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On the Rational Reconstruction of Discrete and Continuum in the History of Mathematics, An Essay

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Abstract

Continuum and Discrete hypotheses are the only ones that have not been abandoned, even if they have been, up to now, exposed to many falsifications. We will use Lakatos' methodology in order to try a rational reconstruction of the most known theories that had in their core either the discrete or the continuum hypothesis. Moreover, we will try an extension of the Lakatosian terms on sets of theories.

1 Introduction

The conflict between the *Continuum* and the *Discrete Hypothesis* is very old. It is mainly located in mathematics and philosophy, but its consequences are also visible in physics, chemistry and all the other natural sciences. These two hypotheses are the only ones that have not been left out even if they have been, up to now, exposed to many falsifications. We will use *Lakatos'* methodology ([7, 8, 9, 10, 11, 12]), in order to try a rational reconstruction of the most known theories that had in their core either the discrete or the continuum hypothesis. Moreover we will try an extension of the Lakatosian terms. We will use alternatively the terms: discrete (continuum), hypothesis of discrete (continuum), program of discrete (continuum) in order to denote *the set of programs (theories) that have in their core (constructed in an eventually a priori way) the hypothesis of the discrete (continuum).* More details about this construction will be given in Section 6. We will see that the two programs have survived concurrently despite of the signs of their degeneration. Moreover, they have remained progressive in a concurrent way. This fact justifies Lakatos' position

*This article is dedicated to the memory of Dominique Champs-Brunet, former documentalist of LAMSADE, who translated a first version of this article into French.

in his conflict with *Kuhn* ([6]) about th the theories' degeneration. We remark also here that many times those two programs have existed in a complementary way. Elements of the core and of the positive heuristic of the first one, were used to corroborate the other one, or elements of the core of the former are lied in the positive or even in the negative heuristic of the latter.

For example let us mention here the *Euclidean* program. Elements of the Euclidean theory (concerning the infinite universe) that was lied for centuries in the core of the discrete are used in the continuum's theories in order for some "difficulties" of those theories to be overcome (uniform convergence, discretization of the continuum, etc.).

In general, when we try to apply an epistemologic method in mathematics this application is problematic. This is due to the fact that the definition of the notion of experiment is not defined in mathematics in such a coherent way as in the natural sciences. The objects in mathematics are defined in a definition-depended way, while the experiment preconditions an empirical frame. Then, mathematics are used in a meta-level, in order to explain, or to generalize, the results of a particular experiment or of a class of experiments. The absence of a mathematical equivalent of the experiment impedes the rational reconstruction of mathematical programs and forces us to search for social, idealogical or even metaphysical arguments that analyse those programs and examine their limits concerning their progressiveness or degeneration.

In any case the current work brings up a justification in Lakatos' claims that there is no *"gestalt swhitch"* ([6]) in the evolution of scientific theories. We deduce by this study of the history of mathematics that there are no momentary decisive experiments ([6, 18, 19]) that falsify theories. This falsification is a historically justified phenomenon, so does the definition of a decisive experiment. It is worthy to note here that discrete was a *"seemingly degenerated program"* after *Descartes*. Moreover the falsification of Hilbert's program ([2]) by *Gődel's incompleteness theorem* ([3]) could be something analogous to a decisive experiment for the discrete. However the mathematical models developed in order to describe new computing models show that discrete remains until now a progressive program in all two directions defined by Lakatos. *On the one hand there are problems that are solved by using, as axioms, propositions situated in discrete's core and on the other hand this program augments its empirical content* (progressiveness$_1$ and progresiveness$_2$ respectively, [11, 12]).

2 Discrete before 20th Century

The problem of discrete and continuum appears in *Pythagorians, Prosocratics* and mainly in *Plato*'s philosophy.

For Pythagorians the universe is of discrete nature. The segments of a line are equivalent to every symmetric size. Their mathematical universe contains only symmetric sizes and thus very soon fail to overcome problems raised by their theory and carrying on assymetric quantities, as for example, the quantity $\sqrt{2}$. Despite the multiplicity of problems raised and solved by their program, this has never been placed in the core of the discrete.

On the contrary, *Euclidean* thesis, as described in $\Sigma\tau o\iota\chi\epsilon\acute{\iota}\alpha$ (*Elements*, [4]), has been in the core of discrete for many centuries. What is considered as the core of discrete's program, is exactly the following two axioms:

1. *The Euclidean axiom:*
   There is no point in the plane that is the intersection of more than one parallel lines.

2. *The Eudoxian or Archimedean axiom:*
   Given two comparable sizes a and b the greater one exceeds the smaller one in a size which added to itself can exceed every given size comparable to a and b.

In the work of *Eudoxus* we find a first capture of the notion of *assymetric numbers*, notion formalized later by *Weierstrass* and *Dedekind*. The second axiom of discrete's core is in fact a very strong hypothesis and is potentially falsified only if we add in the mathematical universe the notion of the infinitesimals.

In Plato, discrete appears more concretely and with more constructive details. In his dialogue *Τιμαίος* (*Timaios*, [17]), Plato develops a cosmogony based on discrete. More precisely this cosmogony is brought out on geometric elements, the so-called *Platonic Solids*, which are exactly the *convex solids* demarcated by *equal normal flat polygons*. We can form only three convex solids, namely the *equilateral triangle*, *the square* and the *normal pentagon*. The possible convex solids that can be formed using the previous polygons are the *normal tetraedron*, the *cube*, the *normal octaedron*, the *dodecaedron*, and the *icosaedron*. In this Platonic cosmogony it is claimed that everything sensible is constituted from a particular combination of four basic elementary materials: the *fire*, the *air*, the *water* and the *earth*. In their turn these materials are not primitive but they are composed by elementary platonic solids.

- Earth is constituted from elementary cubes.
- Water from elementary normal icosaedra.
- Air from elementary normal octaedra.
- Fire from elementary normal tetraedra.

These particles can also be analysed in simpler ones. Indeed, the way Plato's world is composed induces that the most elementary constructive materials of this world are some geometric primitive materials, the following two types of triangles:

- *rectangle and isosceles triangle* and
- *rectangle triangle with angle sizes* $90°$, $60°$ *and* $30°$ *respectively.*

It is obvious that in Plato's philosophy the notion of finite divisibility of the sensible world in real, discrete and not infinitely divisible constructive elements is present. His ideas have influenced many philosophers as *Descartes* or *Leibniz*, even if we can consider the latter as a scientist of the continuum.

Although Plato has not formally described his universe, his ideas are formalized in *Dimokritos'* work. Very remarkable are the philosophical conjectures of Plato about the distinction between *the phenomenon (relative) of sensible world and the* *real (absolute) of Ideas which are eternal, exact and independent from their potential connoisseur.* This philosophical position constitutes the core of a mathematical program that appeared in the 20th century, the *Intuitionists*. Even if today the core of discrete is not structured on Platonic Ideas, this notion and its extensions and consequences contribute in enriching discrete's positive heuristic.

To be more precise, in the frame of a natural theory there are entities introduced even if they are not directly observable. These theoretical entities are justified only by their observable consequences. As a recent example we can mention the *quarks* of the *theory of elementary particles*. In the frame of this theory, we can also observe an example of justification of the Lakatosian method concerning the way a scientific program evolves. Without the notion of quark, elementary particle's theory risks to be falsified because of some inexplicable, in its frame, anomalies. Thus, quarks is exactly the notion that helps the scientists working on the theory to direct the modus tolens which could falsify it, not in its core but in its negative heuristic. Following that, they do not only cancel its falsification, but they also enforce the progressiveness of the theory.

After Plato, Eucleides, *Archimides* and Eudoxus the main contribution to discrete's program is due to Descartes.

Descartes tries to falsify the conjectures that began to be formed concerning the continuity principle in the universe. His mathematical results corroborate exactly the Platonic notions of discrete. After him, *Kepler*, *Brahe*, *Copernicus*, influenced by Plato's program, see a mathematical universe *essentially discrete* but, in their works, the notion of infinite is implicitly introduced and also the one of the limit in an even more implicit way.

Berkeley also worked in discrete's frame. He has defined a theory for the universe in order to answer *Leibniz's* theory whose elements we be seen in next session. For Berkeley, real infinite does not exist; indeed, admitting the existence of such a conceptual object would lead to the admission of an infinity of ideas totally articulated in the finite human mind. In the same manner, he denies the *infinite divisibility of finite objects*. By doing so, he is led to a discrete notion for the universe. He considers that space and time are discretely structured and their limits are exactly the conceptual minimum of human mind in space and time.

With Berkeley, ends the prevalence of discrete that seems to be led to the degeneration. Effectively, this program has stopped being progressive for a long time. However, elements of the core of this program and particularly the long Euclidean traditions on mathematics, influence the mathematicians and philosophers, even if they used support the program of the continuum.

In fact, as historically justified, this program has never been refuted. It has simply remained "inactive". This is due to many reasons, seen as elements either of the *external* or of the *internal* history ([7, 9]). Between the ones concerning the external history we can distinguish the strong *Aristotelian* influence on the philosophers and mathematicians up to the beginning of the 20th century (even up to now). Another reason is that the economic, cultural and military organization and development of the societies, mainly in Europe, needed machines. Machines were conceived and developed on mathematical models based upon the continuous mathematics. Moreover, the philosophical systems and economic analyses that have supported, or were opposed to, the given structure of societies have in the core of their conception the notion of the continuity of both the human activity and the construction of the universe.

A reason due to the internal history ([7, 9]) is that the scientists (mathematicians, philosophers and even biologists) after the middle age, required more instruments than those offered by the discrete mathematics of those times. In fact, the study of astronomy, mechanics, physics, chemistry would have never progressed without the introduction of elements of the mathematics, named later *continuous mathematics*. It is not incidental that *Newton* who had constructed one of the most successful research programs in Natural Sciences was the first that introduced the theory of flows that were the primitive version of the infinitesimal calculus.

3 Continuum up to 20th Century

3.1 From Aristotle to Kant

The first who introduced the investigation of the continuum with a certain formalism is Aristotle. He has influenced the conception of the continuum research program in such a way that a lot of his ideas are lying in the core of this program.

The basic characteristic of Aristotle's philosophy is that he does not accept the platonic distinction between the world of ideas and the one of senses. He believes that in every experimental object, its matter and its scheme and appearance, is not separated or separable and their study must be unified. The mathematical objects have two characteristic properties:

1. they are included in the experimental objects and

2. they are not unique.

This second property means that there is a multiplicity of such objects included either as conceptual entities, or as characteristics of the form of the experimental objects and their sets. In other words, Aristotle's point of view on mathematics is rather *extensive* than *intensive* and approaches largely a modern set theoretic point of view, according to which, mathematical objects can be defined as equivalence classes of other objects.

We think that the main contribution of Aristotle to the program of the continuum is that he was the first who alleged that the notion of the *infinity* and that of the continuum are very closely correlated in such a way that the investigation of the continuum leads to the investigation of the infinite and vice-versa.

His conjectures on the nature of infinite were based on the following five claims:

1. Time is infinite.

2. Extended objects can be divided. The number of the divisions applied has not a supremum.

3. The procedure "creation-ruin" for every sensitive object has no end.

4. Everything that has limits is contained in another whose limits are more extended. Thus the totality of things cannot have any limit, because if the opposite is true, these limits would be included in the whole of things. Let us point out here the similarity of this claim with the main idea of the proof that *the power-set of the natural numbers is a non enumerable set (principle of diagonalization*. There is also the result of the undecidability of the *halting problem* in the computation theory ([13]) that refers directly to this thesis of Aristotle.

5. The human mind has no limits. Thus the conceptual constructions of the human being have no more limits.

According to Aristotle there are two procedures that create the infinite: the *addition* and the *division*.

To the question "what is the meaning of adding or dividing some quantities for an infinite number of times, while the time of the human life and human actions are finite concepts", Aristotle answers:

> "Everything sensible can be as large as we want but it is always finite. However nobody can claim that the procedure of the addition is finite. The *nature of the infinite is potential* and only the potential study of that has some meaning".

Hence, although the extended magnitudes are not really infinite, they can be seen as potentially infinite because of their repeated divisibility. Thus the theoretical divisibility of a natural object is equivalently expressible with the divisibility of a finite segment of a line.

One of the main points in the study of the continuum by Aristotle is his conjecture about the nature of the point (of a line) in the structure of the continuum.

The points do not really exist before the actions of dividing a part of a line. This part (length) pre-exists of its points, as their potential carrier. There is an infinite number of potential points that may be reconverted into real ones after the action of an eventual division.

Thus Aristotle's continuum is defined by the possibility of the interminable divisibility on a finite length. Here we have to notice how this correlation of the procedure of the division and the notion of the infinite is similar to Dedekinds' intersection.

The ontology of Aristotle's continuum and infinite includes the notion of time as well as that of space.

Time is potentially infinite and this possibility is also additional and divisional. The addition possibility comes from the property of the time to be the measure of the change that is continuous in the universe. The "divisibility" of the time is analogous to this of the finite part of a line. The only difference between the two divisibilities is that every past segment of time is indivisible. Thus the possibility of dividing a segment of time exists when this segment is in the future and this division is completed when this segment becomes "present".

For Aristotle the intellectual jump into the "mathematicalization" of the infinite is not permitted. Hence the notion of the continuum is rather philosophical than mathematical. However the mathematicalization of his infinite and continuum is present in the works of many mathematicians from Newton to Dedekind.

After Aristotle, and passing through the Hellenistic and Roman centuries up to Middle Ages, the mathematical results and the philosophical conjectures enforce either the program of the continuum, or that of discrete, without any surprising result that could lead one or the other to degeneration. However, during the Middle Age, several philosophers contribute to the enrichment of the positive heuristic of the continuum's program in an a posteriori but impressive way.

Saint Augustin in *Civitas Dei* ([22]), claims that "the whole of the sequence of the integers forms a real infinite". This claim is, according to *Cantor*, the first proposition concerning the "hyperfinite".

Thomas Brawardine ([22]), claims in his turn that "the continuous magnitudes even if they consist of infinite indivisible parts, they have infinite continuums of the same type".

In the conjecture of Brawardine the similarities with Cantor's theses are evident.

A bit later the early-empiricist *Locke*, tries to refute Aristotle's ideas concerning the pre-existence of the part of the line with respect to that of the point. He claims that the simpler mathematical ideas are these of the unit and the point, which are the basic materials for the construction of all the mathematical ideas. This is performed by using the repetition and the combination. The similarity of his claims with the ones of *Kant* concerning the constructiveness of the mathematical objects is, as we will see, very strong. Locke, consequently, defines the infinite to be this mathematical object produced by the unit with the procedure of the "endless" repetition. The main interest of his theory is that he tries to move the continuous ontology from that of time and space to that of the numbers.

The real break through in the continuum hypothesis comes up with *Leibniz*. He introduces many new types of problems on the ontology of the continuum. His results on mathematics (infinitesimal calculus) and his philosophical system are extremely interesting and remarkable. Leibniz is mainly influenced by Aristotle, but one can find in his work "Platonic elements" also.

According to him the present world is timeless. He is constructed from discrete entities, the *units*. These units do not communicate and do not interact. They lie under an internal procedure of representations' interchanging.

The natural (external) world is a sequence of events that are based upon the present world which is the best between all the possible worlds (according to Leibniz there are many possible worlds). For him, the world has a discrete structure because it is constructed by "units", but the space and time are infinitely divisible. This contradiction appearing between the construction of the universe and the possibility of its infinite divisibility is satisfactorily resolved in his philosophical system. An a priori principle is present throughout its work. This *"principle of continuity"* appears under several forms, the main of them being that the nature makes no jump and respects the continuity of what it creates. An other form is that the beings have some properties that are continuous functions of their characteristics.

Leibniz's reasoning for the principle of continuity is the following: "the supreme Being wants the best, which is the best ordered, that is the continuum. Thus, the principle of continuity is true" and, according to Leibniz, "the set of natural organic and inorganic beings forms a chain".

This principle, as well as the *infinitesimals*, are Leibniz's major contributions in the philosophy. In a much more elaborated way, in the "non-standard" analysis of *Robinson*, many of Leibniz's ideas can be found.

The claim of the chain gives to his theory a predictive component based exactly on the continuity principle. It predicts the discovery of other beings (in the chain) unknown until then. In a more modern mathematical metalanguage, we would say that "the chain of the beings is complete".

The infinite and the continuum are closely interdependent. The infinite, according to Leibniz, is potential but also real. This leads to a contradiction that could be summarized as follows: "on the one hand, he denies the existence of such a thing that really has infinite parts; on the other hand, he claims that the most little thing is a world containing an infinite number of beings".

Anapolitanos ([2]) proposes the following conjecture in order for this contradiction to be explained: "Leibniz seems to refute the real infinity of the mathematical continuum, as he considers it as a construction of the mind. Being such a construction, it pre-exists of its points and is potentially divisible and not really divised. On the other hand, he accepts the real infinity of the extended natural objects in the space and time".

Despite its obscurity, the mathematical component of Leibniz's world have largely contributed in the "mathematicalization" of the continuum. In fact, he is the first who gave concrete ideas in the ontologic investigations of both the infinite and continuum.

These two notions can also be found in *Kant*. In his theory we discover influences from Aristotle (potentiality of infinite) and from Leibniz (hypotheses on the chorochronic nature, the frame of which is changeless and constant). According Kant, infinite is *both real and potential*.

Kantian real infinite is a non constructible notion. This notion is opposed to the one of three-dimensional geometric objects for which there is a concrete constructive algorithm whose character, in Kant, is a priori as it is based in the "changeless" of our chorochronic frame.

Here appears the main difference between Kant's and Aristotle's programs. This difference concerns the ontologic status of real infinite. For Aristotle, such a notion can not exist in the frames of our conceptual universe. For Kant, on the contrary, real infinite is an element of this universe, where this notion is included as *not logically impossible even if its representation is a problematic*.

One of the similarities among the two programs is the acceptance of the potential infinite. In order for the relation between the two notions of infinite to be understandable, we will give an example in a temporary mathematical meta-language.

The existence of a constructive algorithm for the production of terms of a sequence of natural or conceptual objects does not imply the existence of this sequence as a real object. Sequence as a real object corresponds to the notion of real infinite, while the existence of the constructive algorithm corresponds exactly to the notion of potential infinite.

In a mathematical set theoretic meta-language we could, according to Kant's thought, define the class of all conceptual constructions as the domain of the notion of potential infinite.

The example given above shows exactly the difference that forces Kant to accept real infinite as a logical and internally consistent notion and potential infinite as a basic instrument for the overcome of non-infinite constructive procedures. Also, in Kant, the notion of infinite is closely related to the notion of continuum. For him, chorochronic frame is continuous in the sense that it is infinitely divisible. This divisibility is potential. Generally the Kantian analysis of the notion of continuum is misleading and it lacks of radical analyses and hypotheses as Aristotelian of Leibnizian philosophy.

After Kant the continuum program is separated in two main directions, the mathematical and the philosophical one. This tendency is not surprising since it is a component of a more general development of human thought that separates philosophy and mathematics.

In philosophy, where continuum is rather a gnosiologic principle than a mathematical construction, ontological researches for the nature of continuum are carried out. In mathematics, the mathematicalization of infinite and continuum allows the development of many instruments and notions, the infinitesimal calculus being the one dominating them. Infinitesimal calculus contributes to the enrichment of continuum's positive heuristic and the implication of its hypotheses not only in philosophical or mathematical programs, but also in natural sciences' ones. Continuum's program remain dominant and progressive, in all senses of progressiveness, ([7, 8, 10, 12]), up to the end of 19th century, as its progressiveness allows to scientists working in its frame, to neglect anomalies emanating from obscure hypotheses and definitions on the nature of infinite and continuum.

3.2 Infinitesimal Calculus. The Positive Heuristic of the Continuum's Program

For a long time the scientific community influenced by the Aristotle's program posed problems that contained the continuum hypothesis as a necessary and sufficient condition or as implied initial constraint (let us imagine what would happen if *Newton* studied the low of the gravity by supposing that the space has a discrete nature). The way the problems were posed and the philosophical tendencies and formations that determined the space of the action of the natural phenomena, needed the appropriate mathematical models in order firstly to describe and secondly to treat the problems.

In the case of scientists of the 17th, 18th and 19th centuries, the continuum hypothesis that dominated the philosophy, has obliged them to take this hypothesis into account in the conception of their mathematical world.

The first that has presented a work on the infinitesimal calculus was *Cavalieri* in his *Geometria Indivisibilibus Continuorum (Geometry of the indivisible continuous)*. According to him a line is produced by the move of a point, the level by the move of a line and the solids by levels overlapping. This primitive method has permitted him to produce results equivalent to integrations of polynomials. After him, a lot of scientists got interested in the completeness of this type of calculus.

Fermat, proposes a method for the determination of the maxima and minima of a function. This method consisted in the application of *slight changes* of the variable of the function and after, on the equalization of these changes with zero.

Fermat's method have been extended by *Hudde* on more general algebraic curves. They could so determine some *tangents*, *volumes of some solids* as well as some *centroids*.

Barrow in his turn has explained in a geometric way, difficult to be understandable, the duality *differentiation-integration*.

We thus see how the hypothesis of the continuum began to be a progressive program, even better, toncreate progressive programs. The aforementioned scientists adopt this hypothesis and find a way (naive at that time) of managing difficult notions as the "very small changes", etc. They do not care about the explanation and the refinement of these notions. The ability to compute quantities uncomputable under the hypothesis of discrete, is sufficient for them.

Pascal, by reasoning on the spirit of tightness more than on the geometric intuition, has proved that

$$(x + dx)(y + dy) - xy = xdy + ydx$$

Huygens conjectures the wavy emission of the light. This work is the first on physics based on the hypothesis of the continuous mean (space).

Newton with the method of *fluxions* is the first, with Leibniz, who handles in a systematic way the *infinitesimal calculus*. He generalizes this method by extending the application of the binomial theorem in the case where exponents are rational numbers or negative integers. The combination of the primitive method of fluxions with the extension of the binomial theorem has allowed the applicability of this method to a large class of algebraic and transcendental functions. Of course, the name of his method (fluxions), shows that Newton views the derivative mainly as velocity. This "erroneous" view has not prevented him from adding surprising results to the positive heuristic of the continuum. The main inconvenience of Newton's work was the absence of clarity concerning the explanation of his symbols. He does not explain, for example, what he means when writing *"infinitely small quantity"*. There is, in his work, an ambiguity concerning this term that could mean either zero or infinitesimals or finite small numbers. Also, his theory of the first and last ratios could not remove the ambiguities of his definitions. The time for a full and concrete removal of the obscurities concerning the infinitesimal calculus had not arrived yet.

Leibniz has worked on differential calculus in a geometrical way and not in a "kinematic" one and the terms *differential* and *integral* calculus, as well as the symbols used today are owed to him. However, the same ambiguities, as those found in Newton's work, have not been removed. It is because of these ambiguities that Marx ([14]) characterizes the differential calculus of these two scientists *mystic differential calculus*. Also, the systematic computation of the extreme of a function is due to Leibniz.

Bernoulli's family, extend the positive heuristic of the continuum's program by solving many problems, for instance problems on geodesic lines on surfaces, or on the motion of a point in a gravity-field. The development of the theory of differential equations with partial derivatives is a work performed by members of this family.

This theory permitted to Daniel Bernoulli to fund a program on hydrodynamics and to *D'Alembert* and *Euler* to study the theory of vibrating chords.

Therefore, we see that, even if all the details troubling the differential calculus have not been removed, the program of the continuum being progressive, it produces new results and permits the extension and implantation of pure mathematical theories in progressive non-directly mathematical programs. All these successful *problemshifts* corroborate the program of the continuum and marginalize temporarily the program of the discrete.

Euler, with his *Introductio in Analysis Infinitorum*, solves many trigonometric problems and introduces symbolisms used up to now. In "Introductio in Analysis Infinitorum", he presents infinite series for $\exp x$, $\sin x$, $\cos x$, as well as investigating the equation

$$\exp \imath x = \cos x + \imath \sin x$$

(investigated under other forms also by Johann Bernoulli). In the same work can also be found the ζ-function and its relation with the prime numbers.

The positive heuristic of continuum's program is enriched by the other works of Euler:

- *Institutiones Calculi Differentialis* and
- *Institutiones Calculi Integralis.*

In these works, we find also a theory of the differential equations with a classification in *linear*, *complete* and *homogeneous* ones, the formula of the *Euler's sum* and the *Eulerian integrals.*

In *Mechanica Sive Motus Scientia Analytice Exposita* and in *Theoria Motus Corporum Solidorum Sen Rigidorum*, Euler extends his mathematical methods to solve natural problems on physics and mechanics by enriching, in parallel, his mathematical work with the presentation of the Eulerian equations concerning the solids that revolve around a point. Finally, in another work entitled *Methodus Inveniendi Lineas Curvas Maxime Mininive Proprietate Gaudentes*, Euler solves major problems of the analytical geometry.

There is something remarkable in Euler. Apart from his contribution in the program of the continuum, he proposes a lot of problems-games that will contribute in the future to the enrichment of the positive heuristic of discrete. For example, the *seven bridges of Kőnixberg* is a famous problem, element of the positive heuristic of discrete, the *graph theory.*

Obviously, even with Euler, the obscurities regarding the differential calculus have not been removed, but, a least Euler, has tried to explain some obscure notions as, for example, the meaning of the infinitesimal changes. In order to remove the obscurity of the division ${}^{dy}\!/_{dx}$ seen at those times by many mathematicians as the division ${}^{0}\!/_{0}$, Euler introduces *types of zeros* and types of equalities, the *arithmetical equality* and the *geometrical* one.

In the arithmetical equality all the zeros are the *same object.* In the geometrical equality *two zeros are equal only if their "ratio" is equal to one.* With this diversification, Euler tries to remove problems on the exact nature of the infinitesimal variations. Even if with this alternative point of view of zeros, things have not been clear, Euler sees a critical point of the differential calculus. This point is exactly the merging of mathematical and metaphysical methods and thoughts. However, with this proposition, Euler opens a discussion, unfinished yet, on the definition of mathematical objects and the critical role of the particular definition in the solution of a problem that uses this object.

In any case, even with the works of Euler, the problem of the "mathematicalization" of infinite and continuum has not been resolved. Euler, despite his prestigious work, treats these two notions in an relatively ambiguous way, for example, his convergence criteria were not so correct. Because of that, he treats the divergent series in such a way that he obtains, many times, false results. Let us recall that under a Euler's reasoning the following result holds:

$$... + \frac{1}{n^2} + \frac{1}{n} + 1 + n + n^2 + ... = 0$$

However, despite the ambiguities and the obsurities in all the programs based on the continuum hypothesis, these programs were progressive and performed very successful problemshifts, producing new and very interesting results.

D'Alembert is the first that introduces the notion of the *limit* of a function. He defines a quantity to be the limit of an other quantity, if this second approaches very (infinitely) closely the first one, and this difference can become smaller than any other given quantity. He introduces also the notion of the infinitesimals of different orders. He also produces results on mechanics published in his book *Traité de Dynamique. In this book we find D'Alembert's principle which is a method for the reduction of the dynamics of solids in statics.* He studies problems of hydrodynamics, aerodynamics and publishes a theory for the vibrating chords. This work contains some interesting mathematical results on the partial differential equations. We see thus, that in parallel to the progress of the pure mathematical programs based on the continuum there is also a remarkable development of programs based on the same hypothesis but working on natural sciences.

Landen proposes the *ellipsoid integrals.* He also proposes new methods for the calculation of the derivatives. On the *complexe functions* he uses infinite series. His works are, in a certain way, precursors of *Lagrange's* algebraic method. The point of view of Landen on the computation of the derivatives appeared in his work, *Residual Analysis.* Even if he does not remove the obscurities on the notions of the "infinitely small changes", he is less metaphysical than Newton or Leibniz, reason for which, when Marx tries to explain the mechanism of the derivation ([14]), he is bases himself more on Landen, than on Newton or Leibniz. In any case, after those two scientists, the accumulated knowledge on the calculus of the continuous quantities has led to big progress. Even if the mathematicians could not take it into account (*there is no instantaneous rationality* says Lakatos in [12]), the accumulated results remove many ambiguities. Landen being a self-educated mathematician could describe a mathematical mechanism in simpler terms than an specialised one. We think that this is the reason that a materialist such as Marx found the hypotheses of Landen more unambiguous.

Despite of his enormous philosophical work, *Berkeley* did not contribute to the differential calculus. He has merged the mathematical formality with the theological metaphysics in order to prove that the results obtained by this calculus were simply a mutual shift of faults. Moreover, we have already seen that he was a philosopher largely accepting the discrete hypothesis.

Maclaurin, in his *Methodus lncrementorum*, presents his famous series, that describes the behaviour of a function under a "small" variation of its variable and is indeed based on Newton's binomial theorem.

Another scientist that referred to this series is *Taylor*. The difference between the two mathematicians is that Maclaurin controls also, in an elementary way, the convergence of those series.

If with Euler the differential calculus begins to find its final (actual) form, *Lagrange* contributes in the elementary notions and the basis of this calculus to begin to be treated in a strictly mathematical way. His objections on the way Euler presents this calculus lead him to write the *Analytic Calculus of the Variations*. This theory, as applied in natural problems, gave rise to the work entitled *Mecanique Analytique*. Although those results contributed to the enrichment of the positive heuristic of the Newtonian research program, on a mathematical level concerning the treatment of the whole problem of the differential calculus, Lagrange's optic is completely in opposition with the one of Newton. His other works, namely *Théorie des Fonctions Analytiques* and *Leçons sur le Calcul des Fonctions* fund the reduction of the differential calculus to the algebra. Lagrange also, while working on the series, neglects their convergence. Thus, the "algebrification" of the calculus has not given solid results. The main contributions of Lagrange consist in the clarification of the basic notions of the differential calculus and on the treatment of the notion of function in an abstract way. These contributions remain always valuable.

Laplace in his turn, contributes in the construction of mathematical models based on the continuum hypothesis, concerning a multiplicity of natural sciences as mechanics, electricity, astronomy. He is the most "faithful" scientist to the continuum hypothesis. We have already seen that all the previous scientists when confronted with the ambiguities of the mathematicalization of the continuum and of the infinite procedures, refuged either in metaphysical hypotheses (Newton, Berkeley, Leibniz) or in a certain "discretization" of the continuum. Laplace on the contrary neglects these "problems". As the continuum program is progressive and gives new results these mathematical-philosophical anomalies can be neglected. But even this, is not sufficient for him. He tries to "continualize" the discrete. Even if elements of the probability theory already existed in Bernoulli's and Legendre's works, Lagrange is the first that formally defines and uses the *generating functions* and demonstrates their use in the solution of the differential equations. The concept of the generating functions is the main example of the passage from the field of the discrete mathematics to this of the continuous ones. Even if Laplace presents his mathematical results in modelling natural problems, these results are very interesting by themselves. Let us only recall here, the pure mathematical value of *Laplacian Equations* as well as this of *Laplacian Transformations*.

Gauss extends the admission of the continuum in several fields. Between his main contributions in differential calculus is the systematic study of the convergence of infinite series. Also, the studies of curves of Gauss strengthen the continuum's program. With his imminent work on the complex numbers, he extends the continuum hypothesis on the plain. Recall that, up to Gauss, the conception of the continuum was limited to the real line. With the conception of the *elliptic functions* he is led in the conception of the non Euclidean geometries. But his abstention from the ideological conflicts prevented him from publishing his discoveries. All these mathematical advances and discoveries pushed Gauss to contest also Kant's refutation that the notion of space is a priori Euclidean. With his works on electricity and especially on the theory of the potential, Gauss enforces the hypothesis that universe is continuous and that the propagation of the electric current is performed in continuous mean. In his works on electricity, we also find purely mathematical results, as some principles for the *spatial integrals.* Between these principles we distinguish under different terms the principle of *Dirichlet.*

Legendre also contests in his turn, the Euclidean ideas in geometry. At the same time other scientists corroborate always the continuum hypothesis by producing surprising results on research programs of natural sciences.

Poncelet and *Coriolis*, explain in a geometric manner the *Mécanique Celeste* of Laplace, conjecture new physical lows and propose a better understanding of the nature.

Malus, demonstrates the polarity of the light and *Fresnel* reformulates the theory of the wavy transmission of the light.

Ampère, always in the frame imposed by the continuum hypothesis founds the *Electromagnetic Theory.*

Fourier, in *Théorie Analytique de la Chaleur*, studies the model of the *Thermal Conductivity.* In this book, except the explanation of this mechanism, Fourier founds also the theory of the integration of the partial differential equations with given boundary conditions by using trigonometric series. His main result in mathematics is the demonstration that an arbitrary function can be developed in terms of a trigonometric series of the form:

$$\sum_{n=0}^{\infty} A_n \cos nx + B_n \sin nx$$

We shall see later that he has also largely contributed to the clarification of the notion of the convergence of mathematical series and, more basically, of the notion of the function.

Cauchy has contributed in the theory of light and in mechanics. But his main success is due to his results in mathematical analysis. With Cauchy the theory of the complex functions have passed from the domain of hydrodynamics and aerodynamics to the domain of pure mathematics. In his *Memoire sur les Integrales Definies Prises Entre des Limites Imaginaires*, we find the computation of integrals through the residuals. His theorem, that every normal function can be developed, around a normal point z_0, to a convergent series in the interior of a cycle having as center z_0, largely contributes in the formalization of the notion of the convergence and the function continuity. A major problem, appeared with the research of Cauchy, is the conflict between the notion of function in Fourier's sense and in Cauchy's sense ([10]). This difference entails a different behaviour of the functions in the case of the Fourier's series and this of Cauchy's power series. Moreover, these questions created a contestation in the nature of the mathematical infinite and continuum. Today, we know very well that this conflict is due to the vague notion of the convergence as it appears in the works of both the two scientists. This conflict can be resolved by the adoption of an other kind of convergence, namely the *uniform convergence*, as this appears in *Riemann* and *Weierstrass*. Under this notion the behaviour of all the trigonometric and power series of a function becomes identical.

Here we can see a justification of the Lakatos point of view. Despite this critical anomaly, the program of the continuum has not been left by the scientists. It had already produced too many results, predicted new phenomena and posed new problems, so this anomaly is put in the margin of the program until another scientist could remove this anomaly by transforming it to a corroboration of the continuum program. We can also see that with the appearance of new results, many results and theories produced previously by adopting the same core of hypotheses, are led to falsification. For example Cauchy's results falsified many points of Laplace's Méchanique Celeste. This does not prevent the program of continuum to remain a progressive research program.

Dirichlet, works on the theory of analytic functions. He gives a formal proof for the convergence of Cauchy's series. With this proof, where we find the notion of the uniform convergence, he contributes in the correct understanding of the notion of function's nature. Also very well known is the Dirichlet's principle in the integral calculus.

Riemann, by working on the theory of complex functions, proposes the so called *conformal representation*. This work completes the formalization (initiated by Gauss) of space's continuity. He applies the previous results on the class of *hypergeometric* and *Abelian* functions. Concerning these last functions, he has also proposed a method for their classification.

We all have beared by *Rieman's Integral*. His works on the integral calculus are very advanced for his times. The proof that the functions defined by Fourier's series have infinitely many minima and maxima, changes the definition of the notion of a function. The notion of Euler's continuous function as a continuously drawn curve begins to be contested. Riemann gives an example of a function that is continuous but has not derivative. The meaning of this counterexample has not been understood as a critical anomaly of the continuum's program, until anomalies in programs of physics (*Michelson - Morley*) that could be explained by these counterexamples have risen up. Then, the critical meaning of this anomaly has been understood. But for the moment, continuum's program is very solid, so does the enthusiasm of scientists.

Weirstrass continues the work of Riemann on complex functions, by founding their theory via the use of power series. He uses also the method of Lagrange with an absolute mathematical formality. His main contribution in the domain of the differential and the integral calculus was the clarification of the notion of the limit, of the minimum of a function, of the derivative and of the function itself. His method of the $\epsilon - \delta$ definitions concerning the limits and the variations of the quantities, offers an explanation free of ambiguities and metaphysical hypotheses. Moreover, this method that clarifies the nature of the limit and the infinitely small quantities, is the precursor of the discretization of the continuum. This evolution is arranged among the successfull problemshifts brought out by Weistrass. Effectively, it happens the contrary. It is the beginning of the end of the imminence of the continuum's program. The scientists unable to understand and consequently to perform the complete mathematicalization of the continuum and of the infinite, take refuge to the safe domain of the discrete means. Weierstrass was the last mathematician that has worked in the heart of the differential calculus. The continuum's program has remained for more than three centuries progressive. The successful problemshifts, the concrete results that were produced, the predictive power of the program, have put in the margin the program of the discrete.

The differential calculus was more than a set of results that enriched the positive heuristic of the continuum's program. It was something like a "core" (not in the Lakatosian sense) for its positive heuristic, while in the same time constituted a principal element of the negative heuristic, in the sense that many times, anomalies appeared in the programs based on the continuum hypothesis, were prevented to falsify them. This was done because anomalies were explained by refereeing to the ambiguities of the hypotheses in the definitions regarding the nature of the infinitesimal calculus. Simultaneously, there is a substantial interaction between natural sciences' programs and mathematical ones, based on the continuum hypothesis. Never after the times of Greek antiquity and its Euclidean hypotheses, a mathematical program has lived so many centuries by performing so many progress as the differential calculus did.

We close this section by discussing how the founders of one of the most successful programs on *Socio-economics* examined the questions of the continuum, the infinite and the infinitesimal calculus.

We do that because, on the one hand, the mathematical programs influence on all the scientific programs having to do with the economy and in general, with the human behaviour and, on the other hand, because the program of Marx seems to be very well structured in Lakatosian terms.

What do we mean by this? This program has a *core* structured on some a priori conjectures concerning the nature of the organization of the civil society as well as the one of the economic laws. His *negative heuristic* has very well functioned for decades by explaining the enormous anomalies appeared in the program. Let us simply recall the contradiction between Marx's conjecture for the revolution in England or in Germany and the reality of the revolution in Russia and the explication of Lenin on this contradiction. Or, the conjectures of many occidental marxists on the failure of the communist system in Soviet Union ([1, 20, 21]). Moreover this program has remained progressive (in the Lakatosian sense) by explaining a lot of social and economical phenomena and by predicting many others. Finally we examine this program because as *Kastoriades* said: *On reflechira encore sur Marx lorsqu'on cherchera peniblement les noms de M.M. von Hayek and Friedman dans les dictionnaires*[1], (Journal *le Monde* 24/4/90).

Marx has worked on the differential calculus even if the relative notions of the continuum and infinite are introduced marginally in his program.

Engels says about the infinite ([14]): *We can know only the infinite. In fact any real and complete knowledge consists simply of reducing every isolated thing in our mind, from the particular to the partial and from this last to the total. We search and we define the infinite through the finite, the eternal through the momentary. How-*

[1] *We will continue to think on Marx when we will search in vain the names of von Hayek and Friedman in the dictionnaries*

ever, the form of the totality is the form of the self completeness and consequently the form of infinite; it is the perception of many "finites" in infinite.

We can see that, in his manner, Engels defines informally infinite in such a way that can be considered almost equivalent to the classical definition of the *induction* in *Logic Positivism.* This notion of infinite allows suppose that for Marx and Engels the universe has a discrete nature. In any case, this is a logical hypothesis if we take into account that Marx opposed to the *Hegelian spirit* and *infinite idea* as well as the to Hegelian conjecture on the *infinity of the human experience as the major form of the infinite motion of matter (substance)*, which refers to the Aristotelian potential infinite.

For Marx the evolution of differential calculus is divided in three periods.

- The period of *mystic differential calculus* of Newton and Leibniz.
- The period of *rational differential calculus* of Euler and D'Alembert.
- The period of *pure algebraic calculus* of Lagrange.

For the first period, Marx sees more the metaphysical hypotheses on the notion of infinitesimals than the theory that begins to be developped. We think that his thesis is due to the absence of an algebraic foundation of the theory of differential calculus defined by Newton and Leibniz. It is true that the infinite variations, particularly their value and nature (are they or not equal to zero?) are not formally defined in the works of the above scientists. But Marx, being opposite to all hypotheses that had metaphysical origins, ignores the power of this type of calculus. For him, the most obscure point in the procedure of differentiation is the "division" ${}^{dy}\!/_{dx}$ seen as a division ${}^{0}\!/_{0}$. Euler, by having explained more satisfactorily (even not formally) the nature of variations by introducing the "types of zeros", allows Marx to discover a rationality in his works. The other point that was hardly explicable for Marx, was the "algorithmicity" of the differantiation method which is due to his conception of a discrete world. Marx's proposition for the differentiation procedure is the following ([14]):

- We take $x_1 \neq x$ and we form the difference $f(x_1) - f(x)$. Let us denote $F(x, x_1) = f(x_1) - f(x)$. Of course F is a 2-variable function. We express $F(x, x_1)$ as $F(x, x_1) = (x - x_1)G(x, x_1)$. In function G we take $x_1 = x$. This quantity $G(x, x)$ is $f'(x)$. In this way we cancel all the difficulties emanating from our game with infinitely small quantities.

A mathematical remark for the above procedure is that Marx does not consider the continuity of G in the neighbourhood of x as a sufficient condition for the differentiation of f. Another remark also is that he introduces, in his own terms, a notion of discretization of the continuum by capturing in his proper informal way, the the meaning of "infinite variation". This way is intuitively very similar to the one of Weierstrass.

Marx sees the world under a discrete form. Moreover, most of his remarks on *Lacroix's* works were carried on the conjecture of the latter that the form $\varphi(x_0) = {}^{f(x_0)}\!/_{g(x_0)}$ was mathematically legal even in the case where $f(x_0) = g(x_0) = 0$. Of course today this conjecture is seen as a consequence of the continuity on the line of the real numbers, however Marx found it metaphysical, hence rejectable. Anyway, the critics of Marx on differential calculus and on continuum shows how this program with his rich positive heuristic has influenced non-mathematical theories.

4 Continuum and Discrete in 20th Century

The main mathematical research programs in 20th century are *Logical Positivism* with main representatives *Russel* and *Frege*, *Formalism* with *Hilbert* and *Intutionism.*

Logical Positivism has no explicitely viewed itself in the frame, neither of continuum, nor of discrete. But Russel's philosophical influences have an important Aristotelian and Leibnizian origin. We think that it would not be wrong to say that Russel's overall (and general) view of mathematical universe was closer to continuum's than to discrete's program.

Hilbert by founding Formalism, has tried to model a philosophical approach for mathematical activity called *Peratokratism.* The core of his program is structured on two theses:

- Mathematical activity must be continuous and without limits imposed by ontologic views concerning the nature of real infinite.
- The safe supporting of mathematical creativity has to be the main query of every effort for an overall philosophical view of this creativity.

His basic ideas on infinite and discrete are included in his article *Űber das Unendliche* dedicated to Weierstrass ([2]). In this article, Hilbert shows his preferences in Weierstrass' point of view in what concerns discretization of continuum, that is the introduction of $\epsilon - \delta$ techniques in defining the notion of the limit and the treatment of the continuum as a sequence of "very small" discrete quantities. According to Hilbert, two are the sources for searching for an ontologic justification of the notion of infinite: the *nature* and the *intellection.*

Concerning nature, Hilbert looks for support in programs of physics, where, the dominating theories contest (even today) the infinite divisibility of an extended object, given that Inductivism, which is the dominating principle, has a non continuous character. This means that, when some anomaly appears to a physical program, one searches to explain it, by supposing the existence of new elementary particles in order to constitute a new non-inducible material reality. On the other hand, the temporal theories suppose the universe to be finite (the notion of a Euclidean universe was not prevalent even in Hilbert's ages). It is for these reasons that the nature (for Hilbert) does not justify the existence of real infinite.

Essentially by using as an analogical model, the model of the real line founded by Weierstrass, Hilbert tends discretization of the continuum by liberating it from "ontologically doubtful" notions (the notion of real infinite).

Main thesis of Hilbert's program is that *the set of natural numbers provided with the class of elementary recursive functions, as well as with the possibility of using the axiomatic schema of Peano's arithmetic without quantifiers ([3]), is exactly the intuitively safe domain upon which all the rest of mathematics can be based.*

Hilbert's programm has been falsified, the critical experiment being *Gődel's incompleteness theorems* ([3]). In fact, *Gődel* has proved the existence of an *undecidable* well formed formula in *Peano's* arithmetic. In other words, he has constructed such a formula that neither it, nor its negation are theorems deduced by Peano's axioms. It is well known the method of *Gődel's numbers* with which he "arithmeticalizes" the syntactic part of the arithmetic language.

Certainly, there was not only the "by default" adoption of discrete hypothesis that was problematic in Hilbert's program. We know today that if the formal system of *Zermelo-Fraenkel set theory* contains an undecidable proposition, then the addition in the system of the continuum hypothesis creates a new formal system that contains always an undecidable proposition.

But Gődel's theorems instead of constituting a critical experiment for the whole program of discrete, has constituted on the contrary a regenerative factor for this program as we will see in the next section.

The *Intuitionism*, in its turn, contains some contradictions in what concerns the dual discrete-continuum. On the one hand it adopts as prevalent, for the acceptance of a mathematical object, the existence of a property that constitutes a constructive algorithm for this object, adopting therefore a discrete mathematical universe and, on the other hand, it accepts the totality of real numbers by accepting the existence of the real line even if there is no algorithm that constructs this line. In any case, because of the nature of Intuitionist's program, the ontologic problems on discrete and continuum are not posed in a so acute manner as in the frame of the classical mathematics.

5 The Discrete and the Information Technology

Today the program of discrete predominates with respect to the one of continuum and this, we think, is due to the development of the *information technology* based upon the *mathematicalization of computing models*, *algorithms*, *Markov chains*, etc. Of course, this evolution is not the only reason for the relative degeneration of the continuum which, in fact, has never resolved a number of problems posed on the nature of the hypotheses adopted.

Already, from the time of Weierstrass, the only way to establish some explanations concerning the obscurities of infinitesimal calculus, was the discretization of the continuum. Also in the middle of 19th century *Kronecker* has worked on the domain of *number theory*. Moreover, he has treated mathematics in such a way that a prevalence of natural numbers on every other mathematical object was obvious. He accepted the definition of such an object only if it is algorithmically (hence finitely and discretely) producible.

Moreover, even if Dedekind accepted the real infinite and continuum, his works used Eudoxean elements. Cantor in his turn, by trying to "mathematicalize" the infinite and the continuum, has created an *arithmetic of hyper-finite numbers* with many characteristics in common with the well known discrete arithmetic. Finally, all these scientists, have tried to justify the continuum hypothesis partially based upon discrete's principles.

Imminent among those "strikes", Gődel's incompleteness theorem ([15]). There is something interest with his results. Although they are supposed to refute Hilbert's conjectures, which accept a certain prevalence of discrete's principles, they turn finally against continuum. In Gődel's proof the formal definition of the notion of algorithm is introduced and it is exactly this notion that constitutes the *key-notion* of the contemporary discrete mathematical models. Of course the "resurrection" of discrete has not came under a form of justification of Euclidean principles. The *non-Euclidean geometries* are well known today.

The main mathematical instrument of discrete is, today, the so called *discrete mathematics* with their *enumeration methods*, *combinatorial analysis*, *graph theory*, *languages and finite state machines*, that operate on *discrete mathematical structures.*

In the *theory of algorithms and computation* ([5, 16]), a problem is a general question to be answered, usually possessing several parameters or free variables, whose values are left unspecified. A problem Π can be formally described by giving:

1. a general description of all its parameters and
2. a statement of what properties the answer (solution) is required to satisfy.

An instance I of Π is obtained by specifying particular values for all problem's parameters.

An algorithm is supposed to solve Π, if it can be applied to any instance of Π and if it guarantees always the construction of a solution for this instance. Thus, the first condition that characterizes the good "behaviour" of an algorithm is the *convergence* of the method, that is its ability in providing solutions for every instance of a given problem in finite time. The study of this behaviour uses exclusively elements of combinatorial analysis, set theory, graph theory and enumeration techniques, all these instruments operating under the hypothesis of a discrete universe.

Turing and *Church* conjecture that *the class of problems that are algorithmically solvable, is exactly the class of partial recursive functions*[2]. In other words, some problems are so hard, that are not solvable by algorithmic methods (*undecidable problems*).

This is the critical notion of Gődel's incompleteness proofs. In fact Gődel proves the undecidability of at least one well formed formula of *first order logic.*

The quality of an algorithm is characterized not only by its convergence but also by its *efficiency.* This term reflects the time requirements for the convergence and are conveniently expressed in terms of a single variable, the *size* of a problem's instance, which is intended to reflect the amount of input data needed to describe this instance, whose description can be viewed as a finite string from a *finite* alphabet. We thus have a particular encoding scheme that maps the natural world (instances of the problem) to strings describing this world.

The length of an instance for a given problem is defined as the number of distinct alphabet symbols used for the description of this instance. The time complexity function of an algorithm expresses exactly its time requirements, by giving for every input length the greater amount of time required by the algorithm solving an instance of this length, and this function is well defined when given:

[2]This is the famous *Turing-Church Thesis.*

1. the encoding schema used to define the input length and

2. the computational model used to determine the execution time.

It is clear that all these definitions refer to discrete mathematical entities.

In general, data are discrete objects represented, for example, by the vertices of some graph, or by the members of an (extensively or intensively) defined finite, or at least enumerable, set. Thus, the manipulation and analysis of these objects is based on discrete mathematical models. Moreover, every individual datum is treated as a self-completed entity and it is supposed to be processed in a time period expressed by a natural number.

In fact, there is also some other intuition behind the use of discrete mathematics in the frame of computer technology. Computer is a machine constructed to operate in discrete steps (like all machines up to now). Electrical signals, interpreted to information of any kind, are of discrete level. Thus, the conception of any model that controls computer's operation or efficiency of computation's method, has to be based on discrete mathematical entities. Moreover, the communication of a computing machine with either the human environment, or with other computing machines is expressed and controlled by discrete models as Markov chains, finite state machines, etc.

According to Turing-Church thesis ([3, 13]), the mathematical equivalent of an algorithm is the class of partial recursive functions whose domain is $I\!N$ (the set of natural numbers), the model for mathematicalization of discrete. On the other hand the definition of the notion of algorithm (expressed in the beginning of 20th century) is to be *a "machine" for solving equations (problems more generally) proceeding by succesive elementary discrete steps*. Thus, even by its definition, the notion of discrete is present in the notion of algorithm. Moreover, use of any kind of mathematical objects referring to the so-called continuous mathematics, has not led, until now, in the conception of reliable computational models. Everything concerning computer's operation is based on the "arithmeticalization" of every kind of data concerning either *hardware* (electrical signals and their level, memory units, etc.) or *software*.

Moreover, in what concerns numerical problems carried on continuous mathematics, numerous between them can satisfactorily been solved into the discrete mathematical universe. Let us take, for example, numerical methods as *Runge Cuta's method*, or the *method of Simpson*, or more generally all the *finite elements techniques*. All these methods solve equations with functions defined on $I\!R$ (Runge Cuta's method) or evaluate integrals (method of Simpson) or solve differential equations. These mathematical problems are classically considered as continuous mathematics' problems, but can be solved by techniques that "discretize" continuum. In general, there are numerous methods that solve problems of continuous mathematics in a discrete environment. Those techniques are step-by-step techniques approximating discretely the solution under research. Here is a case where problems lying in the negative or even in the positive heuristic of continuum are coming to be added to the positive heuristic of discrete.

We see thus how, for example, the discretization of the continuum has advanced more than the $\epsilon - \delta$ definitions of Weierstrass or the hyperfinite numbers of Dedekind.

On the contrary the "continualization" of the discrete, the only "counter-attack" that has to demonstrate is the use of *generating functions* in the theory of *arithmetic functions*. Moreover, the major theoretical concept of information technology: the study of algorithm's efficiency and the completeness of algorithmic problems, is closely related to nature and size of input data. The main instruments used in modelling this concepts, are borrowed from graph theory and combinatorial methods and operate on the hypothesis of a discrete universe. This is because even the notions of datum, or of size input, are discrete natural or mathematical entities that can be represented by means of graph theory, or of strings of languages over an alphabet, or of array's or list's elements. Obviously, the techniques of elaborating such entities (for example enumeration techniques, etc.) come from the domain of discrete mathematics.

The mathematical principles of enumerable sets, constitute the core of discrete program; the information technology, that is a major part of its positive heuristic, assumes discrete mathematical models, operating in a discrete universe.

Hence, discrete is a progressive program. It has performed many progressive problemshifts which have resolved numerous open problems and, on the other hand, *pose* many other problems and *open* many research domains. Let us simply think that a simple core of a priori hypotheses on discrete universe have bolstered a scientific revolution whose impacts have changed not only the human development but also the the social structure of the world ([23]). The intelligent cells of *von Neuman* have been transformed today to *supercomputers* and *artificial neural networks*. The simple idea of a decidable problem in Gődel's proofs, has open the domain of the *complexity theory* (*NP-completeness*, *polynomial approximation*, etc.), of the *parallel computation*, etc. The "games" of Euler and Hamilton are today famous problems and have contributed to the development of the powerful domain of graph theory. The essays on probabilities of Fermat have became enumeration techniques on discrete structures and *random graph theory*.

We think that the previous thoughts explain, in a certain way, the break through on the research on discrete mathematics, that implies the re-appearance of discrete's program. This arrives after many centuries of continuum's scientific prevalence. Of course, claiming that information technology is the crucial experiment for the continuum, is either erroneous or at least pre-mature. Although during centuries continuum's prevalence was almost total, discrete was not a "dead" scientific program, it was simply in crisis, possibly acute but not fatal. Perhaps this is exactly the case of the continuum.

We have to keep in mind that discrete has not satisfactorily answered to problems as this of infinite or continuum. But it behaves as every (temporarily) successful program. It borrows from the continuum, notions (as for example the notion of limit) defined in the proper frame of its opponent. Let us recall that the same questions (via the duality) were the crucial points of continuum's degeneration.

6 Discussion

We have tried to examine the history of the evolution of the continuum and discrete hypotheses in mathematics. We have seen that the limits of progression or degeneration of these programs are hardly discriminated because of their nature and of the non formal and concrete definition of the notion of critical experiment in mathematics. There were always an interaction between mathematics and natural sciences, in the sense that: mathematical models were applied on natural science's problems in order, on the one hand, to produce results for these problems and, on the other hand, in order that these mathematical models be tested. This means that if we have an instrument to decide the progress or the degeneration of a mathematical program, this instrument is provided by the association of this program with the corresponding natural science programs based on it. This association is not straightforward. There is a multiplicity of "natural" programs, based upon a given mathematical program and, moreover, some of those programs become degenerated while other ones remain progressive. Here is exactly the strong point of Lakatos' approach on mathematical theories, since the "historicity" on the evolution of those theories is much more clear and evident than the one of "natural" theories.

When dealing with mathematics, in order to associate the term program to each distinct theory, it seems better to associate it to sets of theories. In fact throughout the paper we have applied the following constructing schema:

> A mathematical program is a set of mathematical theories.
> The core of the program is the intersection of the a priori hypotheses of the theories deriving this program.
> Its positive (negative) heuristic is the union of the positive (negative) heuristics of the deriving theories.

It is easy to see that the above construction, applied to the set of theories presented above, implies the a priory hypotheses either of discrete or of continuum mathematical (and not only) universe.

Thus, the discrete, for example, is the essential matter in the core of all theories up to Aristotle. This phenomenon is quite reasonable. The objects of the surrounding world are discrete and finite and hence all conjectures in the "nature of things" are based in this view of the world. The notion of infinite that produces effectively the continuum is not directly supervised.

Pythagorians, Euclides, Plato and Archimedes give each one in his domain, explications or conjectures concerning phenomena of their known and limited universe and moreover they foresee a lot of new phenomena. Even after the degeneration of Pythagorian theories, the ones of Euclides or Archimedes remain progressive for centuries.

With Aristotle, human mind and vision become deeper and more complex. Notions more abstract and not directly supervised, enter in the conception of the universe. This abstraction leads to the foundation of another program based on the continuum hypothesis. But even with the foundation of this new program that can predict and explain more new phenomena, discrete does not disappear. On the contrary, even if continuum is more powerful, discrete remains more "popular" among the members of the scientific community up to Alexandrian times, this fact having as consequences that continuum program appears to be almost inactive. But the accumulated scientific progress of discrete, accumulates also open problems and unexplained phenomena.

Moreover, Christian dogma, full of metaphysical and obscure hypotheses, politically strong and socially popular, changes the beautiful and symmetric conception of discrete universe, with its finite mathematical objects and its Gods who constituted almost a partition on human activities. For Christian ontology, continuum is more convenient. For mathematics, the theory of *irrational* and *asymmetric* numbers destroys the conviction for the universal constructibility by using the rule and the pair of compasses.

The hypothesis on continuous universe, obliges the scientists to use new mathematical instruments containing and using notions of continuum and infinite. The main impact of the continuum hypothesis, the infinitesimal calculus, add more and more new results, solves open problems and predicts new phenomena. During this evolution, discrete passes in a second plan but does not disappear.

On the contrary, it exercises a strong influence in the conception of continuum. The Euclidean, Archimedean and Democritean worlds are present in the most of hypotheses made about the nature of continuous universe. Let us remember that the structure of Democritean microcosm is omnipresent even in programs that refute this structure. Let us also remember that the negative heuristic of Cauhy's program has used the "bad Euclidean influence" in order to explain the anomalies of this program indicated by Fourier. In fact, the introduction of the notion of "uniform convergence" is the negation of some Euclidean conjectures but, almost in the same time, Euler and, later, Weirstrass introduce some other Euclidean hypotheses in order to explain the "infinitely small changes".

After Middle Ages, begins a very strong interaction between mathematical and natural programs. The results of natural programs are directly added in the positive heuristic of the corresponding mathematical programs. This, as we have seen, has constituted the triumph but also the "calamity" of the continuum.

The main problem of discrete program were that has largely been based upon the constructiveness of objects. This main anomaly, constitutes his strong point today, where the constructiveness of an object is the main measure of its value. On the contrary, in continuum's program there is a lot of abstraction. The "game" is played more on formalism and "intuition" than on constructiveness and "algorithmicity".

The hypotheses of the continuous world have given satisfactory explications in mechanical problems as well as in problems concerning the outer-earth universe and the "visible" world, but failed out in explaining the microcosm. There, discrete program seems to be more successful and progressive. Theories of quantization of energy like these of *De Broglie*'s or *Plunck*'s hypotheses on the existence of new particles in the microcosm, even if those particles are not directly observable, support a discrete notion of the universe. In those scientific streams, we have to add the new convictions and claims on the finiteness of the universe, as well as an external factor, the vertiginous development of the information technology based in discrete mathematical models, discrete (of course) materials and discrete conception of the world (let us think that the only primitive information and notion in the whole conception of information technology are the quantities 0,1).

We can speak today for temporal "victory" of the discrete program, as also temporal was the prevalence of the continuum one. The two rivals remain in the field of the scientific war. Only the history can nominate the winner and the "final countdown" is not yet played. Contrary to the current opinion of some naive "scientists", History has not come to an end.

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