_i^* satisfying the system of inequalities (15).

$$\begin{aligned} &\mathrm{C}(R_i^*) - \mathrm{C}(R_i^* + 1) < 0 \\ &\mathrm{C}(R_i^*) - \mathrm{C}(R_i^* - 1) < 0 \end{aligned} \tag{15}$$

The determination of R_i^* and thus of α_i^* is achieved through the study of the function $\mathrm{C}(R_i) - \mathrm{C}(R_i + 1)$. After development and replacement, we obtain relation (16) if more than one emergency transport is possible, and we obtain relation (17) otherwise.

$$\mathrm{C}(R_i) - \mathrm{C}(R_i + 1) = -p_i + (c_{V_i} + p_i) \cdot \mathrm{P}(Y_i \geq R_i + 1) + c_{F_i} \cdot \sum_{u=1}^{\infty} \mathrm{P}(Y_i = R_i + 1 + u \cdot W) \tag{16}$$

$$\mathrm{C}(R_i) - \mathrm{C}(R_i + 1) = -p_i + (c_{V_i} + p_i) \cdot \mathrm{P}(Y_i \geq R_i + 1) + c_{F_i} \cdot \mathrm{P}(Y_i = R_i + 1) \tag{17}$$

In the continuous case, the optimum is defined by $d\mathrm{C}(R_i)/dR_i = 0$. Then, if the use of more than one emergency transport capacity is considered, we obtain the relation (18). If this situation is regarded as highly improbable, then the relation (19) must be considered.

$$p_i \cdot \mathrm{P}(Y_i < R_i^*) - c_{V_i} \cdot \mathrm{P}(Y_i > R_i^*) - c_{F_i} \cdot \sum_{u=0}^{\infty} f(R_i^* + u \cdot W) = 0 \tag{18}$$

$$p_i \cdot \mathrm{P}(Y_i < R_i^*) - c_{V_i} \cdot \mathrm{P}(Y_i > R_i^*) - c_{F_i} \cdot f(R_i^*) = 0 \tag{19}$$

Equations 16 and 18 may be used to calculate the optimal solution for any problem, provided that the distribution of Y_i is known. Thereafter, we will use equations (17) and (19), considering that the possibility of requiring more than one emergency transport capacity is highly improbable (for the reasons listed above). We now distinguish three cases according to the values assumed by c_{F_i} and c_{V_i} .

3.1.1 Case 1: variable emergency supply cost $(c_{F_i} = 0)$

The use of equations 17 and 19 leads to the classical formulation of the newsvendor problem in which the optimal stock-out risk value is given by (20) in the discrete case and (21) in the continuous case.

$$\mathrm{P}(Y_i \geq R_i + 1) < \frac{p_i}{c_{V_i} + p_i} < \mathrm{P}(Y_i \geq R_i) \tag{20}$$

$$\alpha_i^* = \frac{p_i}{c_{V_i} + p_i} = \frac{1}{(c_{V_i} / p_i) + 1} \tag{21}$$

With a unit purchase cost c_{u_i}, the holding cost $p_i = cu_i \cdot \tau \cdot T$ depends on the parameters τ and T that are shared by the other part of the supply. Then, the optimal stock-out probability α_i^* depends primarily on the relative cost structure c_{V_i} / c_{u_i}, as illustrated by Figure 1 (this figure and the following ones are drawn to illustrate a numerical example). The order-up-to level R_i^* is the fractile associated with α_i^*. The inverse functions of the major probability distributions are available in spreadsheet applications for continuous and discrete distributions.

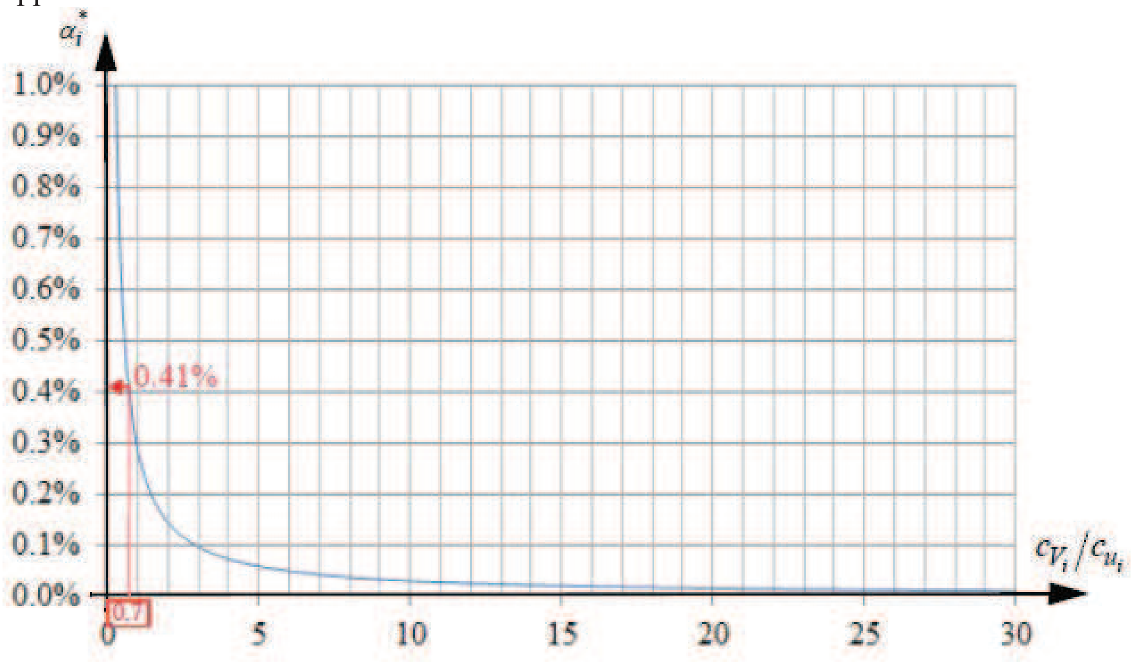

Figure 1 : α_i^* as a function of c_{V_i} / c_{u_i}, with $c_{F_i} = 0$, $T = 1$ week and $\tau = 15\%$

3.1.2 Case 2: fixed emergency supply cost $(c_{V_i} = 0)$

In this case, (17) is replaced by (22).

$$\mathrm{C}(R_i) - \mathrm{C}(R_i + 1) = c_{F_i} \times \mathrm{P}(Y_i = R_i + 1) + p_i \times \mathrm{P}(Y_i < R_i + 1) \tag{22}$$

The optimality is reached when the relation (23) is satisfied.

$$\frac{\mathrm{P}(Y_i = R_i^* + 1)}{\mathrm{P}(Y_i < R_i^*)} < \frac{p_i}{c_{F_i}} < \frac{\mathrm{P}(Y_i = R_i^*)}{\mathrm{P}(Y_i < R_i^* - 1)} \tag{23}$$

In the continuous case (equation (19)), we obtain the relation (24) in which f is the probability density function of the random variable representing the demand. We observe that if the density function is symmetrical, then the first member of equation (24) corresponds to a hazard function[3].

$$f(R_i^*) \big/ \mathrm{P}(Y_i < R_i^*) = p_i \big/ c_{F_i} \tag{24}$$

Whether we are in a discrete case or in a continuous case of a normal distribution, the numerical determination of the optimal solution is relatively simple. When the demand Y_i is a weighted sum of binomial distributions, the solution can be obtained using the Monte Carlo method. When a normal approximation of Y_i is possible, then the resolution becomes considerably simpler with an abacus using a standardized normal distribution. With $Y_i \to \mathcal{N}(\mu_{Y_i}, \sigma_{Y_i})$ and $U_i = \left(Y_i - \mu_{Y_i}\right) / \sigma_{Y_i} \to \mathcal{N}(0,1)$, the relation (24) can be replaced by the equation (25), after a permutation of numerator and denominator in which u_i^* equals $(R_i^* - \mu_{Y_i}) \big/ \sigma_{Y_i}$, and Φ is the cumulative distribution function of the standardized normal distribution.

$$\frac{\Phi(u_i^*)}{f(u_i^*)} = \frac{c_{F_i}}{p_i.\sigma_{Y_i}} = \frac{c_{F_i}}{T.\pi_i.\sigma_{Y_i}} = \frac{c_{F_i}}{cu_i}.\frac{1}{\sigma_{Y_i}}.\frac{1}{T.\tau} \tag{25}$$

The second member of equation (25) decomposes the factors that influence the optimal solution. The tabulation of the hazard function allows the finding of u_i^* and α_i^*. For given values of τ and T, the abacus of Figure 2 yields α_i^* for different values of $c_{u_i} \big/ c_{F_i}$ and σ_{Y_i}.

---

[3] If the density function is symmetrical to the mean, as it is for the Normal Distribution, $\mathrm{P}(Y_i < R_i) = \mathrm{P}(Y_i > 2\overline{Y}_i - R_i)$, then $f(R_i) = f(2\overline{Y}_i - R_i)$, and $f(R_i) / \mathrm{P}(Y_i < R_i) = f(2\overline{Y}_i - R_i) / \mathrm{P}(Y_i > 2\overline{Y}_i - R_i)$, which is the definition of the hazard distribution.

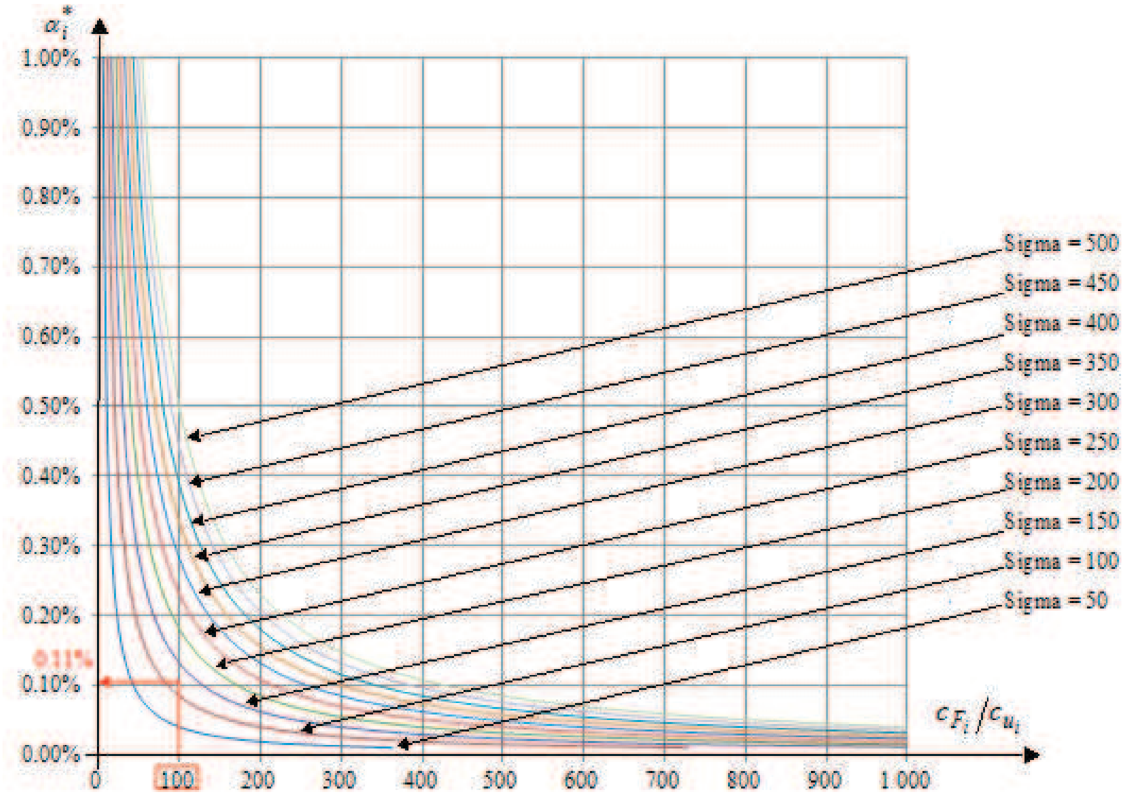

Figure 2 : α_i^* as a function of c_{F_i}/c_{u_i}, with $c_{V_i}=0$, $T=1$ week and $\tau=15\%$.

3.1.3. Case 3: variable and fixed emergency costs ($c_{V_i}\neq 0$ *and* $c_{F_i}\neq 0$)

In the discrete case, the equation (17) cannot be simplified. The optimal solution can be reached quickly using a dichotomy approach.

In the continuous case, the solution given by equation (19) can be simplified if an approximation of the demand Y_i by a normal distribution is possible. Then, the optimal solution is given by equation (26).

$$\frac{f(u_i^*)}{1-\Phi(u_i^*)}=\sigma_{Y_i}\left[(T\cdot\tau)\cdot\frac{cu_i}{c_{F_i}}\cdot\left(\frac{\Phi(u_i^*)}{1-\Phi(u_i^*)}\right)-\frac{c_{V_i}}{c_{F_i}}\right] \tag{26}$$

For given values of τ, T and σ_{Y_i}, areas representing α_i^* as a function of c_{F_i}/c_{u_i} and c_{F_i}/c_{V_i} can be drawn, as shown by Figure 3.

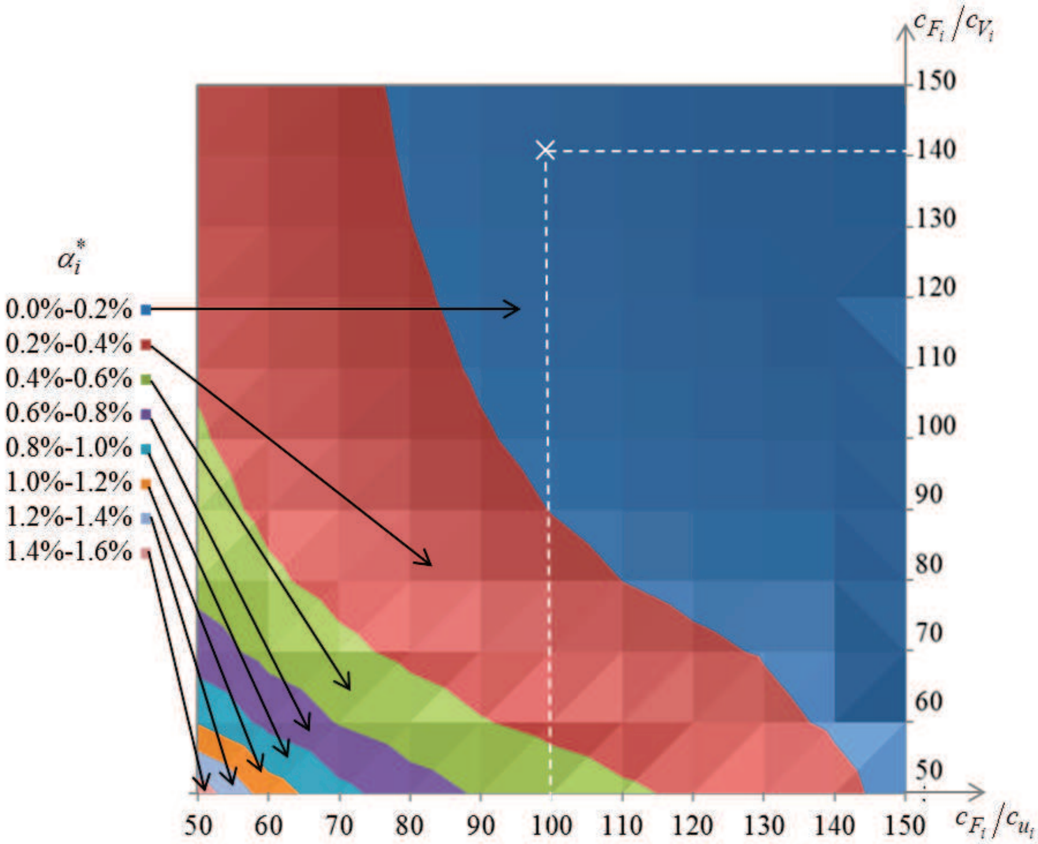

Figure 3: α_i^* as a function of c_{F_i}/c_{u_i} and c_{F_i}/c_{V_i}, with $T = 1$ week, $\tau = 15\%$ and $\sigma_{Y_i} = 123.84$.

3.2. The choice between emergency supply systems

Of the three cases of emergency supply, the last one is the least common. Most often, a company must select one of the first two policies.

- In the *variable cost policy* ($c_{F_i} = 0$), an agreement is settled with a company specializing in international express freight, with a guarantee of a short delivery time and a constant transportation cost c_{V_i} per component to deliver.
- In the *fixed cost policy* ($c_{V_i} = 0$), a mean of emergency freight transportation (plane, truck), which is entirely dedicated to emergency transportation, is used. Its cost c_{F_i} does not depend on the number of transported units, if the unit transportation capacity is sufficiently large or the shortage risk is low (as discussed previously).

We propose a simple rule to help managers determine the best option when confronted with these two possibilities for emergency supply.

We suppose that the unit purchase cost cu_i of the component, the annual holding rate τ, and the duration T of the review period are known. For a given order-up-to level R_i, the expectation of the periodic holding cost $\mathrm{CH}(R_i)$ is identical for both policies. Let us reformulate the expected periodic cost.

- The expected periodic cost $\mathrm{C}_1(R_i,c_{V_i})$ of the *variable cost policy* depends on the variable cost c_{V_i} and can be derived from equation (17) and (19), with $c_{F_i}=0$ (cf. equation (27)).

$$\mathrm{C}_1(R_i,c_{V_i})=\mathrm{CH}(R_i)+\mathrm{CS}_1(R_i,c_{V_i})$$
$$=\begin{cases}\mathrm{CH}(R_i)+c_{V_i}\cdot\sum_{y_i>R_i}(y_i-R_i)\cdot\mathrm{P}(Y_i=y_i) & \text{discrete case}\\ \mathrm{CH}(R_i)+c_{V_i}\cdot\int_{R_i}^{+\infty}(y_i-R_i)\cdot f(y_i)dy_i & \text{continuous case}\end{cases} \quad (27)$$

Note that $\forall c'_{V_i}<c_{V_i}\Rightarrow \mathrm{C}_1(R_i,c'_{V_i})<\mathrm{C}_1(R_i,c_{V_i})$.

- Similarly, the expected periodic cost $\mathrm{C}_2(R_i,c_{F_i})$ of the *fixed cost policy* depends on the fixed and is given by equation (28).

$$\mathrm{C}_2(R_i,c_{F_i})=\mathrm{CH}(R_i)+\mathrm{CS}_2(R_i,c_{F_i})=\mathrm{CH}(R_i)+c_{F_i}\cdot\mathrm{P}(Y_i>R_i) \quad (28)$$

Note that $\forall c'_{F_i}<c_{F_i}\Rightarrow \mathrm{C}_1(R_i,c'_{F_i})<\mathrm{C}_1(R_i,c_{F_i})$.

Two sufficient conditions of dominance can be proved. The conditions permit choosing between the two emergency policies, in most situations, assuming that such a choice is possible.

- Let us start with the optimal policy found for the *fixed cost policy* (case 2) characterized by R^*_{2i}. We introduce $\tilde{c}_{V_i}$ the variable cost that yields, in case 1, an identical stock-out cost $\mathrm{CS}_1(R^*_{2i},\tilde{c}_{V_i})=\mathrm{CS}_2(R^*_{2i},c_{F_i})$ and thus an identical periodic total cost for the two policies, thus yielding $\tilde{c}_{V_i}\int_{R^*_{2i}}^{+\infty}(y_i-R^*_{2i})\cdot f(y_i)dy_i=c_{F_i}\cdot\mathrm{P}(Y_i>R^*_{2i})$, with a unique positive value $\tilde{c}_{V_i}$. Then, $\tilde{c}_{V_i}=c_{F_i}\cdot\mathrm{P}(Y_i>R^*_{2i})/\int_{R^*_{2i}}^{+\infty}(y_i-R^*_{2i})\cdot f(y_i)dy_i$. If Y_i is normally distributed, with $u^*_{2i}=(R^*_{2i}-\mu_{Y_i})/\sigma_{Y_i}$, and knowing that $\int_{R_i}^{+\infty}(y_i-R_i)\cdot f(y_i)dy_i=\sigma_i\left[f(u_i)-u_i\cdot P(u>u_i)\right]$ according to Hadley & Whitin (1961), the definition of $\tilde{c}_{V_i}$ is given by equation (29).

$$\tilde{c}_{V_i}=c_{F_i}/\left\{\sigma_i\left[\left(f(u^*_{2i})/\mathrm{P}(U>u^*_{2i})\right)-u^*_{2i}\right]\right\} \quad (29)$$

With $\mathrm{C}_1(R^*_{2i},\tilde{c}_{V_i})=\mathrm{C}_2(R^*_{2i},c_{F_i})$, and $\forall c'_{V_i}<c_{V_i}\Rightarrow \mathrm{C}_1(R_i,c'_{V_i})<\mathrm{C}_1(R_i,c_{V_i})$, it can be deduced that $\forall c_{V_i}<\tilde{c}_{V_i}\Rightarrow \mathrm{C}_1(R^*_{2i},c_{V_i})<\mathrm{C}_1(R^*_{2i},\tilde{c}_{V_i})$. The *variable cost policy* is more efficient than the *fixed cost policy* if $c_{V_i}<\tilde{c}_{V_i}$. A curve, linking all possible values of c_{F_i} with the corresponding values $\tilde{c}_{V_i}$, can be drawn for decision-making. This curve delimits the range of c_{V_i} for which the *variable cost policy* is preferred to the *fixed cost policy*.

- Now, let us start from the optimal policy of the *variable cost policy* (case 1) characterized by R_{1i}^*. We introduce $\tilde{c}_{F_i}$ as the fixed cost that yields, in case 2, an identical stock-out cost $\mathrm{CS}_2(R_{1i}^*, \tilde{c}_{F_i}) = \mathrm{CS}_1(R_{1i}^*, c_{V_i})$ and thus an identical periodic total cost for the two policies, thus yielding $\tilde{c}_{F_i} \cdot \mathrm{P}(Y_i > R_{1i}^*) = c_{V_i} \int_{R_{1i}^*}^{+\infty} (y_i - R_{1i}^*) \cdot f(y_i) dy_i$, with a unique positive value $\tilde{c}_{F_i}$. Then, $\tilde{c}_{F_i} = c_{V_i} \cdot \int_{R_{1i}^*}^{+\infty} (y_i - R_i) \cdot f(y_i) dy_i / \mathrm{P}(Y_i > R_{1i}^*)$. If Y_i is normally distributed, then $\tilde{c}_{F_i}$ is given by equation (30)

$$\tilde{c}_{F_i} = c_{V_i} \cdot \sigma_i \left[f(u_{1i}^*) / \mathrm{P}(u > u_{1i}^*) - u_{1i}^* \right] \tag{30}$$

  Given $\mathrm{C}_2(R_{1i}^*, \tilde{c}_{F_i}) = \mathrm{C}_1(R_{1i}^*, c_{V_i})$, and $\forall c'_{F_i} < c_{F_i} \Rightarrow \mathrm{C}_1(R_i, c'_{F_i}) < \mathrm{C}_1(R_i, c_{F_i})$, it can be deduced that $\forall c_{F_i} < \tilde{c}_{F_i} \Rightarrow \mathrm{C}_2(R_i, c_{F_i}) < \mathrm{C}_2(R_i, \tilde{c}_{F_i})$. The *fixed cost policy* is more efficient than the *variable cost policy* if $c_{F_i} < \tilde{c}_{F_i}$. A curve, linking all possible values of c_{V_i} with the corresponding values of $\tilde{c}_{F_i}$, can be drawn for decision-making. This curve delimits the range of c_{F_i} for which the *fixed cost policy* is preferred.

Two rules, which address most cases, can be formulated for choosing between the two emergency transport policies without comparing overall costs:

- Rule 1: Policy 1 (variable cost of emergency transport) is better than Policy 2 (fixed cost of emergency transport) if $c_{V_i} < \tilde{c}_{V_i}$ or if $c_{V_i} / c_{u_i} < \tilde{c}_{V_i} / c_{u_i}$.
- Rule 2: Policy 2 (fixed cost of emergency transport) is better than Policy 1 if $c_{F_i} < \tilde{c}_{F_i}$ or $c_{F_i} / c_{u_i} < \tilde{c}_{F_i} / c_{u_i}$.

Two curves are drawn in Figure 4, the first curve with $\tilde{c}_{V_i} \big/ cu_i$ as a function of $c_{F_i} \big/ cu_i$ and the second curve with $\tilde{c}_{F_i} \big/ cu_i$ as a function of $c_{V_i} \big/ cu_i$. These two curves are nearly mixed up. Two enlargements are conducted to highlight each curve. Any point whose coordinates $(c_{F_i} / c_{u_i}, c_{V_i} / c_{u_i})$ are "below" the upper curve respects Rule 1, not Rule 2. In that case, Policy 1 (variable cost of emergency transport) is better than Policy 2. Conversely, any point whose coordinates are "above" the lower curves respects Rule 2, not Rule 1. In that case, Policy 2 (fixed cost of emergency transport) is better than Policy 1. We can add three comments.

- This dominance depends also on the variance of Y_i, observing that the review period T and the holding rate can be considered as determined, once for all.

Therefore, a 3D representation can be used for analyzing the influence of that last factor on the relative dominance of those two policies.

- This dominance does not depend on the ratio c_{F_i} / c_{V_i}, as c_{u_i} contributes to the periodic cost $\mathrm{C}(R_i)$ through the holding cost $\mathrm{CS}(R_i)$.

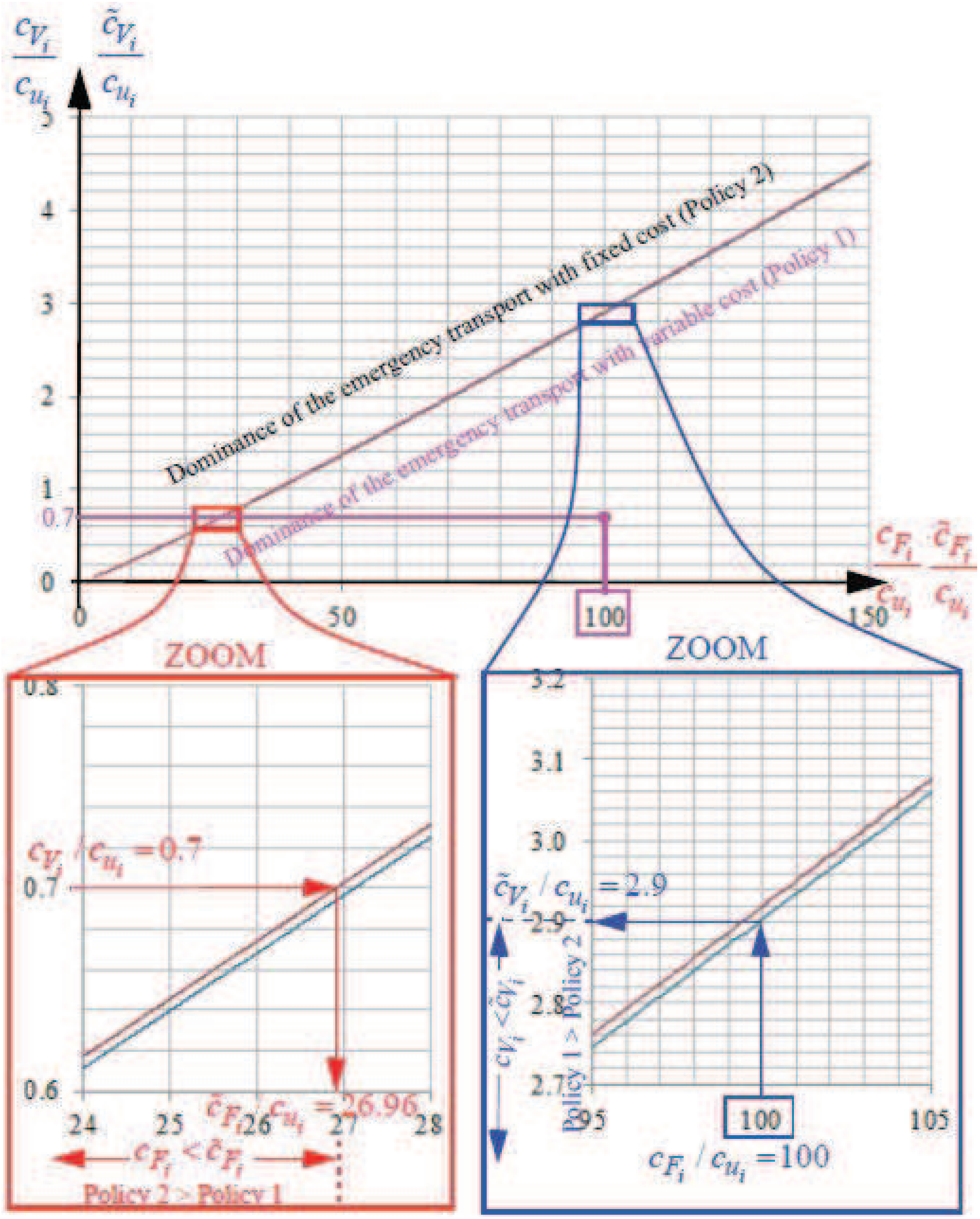

Figure 4: Dominance analysis of emergency transport policies: variable cost (Policy 1) versus fixed cost policy (Policy 2) with $c_{V_i} = 0$, $T = 1$ week, $\tau = 15\%$ and $Y_i \rightarrow \mathcal{N}(6086.4, 123.84)$

3.3 Numerical example

Let us now illustrate numerically the calculation of the optimal stock-out risk α_i^* for a component i in the first two cases mentioned above.

We use the numerical example presented in (Giard and Sali, 2012) that considers the procurement of piston crowns for automotive assembly plants. The application of the MRP mechanism, as discussed in §2, provides a demand Y_i for this component, following a weighted sum of binomial random variables.

$$Y_i \to 4\times\mathscr{B}(960,0.2)+4\times\mathscr{B}(1840,0.54)+4\times\mathscr{B}(960,0.2)+6\times\mathscr{B}(960,0.1)$$

The normal approximation of Y_i yields the formulation (31).

$$Y_i \to \mathscr{N}(6086.4,123.84) \quad (31)$$

The costs used here are fictitious: the unit purchase cost is $c_{u_i}=10\,€$; the annual holding rate is $\tau=15\%$ and a periodic review T of one week lead to a weekly unit holding cost of $p_i=0.029\,€$. The fixed cost for an emergency transport is $c_{F_i}=1000\,€$ and the variable cost is $c_{V_i}=7\,€$. If emergency transport is triggered, it is assumed that, a second transport is not needed. Those parameters were used in the establishment of figures 1 to 4. First, we illustrate the three policies of emergency supply. Then, we analyze (§3.3.4) the dominance of emergency transport policies.

3.3.1. Case 1: variable emergency supply cost $(c_{F_i}=0)$

When no fixed cost is considered, the calculation of α_i^* using the relation (21) yields an optimal risk $\alpha_{1i}^*=0.41\%$, an order-up to level $R_{1i}^*=6413$ and a safety stock $\mathrm{SS}(R_{1i}^*)=327$. A simple reading of the abacus of the Figure 1 yields this result.

3.3.2. Case 2: fixed emergency supply cost $(c_{V_i}=0)$

In the second case, the use of the equation (25) yields a risk $\alpha_{2i}^*=0{,}1115\%$, which corresponds to an order-up-to level $R_{2i}^*=6467$ and a safety stock $\mathrm{SS}(R_{2i}^*)=380$.

Figure 2 does not include a curve for $\sigma_{Y_i}=123{,}84$. A linear interpolation between the curves drawn for $\sigma_{Y_i}=100$ and $\sigma_{Y_i}=150$ yields a risk whose value is approximately 0.11%.

3.3.3. Case 3: variable and fixed emergency costs ($c_{V_i}\neq 0$ *and* $c_{F_i}\neq 0$)

A dichotomy search using (26) yields $\alpha_i^*=0{,}01\%$. This result can be found approximately by a direct reading of Figure 3.

3.3.4. Dominance analysis of emergency supplies

With $c_{F_i}=1000\,€$ ($\to c_{F_i}/c_{u_i}=100$), we obtain $\alpha_{2i}^*=0{,}11\%$ and $R_{2i}^*=6467$. To obtain the same shortage cost in the variable cost policy with the same order-up to level, a variable cost $\tilde{c}_{V_i} = 29{,}03\,€$ must be used for the emergency transport. Given $\forall c_{V_i} < \tilde{c}_{V_i}$, and $\mathrm{C}_1(R^*_{1i}, c_{V_i}) < \mathrm{C}_2(R^*_{2i}, c_{F_i})$ we obtain $\mathrm{C}_1(R^*_{1i}, 7) < \mathrm{C}_2(R^*_{2i}, 1000)$, which supports the economic superiority of the variable cost policy.

If we plot the point of coordinates "$c_{F_i} = 1000$ and $c_{V_i} = 7$" in Figure 4, we observe that it is "above" the lower curves. We conclude that the variable cost policy is dominant.

4. Conclusions

We demonstrated how to determine the optimal stock-out risk in the case of an emergency supply in the upstream portion of a supply chain dedicated to the mass production of highly diversified products. We addressed cases of emergency supply in which the stock-out cost is the sum of a fixed cost and a variable cost, depending on the amount of components demanded. Analytical solutions were established and illustrated with numerical examples using a normal approximation of the demand relevant to the studied case. The demonstrated approach can be extended to any supply chain for which the MPS requirement beyond the frozen horizon can be modeled with probability distributions. For an emergency supply, a tactical choice is often possible between a solution in which the transportation cost is proportional to the number of carried units and a solution characterized by a fixed cost. In this study, the foundations of the dominance analysis were established and illustrated.

5 Appendices

5.1 Appendix 1: List of the used parameters and variables

i	Index of a component of level n in the BOM
k	Index of a component of level 0 in the (all components of level 0 are alternative modules)
t	Index of a period used in the MRP
l	Index of a final assembly line
$\mathcal{C}_i^l$	Subset of alternative modules that use component i in assembly line l
H_F^l	Frozen horizon of assembly line l
H_P^l	Planning horizon of assembly line l
a_{ik}	Number of units of component i, included in module k that belongs to the subset $\mathcal{C}_i^l$ of alternative modules
L_i	Lead-time of component i
N^l	Periodic production on line l
c_k^l	Coefficient of the PBOM associated with the alternative module k mounted on the line l
T	Length of the review period used in the MRP
λ_{ik}	Lag between the period t of the production launch of reference unit i and the period $t + \lambda_{ik}$ of the requirements of the module k in the MPS
$MPS_{kt'}^l$	Requirement of the module k for the period $t' > t$ in the Master Production Schedule of the assembly line l
$X_{k,t'}^l$	Stochastic requirement of the alternative module k for the period $t' > H_F^l$
GR_{it}	Gross Requirement of a reference i (level n of the BOM) at period t

NR_{it}	Net Requirement of component *i* for the period *t*
PO_{it}	Planned Order of component *i* at the beginning of the period *t*
$R_{i,t+L_i}$	Order-up-to level of component *i* for the period $t+L_i$
R_i	Order-up-to level of component *i* in the steady state
α_i	Probability of a stock-out of component i, with an order-up-to level R_i
$Y_{i,t+L_i}$	Random demand of component *i* from the period *t* to the period $t+L_i$
p_i	Holding cost of component *i* between two consecutive deliveries
c_{u_i}	Unit production cost of component *i*
π_i	Annual holding cost of component *i*
τ	Annual holding rate
$\mathrm{C}(R_i)$	Expectation of the periodic cost associated with the order-up-to level R_i
$\mathrm{CH}(R_i)$	Expectation of the holding periodic cost associated with the order-up-to level R_i
$\mathrm{CS}(R_i)$	Expectation of the shortage periodic cost associated with the order-up-to level R_i
c_{V_i}	Unit variable cost of an emergency supply of component *i*
c_{F_i}	Unit variable cost of an emergency supply of component *i*
$\tilde{c}_{V_i}$	Unit variable emergency cost that yields $\mathrm{CS}_1(R_i,\tilde{c}_{V_i})=\mathrm{CS}_2(R_i,c_{F_i})$
$\tilde{c}_{F_i}$	Fixed emergency cost that yields $\mathrm{CS}_2(R_i,\tilde{c}_{F_i})=\mathrm{CS}_1(R_i,c_{V_i})$
W	Capacity of the charter used in the emergency transport of component *i* expressed as the number of units of component *i*

5.2 Appendix 2: Conditions of approximation of a binomial distribution by a normal distribution

Several conditions of approximation of the Binomial distribution $\mathscr{B}(n,p)$ by the Normal distribution $\mathscr{N}(n\cdot p,\sqrt{n\cdot p\cdot(1-p)})$ are given in statistics textbooks. This analysis is based on the condition $\left|\sqrt{\frac{p}{1-p}}-\sqrt{\frac{1-p}{p}}\right|\cdot\frac{1}{\sqrt{n}}<0{,}3$ from which the relation used to draw the limits values of p depending on n can be established, as shown in Figure 5. In our case, n is the number of units produced by the assembly line during the period used to define the MPS, and p is the probability of using a given alternative module in a given station of the line.

$$\frac{\dfrac{\left(2+0.3^2\cdot n\right)-\left[\left(2+0.3^2\cdot n\right)^2-4\right]}{2}}{1+\dfrac{\left(2+0.3^2\cdot n\right)-\sqrt{\left(2+0.3^2\cdot n\right)^2-4}}{2}}<p<\frac{\dfrac{\left(2+0.3^2\cdot n\right)+\left[\left(2+0.3^2\cdot n\right)^2-4\right]}{2}}{1+\dfrac{\left(2+0.3^2\cdot n\right)+\sqrt{\left(2+0.3^2\cdot n\right)^2-4}}{2}}$$

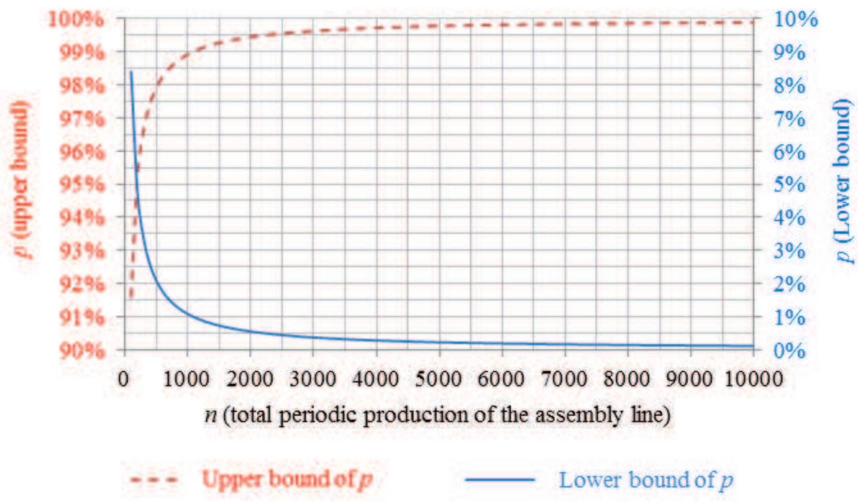

Figure 5: Conditions of approximation of a binomial distribution by a normal distribution

6 References

Aneja Y., Nouri H. A., "The optimality of (s, S) policies for a stochastic inventory problem with proportional and lump-sum penalty cost", *Management Science*, Vol. 33 n° 6, p750-755. 1987.

Bell, P. C., Hamidi-Noori A., "A one-period stochastic inventory problem with a lump-sum penalty cost", *Decision Sciences*, Vol. 13 n° 3, pp. 440-449. 1982.

Bell, P. C., Hamidi-Noori A., "A one-period stochastic inventory problem with a lump-sum penalty cost", *The Journal of the Royal Operational Research Society*, Vol. 35 n° 6, pp. 513-525. 1984.

Benkherouf L., Sethi S. P., "Optimality of .s; S/ policies for a stochastic inventory model with proportional and lump-sum shortage costs", *Operations Research Letters*, vol. 38, pp. 252-255. 2010.

Giard V., Sali M., Pilotage d'une chaîne logistique par une approche de type MRP dans un environnement partiellement aléatoire, *Journal Européen des Systèmes Automatisés,* Vol. 46 n°1, pp. 73-102, 2012.

Giard V. et Sali M., "Monitoring the upstream part of a supply chain dedicated to the customized mass production with a revisited version of MRP", *4th International conference on information systems logistics and supply chain*, Quebec 2012.

Hadley G., Whitin T.M., *Analysis Of Inventory Systems*, Prentice Hall, 1961.

Hill A. V., Giard V., Mabert V., "A decision Support System for determining optimal retention stocks for service parts inventories", *IIE Transaction*, Vol. 21 n° 3, p. 221-229, 1989.

Naddor E., *Inventory Systems*, Wiley, 1966.

Noori H. A., Bell, P. C., "A one-period stochastic inventory problem with a lump-sum penalty cost", *Decision Sciences*, Vol. 13 n° 3, pp. 440-449. 1982.

Sali M., "Exploitation de la demande prévisionnelle pour le pilotage des flux amont d'une chaîne logistique dédiée à la production de masse de produits fortement diversifiés", *PhD Dissertation*, Université Paris Dauphine, 2012.

Wagner H.M., *Principles of Operations Research*, Prentice-Hall, 1975.