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Polyhedral Study for the Capacitated Multi-Layer Network Design with Unsplittable Demands

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Abstract

We consider the *Capacitated Multi-Layer Network Design with Unsplittable demands* (CMLND-U) problem. Given a two-layer network and a set of traffic demands, this problem consists in installing minimum cost capacities on the upper layer so that each demand is routed along a unique "virtual" path (even using a *unique* capacity on each link) in this layer, and each installed capacity is in turn associated a "physical" path in the lower layer. This particular hierarchical and unsplittable requirement for routing arises in the design of optical networks, including optical OFDM based networks. In this paper, we give an ILP formulation to the CMLND-U problem and we take advantage of its sub-problems to provide a partial characterization of the CMLND-U polytope including several families of facets. Based on this polyhedral study, we develop a branch-and-cut algorithm for the problem and show its effectiveness though a set of experiments, conducted on SNDlib-derived instances and also on real instances.

Keywords: Multi-layer network design, optical networks, polytope, facet, branch-and-cut.

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1. Introduction

User demand in traffic has increased significantly during the last decades. Nowadays telecommunication networks are already reaching their limits, and it is necessary to upgrade their transport capacity. Indeed, the advent of new services, mainly driven by internet applications and multimedia contents, requires more flexible and cost-effective network infrastructures. To overcome this explosive growth of traffic (estimated at 45 % per year in average [26]), telecommunication industry actors investigate new technologies that provide a solution to the increasing capacity requirements, as well as the flexibility needed to use smartly this capacity.

Telecommunication networks can be seen as an overlapping of multiple layers, upon which different services may be furnished. In particular, optical fibers networks consist of two layers: a physical layer and a virtual one. The physical layer is based on optical fibers, while the virtual one supports the WDM (Wavelength Division Multiplexing) technology. Such a process is based

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on a set of devices referred to as multiplexers, interconnected by optical links, made of several wavelengths. Both layers are connected, as the wavelengths of the virtual layer use the optical fibers of the physical layer as a support to carry the customers traffic.

Although WDM technology is currently used to transport informations over long distances (metropolitan areas, submarine communication cables), with wavelength capacities of 2.5, 10 or 40Gb/s, it is not possible to reach similar distances with higher capacities. In fact, the existence of physical phenomena, also called transmission impairments [18], that affect the optical fibers, highlights the difficulty of setting up higher capacitated wavelengths on long distances. Recent innovations in optical fibers comunications concerning a new technology called Multi-band Orthogonal Frequency Division Multiplexing (OFDM) have shown very promising results, and should enable the transition of WDM-based infrastructures to high capacitated wavelengths (100 Gb/s and more) over long distances. OFDM is based on the division of each available wavelength into many subwavelengths, also called subbands, this is known as *Optical Multi-band OFDM network*.

Now consider an optical multi-band OFDM network that consists of an OFDM/WDM network over a fiber layer. The OFDM/WDM layer is called *virtual layer* and the fiber layer is called *physical layer* as well. The OFDM/WDM layer is composed of devices called *Reconfigurable Optical Add-Drop Multiplexers* (ROADM), which are interconnected by virtual link. A virtual link may receive one or many OFDM subbands. Note that, although a subband is said to be installed over a virtual link, it is in fact generated by a pair of ROADMs at the extremities of the link. The physical layer is composed of several transmission nodes interconnected by physical links. Each physical link contains two optical fibers, so that the traffic can be transported in both directions. The physical and virtual layers are communicating via an interface referred to as OEO (Optical-Electrical-Optical) interface.

Each ROADM in the virtual layer is associated with a transmission node in the physical layer. And every link in the virtual layer carries one or several subbands. We suppose that there exists a link between each pair of ROADMs in the virtual layer, as one or many subbands may eventually be installed between any pair of devices. Each subband installed over a virtual link is assigned a path in the physical layer. A link in the physical layer can be assigned to several different subbands. However, due to technical aspects of OFDM technology, a physical link can be assigned at most once to an installed subband. In practice, one or many ROADMs may be installed upon a transmission node. However, we assume that all the subbands installed over a virtual link are produced by a unique pair of ROADMs, set up on the extremities of this link. In addition, establishing a subband yields a certain cost, which is the cost of ROADMs that generate this subband. We assume that we have a traffic matrix, where each entry is a point-to-point traffic demand that may correspond to a given service, internet application or a multimedia content. This traffic demand has a value that is an amount of informations measured in Mb/s or Gb/s.

Figure 1 shows a bilayer network. The virtual layer includes four ROADMs denoted R_1, R_2, R_3 and R_4, while physical layer contains six transmission nodes denoted T_1 to T_6. We can see that R_1, R_2, R_3 and R_4 are connected to T_1, T_2, T_3 and T_4 via OEO interfaces. In addition, there exists a link between each pair of installed ROADMs. Remark that nodes R_5 and R_6 have not been represented in the figure, as they do not carry any ROADM. Furthermore, three subbands are represented in the figure, respectively installed on the links (R_1, R_2), (R_1, R_4) and (R_3, R_4). The traffic using these virtual links is in fact transmitted through paths made of optical fibres in

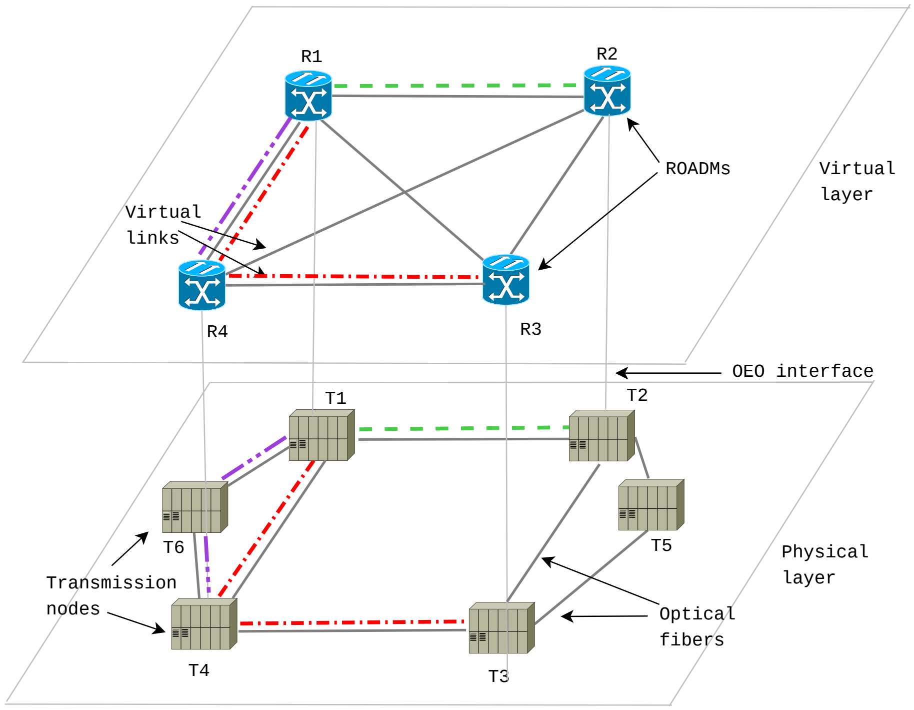

Figure 1: Example of multilayer network

the physical layer. Indeed, the link (R_1, R_2) is associated with the path (T_1, T_2), while (R_1, R_4) is assigned the path (T_1, T_6), (T_6, T_4) and (R_4, R_3) is physically routed by (T_4, T_3). It should be pointed out that there are two levels of routing in such networks. The traffic is routed using subbands installed on the virtual links, and the subbands themselves may be seen as demands for the physical layer. **Both the traffic and subbands may possibly be routed through multi-hop paths**. Thus, when given those two layers of network and a traffic matrix, one may determine the set of virtual links that will receive the subbands, and the set of physical links involved in the routing of those subbands, and establish the traffic commodities routing.

In this context, we are interested in a problem related to the design of OFDM/WDM networks. Thereby, assume that we are given an optical fiber layer, an OFDM/WDM layer and a traffic matrix. The *Capacitated Multi-Layer Network Design with Unsplittable Demands* (CMLND-U) problem consists in determining the number of subbands to be installed over the virtual links, and their physical path as well, so that the traffic can be routed on the virtual layer and the cost of the design is minimum. This work was initially motivated by a collaboration with Orange Labs, whose engineers are also interested in evaluating the performances of OFDM-based networks. For this reason, and throughout the paper, we will use this context to explain our model and the results we will provide.

Actually, the problem of designing layered networks have been studied first by [20]. Authors wish to set up a set of virtual links referred to as ”pipes” on the physical layer. They propose an integer linear programming formulation based on cut constraints for the problem. They study the associated polytope and provide several classes of valid inequalities that define facets under some conditions. They also provide a cutting planes based algorithm embedding their theoretical results. Further works consider exact methods for different variants of the multilayer network design. In fact, in [30], Orlowski et *al.* propose a cutting planes approach for solving two-layer network design problem, using different MIP-based heuristic allowing to find good solutions early in the Branch-and-Cut tree. Belotti et *al.* [9] investigate the design of multilayer networks using MPLS[2] technology. They propose a mathematical programming formulation based on paths, then apply a Lagrangian relaxation working with a column generation procedure to solve their model. We also cite a more recent work of Raghavan and Stanojević [34] that studies the two-layer network design arising in WDM optical networks. The authors consider non-splittable traffic demands and propose a path based formulation for the problem. They provide an exact Branch-and-Price algorithm which solves simultaneously the WDM topology design and the traffic routing subproblems. In [31], the authors address the problem of planning multilayer SDH[3]/WDM networks. They consider the minimum cost installation of link and node hardware for both layers, under various practical constraints such as heterogeneity of traffic bit-rates, node capacities and survivability issues. They propose a mixed integer programming formulation and develop a Branch-and-Cut algorithm using non-trivial valid inequalities, from the single-layer network design problem, to solve it. In [21], the authors study the multi-layer network design problem. They propose a Branch-and-Cut algorithm to solve a capacity formulation based on the so-called metric inequalities, enhancing the results obtained in [25] for the same formulation. In [28], Mattia studies two versions of the two-layer network design problem. In particular, the author proposes capacity formulations for both versions and investigates the associated polyhedra. Some polyhedral results are provided for both versions of the problem, specifically proving that the so-called tight metric inequalities, introduced in [6], define all the facets of the considered polyhedra. The author also shows how to extend these polyhedral results to an arbitrary number of layers. In [16], Borne et *al.* study the problem of designing an IP-over-WDM network with survivability against failures of the links. They conduct a polyhedral study of the problem, give several facet defining valid inequalities, and propose a Branch-and-Cut algorithm to solve the problem. Finally, the CMLND-U is studied in [12] where the authors propose a flow-based formulation for the problem. They further propose two path-based formulations leading to a branch-and-price approach to solve it.

Our contribution

The capacitated design of single-layer networks has received a lot of attention in the literature, and a big amount of research has been conducted on the associated polyhedron (see [17], [5], [4] and the references therein). Yet the investigation of capacitated multilayer network design **with unsplittable demands** problems received only a limited attention, specifically in a polyhedral point of view. The objective of this paper is to investigate the CMLND-U problem within a polyhedral framework, and to provide an efficient Branch-and-Cut algorithm to solve it. In this context, we give an integer linear programming formulation for the problem and study the polyhedron associated with its solutions. We then introduce further classes of valid inequalities and study their facial structure. These inequalities are used within an efficient Branch-and-Cut algorithm for the CMLND-U problem.

[2]MultiProtocol Label Switching.

[3]Synchronous Digital Hierarchy.

The rest of the paper is organized as follows. In Section 2, we describe the CMLND-U problem in terms of graphs and give an ILP formulation to model it. In Section 3, we present the CMLND-U polyhedron and study its basic properties. We then introduce several classes of facet-defining valid inequalities. These results are used to devise a Branch-and-Cut algorithm which is described in Section 4. Several series of experiments are conducted and Section 5 is devoted to present a summary of the obtained numerical results. Finally, we give some concluding remarks in Section 6.

2. The capacitated multi-layer network design problem with unsplittable demands

2.1. Definition and notations

In terms of graphs, the CMLND-U problem can be presented as follows. We associate with the virtual layer, a directed graph $G_1 = (V_1, A_1)$. G_1 is a complete graph where V_1 is the set of nodes and A_1 the set of arcs. Each node $v \in V_1$ corresponds to a ROADM and each arc $e \in A_1$ corresponds to a virtual link between a pair of ROADMs. In addition, G_1 is a bi-directed graph, i.e. there exists two arcs $(u, v) \in A_1$ and $(v, u) \in A_1$, connecting each pair of nodes u and v of V_1. Consider the directed graph $G_2 = (V_2, A_2)$ that represents the physical layer of the optical network. V_2 denotes the set of nodes and A_2 the set of arcs. Each node $v' \in V_2$ corresponds to a transmission node and each arc $a \in A_2$ corresponds to an optical fibre. Every node u in V_1 has its corresponding node u' in V_2. The graph G_2 is such that if there exists a link between two nodes u' and v' of V_2, then, both arcs (u', v') and (v', u') are in A_2. In this way, the link can be used in both directions between u' and v'.

Suppose that we have $n \in \mathbb{Z}^+$ available subbands. We denote by $W = \{1, 2, ..., n\}$, the set of indices associated with these subbands. Every subband $w \in W$ has a certain capacity C and a cost $c(w) > 0$. Moreover, a subband installed over an arc $e \in A_1$ can be seen as a copy of this arc. Each pair (e, w) such that w is installed over the arc $e = (u, v)$, is associated with a path in G_2 connecting nodes u' and v'. The same path in G_2 may be assigned to different subbands of W. Nevertheless, an arc $a \in A_2$ can be associated at most once with a given subband w. In other words, if the subband w is installed p times, $p \in \mathbb{Z}^+$, over different arcs $e_1, \ldots, e_p$ of A_1, then the pairs (e_i, w), $i = 1, \ldots, p$, have to be assigned p paths in G_2 that are arc-disjoint. This comes from an engineering restriction and will be called disjunction constraint. In addition to the design cost, we will also attribute a physical routing cost denoted $b^{ew}(a)$ for every arc a of A_2 involved in the routing of a pair (e, w) such that w is installed on e.

Now let K be a set of commodities in G_1. Each commodity $k \in K$ has an origin node $o_k \in V_1$, a destination node $d_k \in V_1$ and a traffic value $D^k > 0$. We suppose, that $D^k \leq C$, for all $k \in K$. Note that there might exist different commodities with the same origin and destination. A routing path in G_1 has to be assigned to each commodity $k \in K$ connecting its origin and its destination. Every section of a routing path uses the subbands installed over the arcs of A_1. Thereby, we will say that a pair (e, w), $e \in A_1$, $w \in W$ is used by a commodity k, if w is installed on e and (e, w) is involved in the routing of k. Furthermore, several commodities are allowed to use the same subband (e, w), if they fit in its capacity. However, one commodity can not be split into several subbands or several paths.

Definition 1. *Capacitated Multi-Layer Network Design with Unsplittable demands (CMLND-U) problem: Given two bi-directed graphs G_1 and G_2, a set of subbands W, the installation cost $c(w)$ for each subband w, a routing cost b^{ew} for each subband installed on a link e and a set of commodities K, determine a set of subbands to be installed over the arcs of G_1 such that*

(i) the commodities can be routed in G_1 using these subbands,

(ii) paths in G_2, respecting the disjunction constraint, are associated with the installed subbands,

(iii) the total cost is minimum.

2.2. *Integer linear programming formulation*

Given a digraph $G = (V, A)$ and a node set $T \subset V$, we denote by $\delta^+_{G(T)}$ (resp. $\delta^-_{G(T)}$), the set of arcs of A having their initial node (resp. terminal node) in T and their terminal node (resp. initial node) in $V \setminus T$, that is to say $\delta^+_{G(T)} = \{a = (u, v) \in A$ with $u \in T$ and $v \notin T\}$.

Now we will present an integer linear programming formulation using three sets of variables. First, let the *design variables* $y \in \mathbb{R}^{A_1 \times W}$ be such that, for each arc $e \in A_1$ and for each subband $w \in W$, y_{ew} takes the value 1, if w is installed on e, and 0 otherwise. Let the *routing in G_2 variables* $z \in \mathbb{R}^{A_1 \times W \times A_2}$ be such that for each arc $e \in A_1$, for each subband $w \in W$ and for each arc $a \in A_2$, z^{ew}_a takes the value 1 if a belongs to the path in G_2 associated with pair (e, w), and 0 otherwise. Finally, we denote by $x \in \mathbb{R}^{K \times A_1 \times W}$ the routing variables such that for each commodity $k \in K$, for each arc $e \in A_1$ and for each subband $w \in W$, x^k_{ew} takes the value 1 if k uses (e,w) for its *routing in G_1*, and 0 otherwise.

An instance of CMLND-U is defined by the quadruplet (G_1, G_2, K, C). Let $\mathcal{S}(G_1, G_2, K, C)$ denote the set of feasible solutions of the CMLND-U problem, associated with an instance (G_1, G_2, K, C). The CMLND-U problem is then equivalent to the following ILP:

$$\min \sum_{e \in A_1} \sum_{w \in W} c(w) y_{ew} + \sum_{e \in A_1} \sum_{w \in W} \sum_{a \in A_2} b^{ew}(a) z^{ew}_a \tag{1}$$

$$\sum_{e \in \delta^+_{G_1}(T)} \sum_{w \in W} x^k_{ew} \geq 1, \qquad \begin{array}{l}\forall k \in K, \forall T \subset V_1, \\ \emptyset \neq T \neq V_1, o_k \in T, d_k \notin T,\end{array} \tag{2}$$

$$\sum_{w \in W} x^k_{ew} \leqslant 1, \qquad \forall k \in K, \forall e \in A_1, \tag{3}$$

$$\sum_{k \in K} D^k x^k_{ew} \leq C y_{ew}, \qquad \forall e \in A_1, \forall w \in W, \tag{4}$$

$$\sum_{a \in \delta^+_{G_2}(T)} z^{ew}_a \geq y_{ew}, \qquad \begin{array}{l}\forall e = (u, v) \in A_1, \forall w \in W, \\ \forall T \subset V_2, \emptyset \neq T \neq V_2, u' \in T, v' \notin T,\end{array} \tag{5}$$

$$\sum_{e \in A_1} z^{ew}_a \leq 1, \qquad \forall w \in W, \forall a \in A_2, \tag{6}$$

$$x^k_{ew} \in \{0, 1\}, \qquad \forall k \in K, \forall e \in A_1, \forall w \in W, \tag{7}$$

$$y_{ew} \in \{0, 1\}, \qquad e \in A_1, \forall w \in W, \tag{8}$$

$$z^{ew}_a \in \{0, 1\}, \qquad \forall e \in A_1, \forall w \in W, \forall a \in A_2. \tag{9}$$

Inequalities (2) are the *cut constraints*. They will also be referred to as *connectivity constraints*. They ensure that a path in G_1 exists for each commodity k between nodes o_k and d_k. Inequalities (4) are the *capacity constraints* for each subband installed over an arc of G_1. They express the fact that the flow using the subband w on arc e does not exceed the capacity of w. They also ensure that the overall capacity installed on arc e is large enough to carry the traffic using e. Inequalities (5) are the *subband connectivity constraints*. They guarantee, for each pair (e, w) where w is installed on $e = (u, v)$, that a path in G_2 is associated with (e, w) between nodes u' and v'. Inequalities (6) are referred to as *disjunction constraint*. Finally, inequalities (7)-(9) are the *integrality constraints*.

Proposition 1. *[10] The formulation* (2)-(9) *is valid for CMLND-U problem.*

3. Associated polyhedron and valid inequalities

In this section, we introduce and discuss the CMLND-U polytope, that is the convex hull of the solutions of problem (1)-(9). In what follows, we will assume that $G_2 = (V_2, A_2)$ is also a complete graph. This is a reasonable assumption, since the problem when G_2 is not complete can be reduced to the case when G_2 is complete by introducing dummy arcs with large costs. We also make the assumption that the number $|W|$ of available subbands is sufficiently large for allowing the routing of all commodities over a single arc $e \in A_1$, if this is necessary. As a consequence, each commodity can be assigned a different subband. Note that such an assumption is resonable because the maximum number of subbands that can be potentially installed in practice is indeed large regarding to the number of commodities. Of course, the costs will prevent the installation of unnecessary subbands.

Given an instance of CMLND-U, defined by the quadruplet (G_1, G_2, K, C), we denote by $P(G_1, G_2, K, C)$ the convex hull of the incidence vectors of $\mathcal{S}(G_1, G_2, K, C)$, that is

$$P(G_1, G_2, K, C) := conv\{(x, y, z) \in \mathbb{R}^{K\times A_1\times W} \times \mathbb{R}^{A_1\times W} \times \mathbb{R}^{A_1\times W\times A_2} :$$

$$(x, y, z) \quad \text{satisfies } (2) - (9)\}$$

In what follows, we will characterize the dimension of polytope $P(G_1, G_2, K, C)$ and investigate the facial aspect of inequalities (2)-(9).

Theorem 1. *$P(G_1, G_2, K, C)$ is full dimensional.*

Proof. See Appendix A. □

In what follows, we will present a first class of valid inequalities arising directly from the capacity requirement of the problem. Similar inequalities have been introduced by [27], [7] and [13] for different variants of the *Capacitated Network Loading Problem*.

3.1. Capacitated Cutset Inequalities

Consider a partition of G_1 nodes in two subsets T and $\overline{T} = V_1 \setminus T$, we denote by $K_{in}(T)$ (respectively $K_{in}(\overline{T})$) the commodities of K having both origin and destination nodes in T (respectively in $\overline{T}$), while the remaining demands of K will be partitioned into two sets $K^+(T)$ and $K^-(T)$ where $K^+(T)$ (respectively $K^-(T)$) is the subset of commodities having their origin node in T (respectively in $\overline{T}$) and their destination node in $\overline{T}$ (respectively in T). We will also denote by $D(K^+(T))$ (respectively $D(K^-(T))$) the total traffic amount of $K^+(T)$ (respectively in $K^-(T)$). In other words, $D(K^+(T)) = \sum_{k \in K^+(T)} D^k$, and $D(K^-(T)) = \sum_{k \in K^-(T)} D^k$. We let $K(u,v)$ be the subset of demands of K having their origin at node u and destination at node v.

Proposition 2. *Let $\emptyset \neq T \subsetneq V_1$. Then the following inequality*

$$\sum_{e \in \delta^+_{G_1}(T)} \sum_{w \in W} y_{ew} \geqslant \lceil \frac{D(K^+(T))}{C} \rceil \tag{10}$$

is valid for $P(G_1, G_2, K, C)$.

Proof. The total capacity of the subbands installed over the cut must be greater than or equal to the traffic amount of the commodities going from T to $\overline{T} = V_1 \backslash T$ and using the arcs of that cut. Then, inequality

$$C \sum_{e \in \delta^+_{G_1}(T)} \sum_{w \in W} y_{ew} \geqslant D(K^+(T))$$

is clearly valid for $P(G_1, G_2, K, C)$. By dividing this by C and rounding up the right-hand side, we obtain inequality (10). □ □

We further denote by $BP(K^+(T))$ (resp. $BP(K^-(T))$) the smallest number of subbands required in $\delta^+_{G_1}(T)$ (respectively in $\delta^-_{G_1}(T)$) to route the commodities of $K^+(T)$ (respectively $K^-(T)$). Actually, this value corresponds to the optimal solution of the *bin packing* problem with $K^+(T)$ (resp. $K^-(T)$) being the set of items to be packed and C the capacity of a bin. For example, if $K^+(T)$ is composed by 3 demands with 6 units of traffic and $C = 10$, then $BP(K^+(T)) = 3$. Again, this happens because the traffic of a commodity can not be split into distinct subbands, even if they are installed in the same arc.

Theorem 2. *Inequalities* (10) *define facets of $P(G_1, G_2, K, C)$ only if*

(i) $K^+(T) \neq \emptyset$,

(ii) $\lceil \frac{D(K^+(T))}{C} \rceil = BP(K^+(T))$.

Proof. (i) If $K^+(T) = \emptyset$, then inequality (10) can be obtained from the non negativity constraints and cannot thus define a facet.

(ii) Given two nodes u and v of V_1, let $K(u,v) = \{k \in K : o_k = u, d_k = v\}$ and T be a subset of nodes such that $u \in T$ and $v \notin T$. Now suppose that $\lceil \frac{D(K(u,v))}{C} \rceil < BP(K(u,v))$. In this case, (10) can not be tight, since the commodities of $K(u,v)$ which need $BP(K(u,v))$ subbands cannot fit in $\lceil \frac{D(K(u,v))}{C} \rceil$ subbands. Therefore (10) cannot define a proper face. □

□

Example 1. *For example, if $K(u,v) = \{k_1, k_2, k_3\}$ with $D^{k_1} = D^{k_2} = D^{k_3} = 6$, while $C = 10$. There is no solution of $P(G_1, G_2, K, C)$ such that all three demands are packed together in $\lceil \frac{D(K(u,v))}{C} \rceil = 2$ subbands and condition (ii) is not satisfied. Thus, inequality $\sum_{e\in\delta^+_{G_1}(T)} \sum_{w\in W} y_{ew} = \lceil \frac{D(K(u,v))}{C} \rceil = 2$ cannot define a facet.*

Theorem 3. *Inequality* (10) *defines a facet of $P(G_1, G_2, K, C)$ if the following conditions hold.*

(i) $K^+(T) \neq \emptyset$,

(ii) $\lceil \frac{D(K^+(T))}{C} \rceil = BP(K^+(T))$,

(iii) $BP(K^+(T) \cup \{k\}) = BP(K^+(T))$, *for all* $k \in K \setminus K^+(T)$,

(iv) for all $k' \in K^+(T)$, there exists $k'' \in K^+(T)$ such that $D^{k'} + D^{k''} \leqslant C$,

(v) for all $k \in K^+(T)$, $BP(K^+(T) \setminus \{k\}) = BP(K^+(T))$ - 1.

Proof. See Appendix B. □

Example 2. *Pick the example 1 and consider a demand in $K^{in}(T)$, say k with $D^k = 7$. The inequality $\sum_{e\in\delta^+_{G_1}(T)} \sum_{w\in W} y_{ew} = 3$ defines a facet for $P(G_1, G_2, K, C)$ even though conditions (iii) and (iv) of Theorem 3 are not satisfied. Indeed, $BP(K^+(T)) = 3 < BP(K^+(T) \cup \{k\}) = 4$ and no pair of demands in $K^+(T)$ verifies condition (iv).*

Conditions (i) and (ii) are, by Theorem 2 necessary conditions for (10) to define a facet. Conditions (iii) to (v) are used to construct feasible solutions from known ones. Condition (iii) permits to route through the cut an extra demand without increasing its capacity. Condition (iv) allows to route two demands of the cut through the same subband. And finally, condition (v) says that if we remove a demand of the cut, its capacity is decreased by 1. All these conditions let the new solutions belonging to the face induced by (10) if the original ones are so.

3.2. Flow Cutset inequalities

In what follows, we will describe a set of valid inequalities for $P(G_1, G_2, K, C)$ that are a generalization of the capacitated cutset inequalities (10). Similar inequalities have been introduced by [19] and were discussed in [3], [14] and [33] for single-layer network design problems where discrete modular capacities are installed on the arcs of the graph.

Proposition 3. *Consider a non empty subset of nodes $T \subseteq V_1$ and a partition F, $\overline{F}$ of the cut $\delta^+_{G_1}(T)$ induced by T. The following flow-cutset inequality*

$$\sum_{w\in W} \sum_{e\in F} y_{ew} + \sum_{w\in W} \sum_{e\in \overline{F}} \sum_{k\in K^+(T)} x^k_{ew} \geqslant \lceil \frac{D(K^+(T))}{C} \rceil. \tag{11}$$

is valid for $P(G_1, G_2, K, C)$.

Proof. It is clear that the following inequalities

$$\sum_{w \in W} \sum_{e \in \delta^+_{G_1}(T)} x^k_{ew} \geqslant 1, \quad \text{for all } k \in K^+(T),$$

are valid for $P(G_1, G_2, K, C)$, as they express the connectivity constraints for the commodities of $K^+(T)$. Multiplying both sides of this inequality by D^k and summing over $K^+(T)$ yields

$$\sum_{w \in W} \sum_{e \in \delta^+_{G_1}(T)} \sum_{k \in K^+(T)} D^k x^k_{ew} \geqslant D(K^+(T)). \tag{12}$$

In addition, we have from the capacity constraints (4), restricted to the commodities of $K^+(T)$ and the arcs of F, that

$$\sum_{k \in K^+(T)} D^k x^k_{ew} - C y_{ew} \leqslant 0, \quad \text{for all } e \in F, w \in W.$$

By summing these inequalities, we obtain

$$\sum_{w \in W} \sum_{e \in F} C y_{ew} - \sum_{w \in W} \sum_{e \in F} \sum_{k \in K^+(T)} D^k x^k_{ew} \geqslant 0. \tag{13}$$

As $\delta^+_{G_1}(T) = F \cup \overline{F}$, by summing (12), (13), and dividing by C, we get

$$\sum_{w \in W} \sum_{e \in F} y_{ew} + \sum_{w \in W} \sum_{e \in \overline{F}} \sum_{k \in K^+(T)} \frac{D^k}{C} x^k_{ew} \geqslant \frac{D(K^+(T))}{C}. \tag{14}$$

Moreover, we have the following trivial inequality

$$\sum_{w \in W} \sum_{e \in \overline{F}} \sum_{k \in K^+(T)} (1 - \frac{D^k}{C}) x^k_{ew} \geqslant 0. \tag{15}$$

By summing (14), (15) and rounding up the right-hand side, we get (11) . □ □

Theorem 4. *A flow-cutset inequality* (11) *defines a facet of $P(G_1, G_2, K, C)$, different from* (10) *only if the following hold*

(i) $F \neq \emptyset \neq \overline{F}$,

(ii) $D(K^+(T)) > C$,

(iii) $D(K^+(T))$ *is not a multiple of* C,

(iv) $\lceil \frac{D(K^+(T))}{C} \rceil = BP(K^+(T))$,

(v) $BP(K^+(T)) < |K^+(T)|$,

(vi) *there exists* $Q \subsetneq K^+(T)$ *such that* $BP(K^+(T) \setminus Q) \leqslant BP(K^+(T)) - |Q|$,

(vii) $BP(K^+(T) \cup \{k\}) \leqslant BP(K^+(T))$, *for all* $k \in K \setminus K^+(T)$.

Proof. See Appendix C. □

There are some connections between the sufficient conditions of Theorem 3 and the necessary conditions of Theorem 4. Condition (v) is the same as condition (ii) in Theorem 3. This is natural since the condition is independent of the partition of $\delta^+_{G_1}(T)$ into F and $\overline{F}$. Also condition (vii) is a generalization of of condition (v) of Theorem 3. Condition (i) of Theorem 3 and Theorem 4 imply that there are demands crossing the cut $\delta^+_{G_1}(T)$. And finally, condition (v) of Theorem 4 can be seen as a generalization of condition (iv) of Theorem 3.

Theorem 5. *A flow-cutset inequality* (11) *defines a facet of* $P(G_1, G_2, K, C)$*, different from* (10) *if the following conditions are satisfied*

(i) *conditions* (i) *to* (vi) *of Theorem 4,*

(ii) *if* $|F|$ = 1*, then for each* $k \in K^+(T)$, $BP(K^+(T) \setminus \{k\}) \leqslant BP(K^+(T))$ - 1*,*

(iii) *there exists* $k' \in K^+(T)$ *such that* $D^k + D^{k'} \leqslant C$.

Proof. The proof is given in Appendix D. □

3.3. Clique inequalities

In what follows, we will study an additional class of inequalities that are valid for $P(G_1, G_2, K, C)$. These inequalities are based on the so-called *clique inequalities* introduced by Padberg [32] for the *stable set polytope*. Similar inequalities have also been studied in [8] for the *Balanced Induced Subgraph problem*. More generally, clique inequalities arise in problems where *conflicts* may occur between objects (see [24, 15]). In order to identify these facet-defining inequalities, we will introduce first the concept of conflict graph.

Definition 2. *Given an instance of the CMLND-U problem, we consider a graph* $H = (V, E)$*, called conflict graph where each node of* V *is associated with a commodity in* K *and two commodities* k_1, k_2 *are connected by an edge* $\{k_1, k_2\} \in E$ *if and only if* k_1 *and* k_2 *cannot be packed in a subbband together. In other words, there exists an edge* $\{k_1, k_2\}$ *in* E *if and only if* $D^{k_1} + D^{k_2} > C$.

A *clique* $\mathcal{C} \subseteq N$ in a graph is a set of nodes such that every two distinct nodes in $\mathcal{C}$ are adjacent. We have the following.

Proposition 4. *Let* $\mathcal{C} \subseteq K$ *be a clique in the conflict graph* H*, and* $(e, w) \in A_1 \times W$*. Then the following clique-based inequality*

$$\sum_{k \in \mathcal{C}} x^k_{ew} - y_{ew} \leqslant 0, \tag{16}$$

is valid for $P(G_1, G_2, K, C)$.

Proof. It is clear that if a subband w is installed on e, then at most one commodity of $\mathcal{C}$ can be routed on e using w. If not, then x^k_{ew} = 0 for all $k \in \mathcal{C}$, and the constraint is trivially satisfied. □

3.4. Min set I inequalities

We introduce here a further class of valid inequalities induced by a subset of commodities for each arc. This class of inequalities has been described first in [11] for the unsplittable non-additive capacitated network design (UNACND) problem. They have been identified using the fact that the single arc UNACND problem reduces to the bin packing problem.

Proposition 5. *Given a subset $S \subseteq K$ of commodities and a non negative integer $p \in \mathbb{Z}^+$, inequalities*

$$\sum_{w \in W} \sum_{k \in S} x^k_{ew} \leqslant \sum_{w \in W} y_{ew} + p, \text{ for all } e \in A_1, \tag{17}$$

are valid for $P(G_1, G_2, K, C)$ if and only if $p \geqslant |S| - BP(S)$.

Proof. The following inequalities

$$\sum_{k \in S} x^k_e \leqslant y_e + p, \text{ for all } e \in A_1, \tag{18}$$

Introduced in [11] for the *Unsplittable non-additive capacitated network design problem* are clearly valid for $P(G_1, G_2, K, C)$, if $p \geqslant |S| - BP(S)$. Indeed, by introducing new "aggregated" variables $x^k_e \in \{0, 1\}$, for all $k \in K$, $e \in A_1$, and $y_e \in \mathbb{Z}^+$, for all $e \in A_1$, we can use the following transformation $x^k_e = \sum_{w \in W} x^k_{ew}$ and $y_e = \sum_{w \in W} y_{ew}$. This is possible since a commodity cannot be split over several subbands installed on the same arc $e \in A_1$. Thus, using the original variables to write (18) yields inequality (17). □

Theorem 6. *Inequality* (17) *defines a facet of $P(G_1, G_2, K, C)$ if and only if the following holds*

(i) $p = |S| - BP(S)$,

(ii) $BP(S \cup \{s\}) = BP(S) = |S| - p$, *where s is the largest element in $K \setminus S$,*

(iii) $BP(S \setminus \{s\}) = BP(S) - 1 = |S| - p - 1$, *where s is the smallest element in S.*

Proof. See Appendix E. □

3.5. Min set II inequalities

Likewise Min Set I, this class of inequalities has been presented first in [11] and originates from the study of the arc-set UNACND polyhedron.

Proposition 6. *Let e be an arc of A_1, S a subset of K, and p and q, two non negative integer parameters such that $q \geqslant 2$. Then, the inequality*

$$\sum_{w \in W} \sum_{k \in S} x^k_{ew} \leqslant q \sum_{w \in W} y_{ew} + p, \tag{19}$$

is valid for $P(G_1, G_2, K, C)$, if $p \geqslant (|S'| - qBP(S'))$ for all $S' \subseteq S$.

Proof. Let S' be the subset of S using arc e. By summing inequalities (3) over S', we get

$$\sum_{k\in S'}\sum_{w\in W} x^k_{ew} \leqslant |S'|, \tag{20}$$

which is valid for $P(G_1, G_2, K, C)$. On the other hand, we also have that

$$\sum_{w\in W} y_{ew} \geqslant BP(S'), \tag{21}$$

is valid for $P(G_1, G_2, K, C)$. Thus we have from (20) and (21)

$$\sum_{k\in S'}\sum_{w\in W} x^k_{ew} = \sum_{k\in S}\sum_{w\in W} x^k_{ew} \leqslant |S'| + q(\sum_{w\in W} y_{ew} - BP(S'))$$

$$= q\sum_{w\in W} y_{ew} + |S'| - qBP(S') \leqslant q\sum_{w\in W} y_{ew} + p.$$

The last inequality comes from the fact that $p \geqslant |S'| - qBP(S')$. □

Theorem 7. *Given a subset of demands $S \subseteq K$, an arc $e = (u, v) \in A_1$, and two non-negative integers q and p, the inequality*

$$\sum_{k\in S}\sum_{w\in W} x^k_{ew} \leqslant q\sum_{w\in W} y_{ew} + p \tag{22}$$

defines a facet of $P(G_1, G_2, K, C)$ if the following holds

(i) There exists an integer $r \in \mathbb{Z}^+$, $p \leqslant r \leqslant |S|-1$ such that for all $S' \subseteq S$ with $|S'| = r$, $BP(S') = \frac{|S'|-p}{q}$,

(ii) for all $s \in K \setminus S$, there exists $S' \subseteq S$ such that $BP(S') = \frac{|S'|-p}{q} = BP(S' \cup \{s\})$.

Proof. The proof is given in Appendix F. □

Note that Proposition 5, Theorem 6 and Proposition 6 are adaptation of results in [11], where the facial structure of both Min Set I and Min Set II inequalities is investigated in details. The authors give necessary and sufficient conditions for these inequalities to define facets for the arc-set unsplittable non-additive capacitated network design polyhedron.

4. Branch-and-Cut algorithm

In this section we present a Branch-and-Cut algorithm for the CMLND-U problem. Our purpose is to substantiate the efficiency of the valid inequalities described in the previous section, and provide exact solutions for realistic and real instances of networks.

4.1. Overview

We describe the framework of our algorithm. Suppose that we are given two graphs $G_1 = (V_1, A_1)$ and $G_2 = (V_2, A_2)$, that instantiate the virtual layer and the physical layer of the network, respectively. Also suppose given a set of commodities K where each commodity k is characterized by a pair $(o_k, d_k) \in V_1 \times V_1$ and a traffic value D^k. We consider a set W of available subbands having a capacity C. A cost vector $c \in R_+^{W \times A_1}$, is given as well.

To start the optimization, we set up the following restricted linear program.

$$
\begin{aligned}
& Min \sum_{e \in A_1} \sum_{w \in W} c(w) y^{ew} + \sum_{e \in A_1} \sum_{w \in W} \sum_{a \in A_2} z_a^{ew} \\
& s.t: \\
& \sum_{w \in W} \sum_{e \in \delta^+_{G_1}(s)} x^k_{ew} \geq 1, && \forall k \in K, s \in \{o_k, d_k\}, \\
& \sum_{w \in W} x^k_{ew} \leqslant 1, && \forall k \in K, \forall e \in A_1, \\
& \sum_{k \in K} D^k x^k_{ew} \leq C y^{ew}, && \forall e \in A_1, \forall w \in W, \\
& \sum_{e \in A_1} z_a^{ew} \leq 1, && \forall w \in W, \forall a \in A_2, \\
& 0 \leq x^k_{ew} \leq 1, && \forall k \in K, e \in A_1, \\
& 0 \leq y^{ew} \leq 1, && \forall w \in W, e \in A_1, \\
& 0 \leq z_a^{ew} \leq 1, && \forall e \in A_1, \forall w \in W, \forall a \in A_2.
\end{aligned}
$$

We denote by $(\overline{x}, \overline{y}, \overline{z})$, $\overline{x} \in \mathbb{R}^{K \times W \times A_1}$, $\overline{y} \in \mathbb{R}^{W \times A_1}$, $\overline{z} \in \mathbb{R}^{W \times A_1 \times A_2}$, the optimal solution of the restricted linear relaxation of the CMLND-U problem. This solution is feasible for the problem if $(\overline{x}, \overline{y}, \overline{z})$ is an integer vector that satisfies all the cut constraints of type (2) and (5). In most of the cases, the solution obtained this way is not feasible for the CMLND-U problem. We then manage to identify, at each iteration of the algorithm, valid inequalities that are violated by the solution of the current restricted linear program. This is referred to as the *separation problem*. Namely, given a class of valid inequalities, the separation problem is to check whether the solution $(\overline{x}, \overline{y}, \overline{z})$ meets all the inequalities of this class, and, if this is not the case, to find an inequality that is violated by $(\overline{x}, \overline{y}, \overline{z})$. The detected inequalities are then added to the current linear program, and such procedure is reiterated until no violated inequality can be identified. The algorithm uses then to branch over the fractional variables.

The Branch-and-Cut algorithm includes the inequalities described in the previous chapter, and their separations are accomplished in an order discussed in Section 5.

Observe that all the inequalities are global (i.e., valid for the whole Branch-and-Cut tree), and several inequalities may be added at each iteration. Furthermore, we move to the next class only if no violated inequalities of the current class are identified. Our strategy is to try to detect violated inequalities at each node of the Branch-and-Cut tree, in order to obtain the best possible lower bound by strengthening the linear relaxation, and thus limit the number of generated nodes.

In the sequel, we describe the separation procedures embedded in our algorithm. We use exact and heuristic algorithms as well, depending on the class of inequalities. Except for cut inequalities (5), all the separation routines are applied on the graph G_1.

4.2. Separation of basic Cut constraints

Algorithm 1: Separation of basic cut inequalities (2)

Data: a vector $(\overline{x}, \overline{y}, \overline{z})$
Result: a set $\mathcal{C}$I of cut inequalities (2) violated by $(\overline{x}, \overline{y}, \overline{z})$
for *each commodity* $k \in K$ **do**
  Associate a weight $c(e, w) = \overline{x}^k_{ew}$ to each pair $(e, w) \in A_1 \times W$;
  Use Goldberg-Tarjan push relabel algorithm [23] to find the min cut separating o_k from d_k regarding to the assigned weights;
  Let $\delta^+(\overline{T})$ denote this cut, where $\overline{T} \subsetneq V_1$ ($\overline{T}$ contains o_k but does not contain d_k);
  if *cut inequality* (2) *induced by* $\delta^+(\overline{T})$ *is violated by* $(\overline{x}, \overline{y}, \overline{z})$ **then**
    add this inequality to $\mathcal{C}$I;
  return the identified cut inequalities $\mathcal{C}$I to be added to the current LP;

We used the implementantion of Goldberg and Tarjan algorithm for max flow/min cut available in LEMON GRAPH C++ library [2]. It has a worst case complexity of $\mathcal{O}(n_1^2\sqrt{m_1})$ where n_1 and m_1 are the number of nodes and arcs of G_1, respectively. Therefore, the exact separation algorithm for cut constraints (2) runs in $\mathcal{O}(n_1^2 t\sqrt{m_1})$, where $t = |K|$.

For the cut constraints (5), we have to solve the separation problem that consists in computing for each pair $(e = (u, v), w) \in A_1 \times W$, such that $\overline{y}_{ew} > 0$, the minimum cut in G_2 separating u' from v' considering $\overline{z}_{ewa}$ as the arc capacities. Using the same Goldberg and Tarjan min cut algorithm, the full exact separation has complexity $O(n_2^2 m_2|W|\sqrt{m_2})$.

4.3. Separation of capacitated cutset inequalities

The separation problem associated with the cutset inequalities has been proven NP-hard in general [13]. In our case, the separation problem related to capacitated cut-set inequalities (10) is also NP-hard. Therefore, we have developed two heuristics to separate these inequalities, one of which is based on the so-called n-cut heuristic, proposed by Bienstock et al. in [13] for the minimum cost capacity installation for multicommodity network flows. We adapt this heuristic in order to make it suitable for our problem.

This heuristic works as follows. For any commodity $k \in K$, we check whether there is a path in G_1 connecting nodes o_k and d_k, and using only pairs (e, w), $e \in A_1$, $w \in W$ with $\overline{y}^{ew} > 0$. Since this can be performed by any path finding algorithm, we use Dijkstra's algorithm. If such a path does not exist, then it is clear that a capacitated cutset inequality is violated. This inequality is induced by a subset of nodes T such that $o_k \in T$ and $d_k \notin T$. If a path between o_k and d_k is identified in G_1 for each commodity k, then we randomly pick a subset of nodes, say $T \subseteq V_1$, $0 \neq T \neq V_1$, and identify the subset of commodities P^+ having their origin node in T and their destination in $V_1 \setminus T$. After that, we compute the right-hand side, and we check if the constraint thus constructed is violated. Since we check the existence of a path for each commodity between its origin and its destination, the worst-case complexity of this procedure is $\mathcal{O}(|K|(m_1|W| + n_1 log(n_1)))$, where $n_1 = |V_1|$ and $m_1 = |A_1|$.

Algorithm 2: Separation of capacitated cutset inequalities (10)

Data: a vector $(\overline{x}, \overline{y}, \overline{z})$
Result: a set $\mathcal{CCS}$ of capacitated cutset inequalities (10) violated by $(\overline{x}, \overline{y}, \overline{z})$
Associate a weight $c(e, w) = \overline{y}_{ew}$ to each pair $(e, w) \in A_1 \times W$;
for *each commodity* $k \in K$ **do**
  Check if there exists a path in G_1 from o_k to d_k using pairs (e, w) with $\overline{y}_{ew} > 0$;
  if *such path does not exist* **then**
    a capacitated cutset inequality induced by a subset T is violated and must be added to $\mathcal{CCS}$;
if *there is a path for each* k *between* o_k *and* d_k **then**
  Randomly pick a subset of nodes T in V_1;
  Construct the subset of commodities $K^+(T)$ such that $K^+(T) = \{k \in K : o_k \in T \text{ and } d_k \notin T\}$ **if** $\sum_{e\in\delta^+(T)} \sum_{w\in W} \overline{y}_{ew} < \lceil \frac{D(K^+(T))}{C} \rceil$ **then**
    A violated capacitated cutset inequality is identified and must be added to $\mathcal{CCS}$
return the identified cut inequalities $\mathcal{CCS}$ to be added to the current LP;

In the second separation heuristic, we use Goldberg-Tarjan max-flow algorithm to find violated capacitated cut-set inequalities (10). We attribute to each pair $(e, w) \in A_1 \times W$ the capacity $\overline{y}^{ew}$, and determine for each $k \in K$ a minimum $o^k d^k$-dicut in G_1, say $\delta^+_{G_1}(T^*)$, with $T^* \subseteq V_1$. We then identify the subset of commodities $K^+(T) \subseteq K$ passing through this directed cut. We finally add inequality

$$\sum_{e\in\delta^+_{G_1}(T^*)} \sum_{w\in W} y_{ew} \geq \lceil \frac{D(K^+(T))}{C} \rceil,$$

in case it is violated. This procedure is based on max-flow computations, thus the worst case complexity is $\mathcal{O}(n_1^2 t \sqrt{m_1})$.

4.4. Separation of flow-cutset inequalities

We now discuss our separation procedure for the flow-cutset inequalities (11). Atamtürk shows in [3] that the separation problem associated with a more general form of flow-cutset inequalities is NP-hard even for one commodity. In case of a multiple commodity set, the complexity of simultaneously determining $K^+(T)$ and F is not known [33]. As we do not know an efficient procedure to separate flow-cutset inequalities in general, we use here a simple heuristic based on Goldberg-Tarjan max-flow algorithm.

Algorithm 3: Separation of flow-cutset inequalities (11)

Data: a vector $(\overline{x}, \overline{y}, \overline{z})$
Result: a set $\mathcal{FCS}$ of flow-cutset inequalities (11) violated by $(\overline{x}, \overline{y}, \overline{z})$
for *each commodity* $k \in K$ **do**
  Associate a weight $c(e, w) = \overline{x}^k_{ew} + \overline{y}_{ew}$ to each pair $(e, w) \in A_1 \times W$;
  Use Goldberg-Tarjan push relabel algorithm [23] to find the min cut separating o_k from d_k regarding to the assigned weights;
  Let $\delta^+(\overline{T})$ denote this cut, where $\overline{T} \subsetneq V_1$ ($\overline{T}$ contains o_k but does not contain d_k);
  if *flow-cutset inequality* (11) *induced by* $\delta^+(\overline{T})$ *is violated by* $(\overline{x}, \overline{y}, \overline{z})$ **then**
    add this inequality to $\mathcal{FCS}$;

return the identified cut inequalities $\mathcal{FCS}$ to be added to the current LP;

The main idea consists in identifying, for each commodity the minimum cut separating its origin and its destination, then derivating the subset of commodities whose origin and destination nodes are separated by the same cut (see Algorithm 3). In other words, for each $k \in K$, we assign the capacity $\overline{y}_{ew} + \overline{x}_{kew}$ to each pair $(e, w) \in A_1 \times W$, and compute the minimum cut separating o^k from d^k in the graph G_1. Let $\delta^+_{G_1}(T^*)$, $T^* \subseteq V_1$, denote this cut. We then pick an arbitrary subset of arcs, say F^* of $\delta^+_{G_1}(T^*)$, such that $\emptyset \neq F^* \neq \delta^+_{G_1}(T^*)$, and we determine the subset of commodities $K^+(T^*) \subseteq K$ using $\delta^+_{G_1}(T^*)$. If $D(K^+(T))/C$ is not integer, we add the succeeding flow-cutset inequality

$$\sum_{e \in F^*} \sum_{w \in W} y_{ew} + \sum_{k \in K^+(T^*)} \sum_{e \in \overline{F}^*} \sum_{w \in W} x_{kew} \geq \lceil \frac{D(K^+(T^*))}{C} \rceil,$$

if it is violated by the current fractional solution $(\overline{x}, \overline{y}, \overline{z})$.

4.5. Separation of clique-based inequalities

Given a fractional solution $(\overline{x}, \overline{y}, \overline{z})$, and a pair $(\tilde{e}, \tilde{w}) \in A_1 \times W$, the separation problem associated with the clique-based inequalities (16) consists in identifying a clique $\mathcal{C}^*$ in the conflict graph H, such that

$$\sum_{k \in \mathcal{C}^*} \overline{x}_{kew} > \overline{y}_{ew},$$

if any. To do so, we use a greedy algorithm introduced by [29] for the independent set problem. This heuristic works as follows. We first construct the conflict graph $H = (V, E)$ where each node $v \in V$ corresponds to a commodity in K and an edge $e \in E$ exists between two nodes $u, v \in V$ if $D^u + D^v > C$. For each pair $(e, w) \in A_1 \times W$, we assign a weight to each node v of V that is $\overline{x}^v_{ew}$, then we choose a node, say u, having the largest weight and we set $\mathcal{C}^* = \{u\}$. We then iteratively add to $\mathcal{C}^*$ the maximum weighted node of $V \setminus \mathcal{C}^*$ whenever it is neighbouring all the nodes of the current clique $\mathcal{C}^*$. We add the clique-based inequality induced by $\mathcal{C}^*$ if it is violated.

4.6. Separation of Min Set I and Min Set II inequalities

Given a fractional solution $(\overline{x}, \overline{y}, \overline{z})$, deciding whether there exists a Min Set I (respectively Min Set II) inequality which is violated by $(\overline{x}, \overline{y}, \overline{z})$ is not an easy problem, since it requires solving the bin packing problem (which is NP-hard in general [22]). We use for the separation of these inequalities heuristic procedures inspired from those proposed in [11] and adapted for the CMLND-U problem. The idea, for both algorithms, is to identify for every arc $a \in A_1$ a subset of commodities S_a that may induce a violated Min Set I (respectively Min Set II) inequality. If such inequality exists, then it is added to the current LP.

5. Computational experiments

We have conducted several series of experiments to test the efficiency of our Branch-and-Cut algorithm. The purpose of this experimental study is first to give an insight of the effectiveness of each of the introduced valid inequalities. Our second objective is to assess the performance of the full algorithm including all inequalities, also identifying which instances features, not only size, make them harder to solve. We have implemented our algorithm in C++ on top of CPLEX 12.6.3 using the cut callback feature and LEMON Graph library for the graph algorithms. Finally, our algorithm was tested on an Intel(R) Xeon(R) CPU E5-2650 v2 2.60GHz machine with 128GB of RAM, running under Linux.

The results shown in this section have been obtained by solving instances adapted from SNDlib [1], as well as from real networks. For all the instances, the graph G_1 representing the virtual layer is supposed to be complete. The cost induced by installing each subband is given by $c(w) = c(1 + w)$, where w is the subband index and c is a fixed cost associated with the ROADMs generating the subband. This cost is justified by our wish to install the subbands progressively on one hand, and for the sake of compliance with practical costs on the other hand. In other words, a subband w_i, $i > 1$, is not used over an arc e before w_{i-1} is installed. We also take into account the length of the routing path in G_2 associated with each installed subband, each arc in the path also costs c.

In the SNDlib derived instances, graph G_2 corresponds to the original instance topology while G_1 is a complete bidirected graph over the same set of nodes. Actually, if the original topology corresponds to a non directed graph, we replace each edge by two anti-parallel arcs in G_2. We have considered two sub-classes of instances. The first one corresponds to randomly generated traffic commodities. For each tested value of $|K|$ we created 3 instances and report average results. The second sub-class not only uses the SNDlib topologies, the traffic matrices are also used. For each tested demand size, we create a single instance by picking the first $|K|$ commodities. The number of available subbands per arc ($|W|$) is set to 5, for both sub-classes. We have considered the SNDlib topologies *pdh*, *polska*, *nobel_us*, *newyork*, and *geant*.

The real instances are provided by Orange (formerly France Télécom, the French historical telecommunication operator). Two topologies were considered here, all related to Bretagne area backhaul network. The traffic commodities, as well as the subbands capacities are also given by Orange. For each topology, we have considered three subband capacities C = 10 Gbit/s, C = 12.5 Gbit/s and C = 25 Gbit/s, so as to compare the performances of each type of OFDM multi-band solution. The number of available subbands per arc ($|W|$) is set to 6.

5.1. *Effectiveness of the Cuts*

Before reporting the complete experiments for the instances described above, we first show preliminary experiments devised to assess the impact of the used valid inequalities within the Branch-and-Cut algorithm. For each topology, we choose the first instance with randomly generated traffic, for 3 different values of $|K|$ (10, 12, and 14). So, 15 instances were used in those tests. The baseline for the comparisons is a Basic Branch-and-Cut that only separates essential inequalities, those that are part of formulation (2)-(9).

Table 1 presents the integrality gaps at the root node (obtained from the root lower bound and from the upper bound given by the optimal or best known solution for the instance) for the Basic B&C and for a B&C that additionally separates a single family of cuts. The results in Table 1 look somehow disappointing. The average gap reductions were quite modest, being zero for Min Set II, the less effective family. However, the next experiments show a much more interesting picture.

Table 1: Impact of using a single family of cuts on the root node gap (%)

Topology	$\|V_2\|$	$\|A_2\|$	$\|K\|$	Basic B&C	MSI	Cutset	Cliques	Flow	MSII
pdh	11	68	10	34.21	31.83	34.21	34.21	34.21	34.21
pdh	11	68	12	27.21	23.63	27.21	27.21	27.21	27.21
pdh	11	68	14	34.36	32.21	33.61	34.36	34.36	34.36
polska	12	36	10	44.36	39.64	42.18	44.36	41.09	44.36
polska	12	36	12	34.79	32.52	32.48	34.79	32.12	34.79
polska	12	36	14	30.92	29.98	28.38	30.92	28.81	30.92
nobel_us	14	42	10	33.77	33.35	33.77	33.77	33.77	33.77
nobel_us	14	42	12	30.95	30.95	30.95	30.95	30.95	30.95
nobel_us	14	42	14	40.49	40.49	39.55	40.49	39.13	40.49
newyork	16	98	10	21.74	20.13	21.74	21.74	21.74	21.74
newyork	16	98	12	35.94	35.94	35.94	35.94	35.94	35.94
newyork	16	98	14	25.50	25.50	24.68	25.50	23.85	25.50
geant	22	72	10	23.20	23.20	23.20	23.20	23.20	23.20
geant	22	72	12	48.07	48.07	48.07	48.07	48.07	48.07
geant	22	72	14	36.88	36.88	35.61	36.88	34.43	36.88
Avg.				33.49	32.29	32.77	33.49	32.59	33.49

Table 2 shows the gaps obtained by running the complete Branch-and-Cut algorithm with a time limit of 5 hours, performing the separation of a single family of cuts over all nodes of the search tree. The reported gaps are obtained from the final lower bound at the time limit or are zero if the instance is solved to optimality. It can be seen that the most effective families of cuts, Min Set I and Cutset inequalities, can more than half the average final gap. Even the less effective families, Cliques and Min Set II inequalities, still have some positive impact in the B&C performance.

Finally, Table 3 shows the combined effect of all families of cuts. The cuts are separated successively in the following order: MSI, Cutset, Cliques, Flow Cutsets, and MSII. This means that the separation routine for a family is only called if separation for previous families can not find violated cuts. The chosen order is largely based in the results of the experiments reported in Table 2, the cuts found to be more effective are separated first. However, there is an inversion, due to relative separation costs: Clique cuts are separated before Flow-Cutsets because the latter separation is much more costly. Columns in Table 3 correspond to final gaps, number of nodes explored and total time (limited to 5 hours). It is clear that the cuts had a large impact in the overall performance. While the basic B&C could only solve 2 instances, the complete B&C solved 12 instances. The additional cut separations make the solution of each node slower, but this (except

Table 2: Impact of using a single family of cuts on the final gap (%) - 5 hours time limit

Topology	$\lvert V_2\rvert$	$\lvert A_2\rvert$	$\lvert K\rvert$	B&C	MSI	Cutset	Cliques	Flow	MSII
pdh	11	68	10	10.17	0.00	0.00	0.00	0.00	6.58
pdh	11	68	12	0.00	0.00	0.00	0.00	0.00	0.00
pdh	11	68	14	11.36	0.00	0.00	0.00	10.25	12.68
polska	12	36	10	24.91	13.36	0.00	27.27	35.09	22.82
polska	12	36	12	21.94	0.00	0.00	20.36	10.55	19.18
polska	12	36	14	22.4	0.00	9.60	14.3	20.9	21.52
nobel_us	14	42	10	21.13	0.00	0.00	20.90	0.00	19.65
nobel_us	14	42	12	19.36	0.00	2.74	18.15	12.85	16.62
nobel_us	14	42	14	31.77	15.81	24.85	27.47	28.15	31.68
newyork	16	98	10	0.00	0.00	0.00	0.00	0.00	0.00
newyork	16	98	12	20.00	0.00	13.87	21.13	18.06	19.65
newyork	16	98	14	11.53	0.00	0.00	10.59	0.00	10.71
geant	22	72	10	11.31	0.00	0.00	12.46	6.74	11.09
geant	22	72	12	37.60	34.30	34.42	38.05	38.91	37.95
geant	22	72	14	30.53	20.92	19.06	29.63	31.08	30.90
Avg.				19.22	7.29	8.58	17.11	15.32	17.4

in topology geant with $|K| = 12$) is more than compensated by the big reduction in the number of explored nodes.

5.2. SNDlib instances with randomly generated traffic

This subsection reports extensive experiments on the instances with SNDlib topologies and randomly generated traffic commodities. The instances considered here have graphs with 10 up to 22 nodes and vary from sparse (like for polska) to highly meshed (like for newyork) topology. The number of commodities ranges from 10 to 20, the distinct origin and destination nodes are random, the D^k values are generated uniformly in the interval $[\epsilon C, C]$, with ϵ = 0.2. For each graph and number of commodities, there is a group of 3 instances. The rows in Table 4 are averages for each of the 30 groups. The columns in that table are:

Opt	:	number of instances solved to optimality
NcI	:	number of Connectivity constraints (2) separated,
NcII	:	number of Subband Connectivity constraints (5) separated,
NMSI	:	number of Min Set I inequalities separated,
NCCS	:	number of Capacitated Cutset inequalities separated,
NC	:	number of Clique inequalities separated,
NFCS	:	number of Flow-Cutset inequalities separated,
NMSII	:	number of Min Set II inequalities separated,

Table 3: The impact of adding all families of cuts in the overall B&C performance

				Basic B&C			B&C with all families of cuts		
Topology	$\|V_2\|$	$\|A_2\|$	$\|K\|$	Final Gap	Nodes	TT	Final Gap	Nodes	TT
pdh	11	68	10	10.17	98167	18000.00	0.00	169	190.55
pdh	11	68	12	0.00	6884	993.94	0.00	215	784.73
pdh	11	68	14	11.36	109514	18000.00	0.00	40	191.42
polska	12	36	10	24.91	175345	18000.00	0.00	2503	1220.38
polska	12	36	12	21.94	247373	18000.00	0.00	1152	820.48
polska	12	36	14	22.40	147090	18000.00	0.00	277	486.86
nobel_us	14	42	10	21.13	86279	18000.00	0.00	846	2393.37
nobel_us	14	42	12	19.36	97318	18000.00	0.00	358	1274.39
nobel_us	14	42	14	31.77	97532	18000.00	13.34	5823	18000.00
newyork	16	98	10	0.00	4147	2478.13	0.00	285	863.25
newyork	16	98	12	20.00	44575	18000.00	0.00	1026	4012.61
newyork	16	98	14	11.53	32605	18000.00	0.00	439	2272.85
geant	22	72	10	11.31	45022	18000.00	0.00	1823	2267.13
geant	22	72	12	37.60	48394	18000.00	44.7	3890	18000.00
geant	22	72	14	30.53	41258	18000.00	12.53	1945	18000.00
Avg.				19.22	85433		4.69	1386	

Nodes	:	number of nodes in the Branch-and-Cut tree,
Root Gap	:	the relative difference between the best upper bound (optimal solution if the problem has been solved to optimality) and the lower bound obtained at the root node of the Branch-and-Cut tree, before branching,
Final Gap	:	the relative difference between the best upper bound and the lower bound at the time limit of 5 hours, zero if solved to optimality,
TT	:	total CPU time, in seconds,
TTsep	:	CPU time spent in performing cut separation, in seconds.

We now discuss the results in Table 4:

- It can be seen that 61 out of 90 instances were solved to optimality within the fixed time. Considering the 30 groups, all the 3 instances were solved in 15 of them. For the groups where some instances were not solved, good feasible solutions are often found. The average final gap is less than 20%, except in 5 groups corresponding to the geant topology.

- The number of commodities impact a lot on the problem difficulty. While all the 15 instances with $|K| = 10$ were solved, only 6 instances with $|K| = 20$ could be solved. This is quite expected since the size of formulation (2)-(9) grows with $|K|$.

- The number of solved instances varied a lot depending on the topology: 17 for pdf, 15 for polska, 12 for nobel_us, 3 for newyork, and only 4 for geant. It seems that the number of vertices have more impact on the difficulty than the number of arcs. For example, pdh instances ($|V_2| = 11, |A_2| = 68$) are much easier than geant instances ($|V_2| = 22, |A_2| = 44$).

Table 4: Branch-and-Cut results for SNDlib instances with randomly generated traffic

Topology	$\|V_2\|$	$\|A_2\|$	$\|K\|$	Opt	NcI	nCII	NMSI	NCCS	NC	NFCS	NMSII	Nodes	Root Gap (%)	Final Gap (%)	TT	TTsep
pdh	11	68	10	3	15.3	16.0	48.7	43.3	18.3	93.7	0.7	259.7	29.75	0.00	166.42	91.81
pdh	11	68	12	3	55.7	0.0	92.3	50.0	34.3	1286.7	2.7	229.3	29.57	0.00	612.32	384.91
pdh	11	68	14	3	92.0	0.0	125.0	59.0	69.3	363.7	3.0	201.0	27.42	0.00	428.28	239.24
pdh	11	68	16	3	192.7	0.0	207.3	71.7	180.0	5936.7	3.0	753.7	24.32	0.00	2657.58	1617.55
pdh	11	68	18	2	129.3	0.0	169.7	66.0	84.7	7365.7	4.0	777.0	23.89	1.96	6559.53	4208.32
pdh	11	68	20	3	230.7	0.3	292.7	74.7	210.0	3360.3	11.3	996.7	23.77	0.00	3448.82	2159.56
polska	12	24	10	3	87.7	70.3	93.0	44.0	27.3	866.3	5.3	1363.3	31.92	0.00	784.27	372.28
polska	12	24	12	3	111.7	84.3	161.3	49.7	65.7	940.7	6.0	1012.7	26.91	0.00	775.26	352.95
polska	12	24	14	3	166.3	72.0	145.7	71.3	107.0	2176.0	2.3	937.0	29.33	0.00	1638.42	934.25
polska	12	24	16	1	225.3	202.3	230.0	87.0	114.3	9977.7	2.0	2514.3	29.20	13.15	12093.44	7089.12
polska	12	24	18	3	329.3	131.3	250.7	50.7	327.7	2673.3	6.3	1939.3	29.66	0.00	2983.04	1317.35
polska	12	24	20	2	551.0	148.3	297.3	85.0	487.0	5933.0	4.7	4715.0	26.49	1.36	10298.53	4014.63
nobel_us	14	28	10	3	121.0	30.7	106.0	101.3	20.3	1494.7	1.3	863.3	32.32	0.00	1710.06	970.37
nobel_us	14	28	12	3	149.3	57.3	167.3	91.3	85.0	1486.3	2.3	636.7	33.29	0.00	1913.95	994.21
nobel_us	14	28	14	1	302.0	81.7	179.3	163.0	221.0	9574.0	1.7	2683.7	34.93	12.96	14381.15	5581.83
nobel_us	14	28	16	3	184.3	83.0	206.7	58.3	141.7	1184.0	2.3	748.7	24.27	0.00	2383.62	1281.57
nobel_us	14	28	18	2	316.0	94.3	279.0	70.3	342.0	4755.0	1.7	2075.0	26.22	4.41	8876.75	4219.13
nobel_us	14	28	20	0	558.3	244.3	447.0	107.3	376.3	4251.0	4.7	4961.7	30.64	12.27	18000.00	6651.82
newyork	16	32	10	3	84.7	0.0	72.7	82.0	40.7	1197.7	0.0	355.7	27.65	0.00	2145.57	1251.74
newyork	16	32	12	3	106.7	1.3	130.7	105.3	60.0	634.0	1.0	625.3	31.17	0.00	2859.72	1690.81
newyork	16	32	14	2	287.3	0.0	153.3	123.3	117.0	2318.0	0.7	1480.3	37.07	16.17	8432.44	3975.11
newyork	16	32	16	2	125.7	0.0	108.0	88.0	17.3	2480.7	0.0	813.3	26.34	5.55	7272.54	3917.47
newyork	16	32	18	2	150.3	2.3	168.7	101.7	139.0	2013.3	3.0	485.0	25.78	3.55	7349.32	4122.91
newyork	16	32	20	1	270.7	9.7	333.3	151.0	384.7	2060.7	2.3	880.0	26.06	9.39	12054.01	7022.06
geant	22	44	10	3	125.0	117.3	75.7	133.3	23.3	300.0	0.3	1100.7	30.81	0.00	7117.77	3931.41
geant	22	44	12	1	275.0	259.7	132.0	186.3	48.7	2557.3	0.7	2211.0	40.26	26.51	16455.93	7825.78
geant	22	44	14	0	320.3	261.0	153.7	219.7	84.0	2470.0	2.0	1300.0	39.97	25.95	18000.00	8570.22
geant	22	44	16	0	173.0	279.3	140.0	201.3	67.7	905.0	0.3	548.7	35.38	39.08	18000.00	10868.47
geant	22	44	18	0	228.3	265.3	179.7	192.7	132.7	847.0	1.0	578.3	41.00	38.84	18000.00	9502.91
geant	22	44	20	0	221.3	189.7	158.0	185.0	111.3	888.0	1.3	459.7	38.43	32.89	18000.00	11125.07
Avg.												1283.5	30.46	8.13	7513.29	3876.12

- On all groups of instances, many Min Set I, capacitated cutset, flow-cutset and clique inequalities are separated along the Branch-and-Cut tree. This is an additional evidence that they do help to solve the problem. However, the number of Min Set II inequalities separated is much smaller. This confirms their marginal effect on B&C performance. Indeed, they were only kept in the final algorithm because their separation is fast, so they do not harm the efficiency.

- On average, 53% of the total time is spent on separation. We noticed that the separation procedure for flow cutset inequalities is the most time consuming. This is why we have chosen to separate then after the clique cuts, even tough the latter cuts are usually less effective.

5.3. SNDlib instances with original traffic

The second series of experiments that we have conducted use the same SNDlib topologies, but now also use the original SNDLib demands (the first $|K|$ commodities) and the original values of C. The results in Table 5 show that the resulting instances turn out to be considerably more difficult. The root gaps are larger and only 8 out of 30 instances (27%) could be solved to optimality, compared to 61 out of 90 (68%) on the previous tests. The main explanation seems to be the relation between the average values of D^k and C, we call that relation α. The random generator used in previous instances produced α values around 0.6. This limits the number of combinations of demands that can be packed in the same subband. The instances with the original SNDLIB demands have smaller average values of α: 0.40 for pdh, 0.38 for polska, 0.32 for nobel_us, 0.48 for newyork, and 0.33 for geant. As expected, as there are fewer demands where $D^k > C/2$, it can be observed that very few clique cuts are separated. Nevertheless, the final gaps are still less than 50%, except on 4 geant instances.

5.4. Real instances

The real instances provided by Orange have two distinct topologies, bretagne9 an bretagne22, with 9 and 22 nodes. The demands were also provided by Orange, with a number of commodities that varies between 15 and 30. On bretagne9, the average traffic value is 6355.5 Mbit/s (for $|K|$ = 30). On bretagne22, the average traffic value is 7406.8 Mbit/s (also for $|K|$ = 30). Orange also indicated the value $|W|$ = 6 for all the instances, and three possible subband capacities, namely C = 10 Gbit/s, 12.5 Gbit/s and 25 Gbit/s. Table 6 shows the obtained results.

It can be seen that 6 out of the 18 tested instances were solved to optimality within the time limit. The algorithm did not performed well for $C = 25$, none of the 6 instances with that capacity value could be solved. This confirms the finding that instances where the average demand values are small with respect to C are harder to the proposed B&C algorithm. On the other hand, the results for the instances with $C = 10$ and $C = 12.5$ were much better. Half of those instances could be solved to optimality, the remaining instances finished with a final gap smaller than 30%, including those with topology bretagne22 and $|K| = 30$, that are fairly large.

6. Concluding remarks

In this paper we have studied a multilayer network design problem with specific requirements arising in optical OFDM networks. The characteristics of the hierarchical routing and the traffic indivisibility makes the problem very difficult to solve, creating a computational challenge.

Table 5: Branch-and-Cut results for SNDlib instances with original traffic

Topology	$\|V_2\|$	$\|A_2\|$	$\|K\|$	Opt	NcI	nCII	NMSI	NCCS	NC	NFCS	NMSII	Nodes	Root Gap (%)	Final Gap (%)	TT	TTsep
pdh	11	68	10	1	258	0	73	179	0	5497	0	5225	60.70	0.00	6790.24	2453.09
pdh	11	68	12	1	160	0	36	136	6	841	0	472	53.09	0.00	727.96	382.56
pdh	11	68	14	1	112	0	65	107	16	1049	0	500	49.23	0.00	881.21	504.36
pdh	11	68	16	1	564	0	184	216	58	6735	1	2607	54.46	0.00	9858.83	4269.88
pdh	11	68	18	0	391	0	200	220	41	35628	0	1423	52.40	21.77	18000.00	8910.63
pdh	11	68	20	1	385	0	152	146	38	4776	3	935	38.89	0.00	5239.48	3447.56
polska	12	36	10	0	214	19	22	130	0	46638	1	1040	58.07	36.85	18000.00	6532.35
polska	12	36	12	0	408	136	61	153	0	39055	1	8999	59.82	35.02	18000.00	6755.4
polska	12	36	14	0	669	185	77	187	0	21222	3	13145	54.70	22.72	18000.00	7518.6
polska	12	36	16	0	615	176	91	166	0	23719	5	8831	56.44	43.24	18000.00	8444.21
polska	12	36	18	0	662	194	167	168	0	20459	22	7025	60.33	46.05	18000.00	9453.77
polska	12	36	20	0	614	187	205	162	0	24838	8	4791	54.56	45.52	18000.00	10396.6
nobel_us	14	42	10	1	86	14	51	105	0	4537	4	1558	48.04	0.00	5055.05	3201.24
nobel_us	14	42	12	0	419	29	62	161	0	18721	4	1663	53.96	38.17	18000.00	11007.1
nobel_us	14	42	14	0	250	28	71	147	0	18630	0	781	52.25	44.25	18000.00	12152.00
nobel_us	14	42	16	0	115	22	67	144	0	25030	0	453	54.56	43.93	18000.00	13548.4
nobel_us	14	42	18	0	189	19	56	138	0	14899	1	440	56.43	47.40	18000.00	14120.4
nobel_us	14	42	20	0	199	15	59	130	0	10242	0	419	58.71	50.00	18000.00	15115.6
newyork	14	98	10	1	9	0	37	31	0	52	0	139	28.69	0.00	539.95	382.81
newyork	14	98	12	1	78	0	137	59	8	398	0	495	30.55	0.00	2453.75	1515.35
newyork	14	98	14	1	271	0	244	85	17	2330	4	2183	35.41	0.00	13960.1	7569.51
newyork	14	98	16	0	520	0	400	98	47	2916	2	3883	44.97	32.46	18000.00	10270.9
newyork	14	98	18	0	433	0	344	109	7	3430	1	3207	32.61	22.36	18000.00	10556.6
newyork	14	98	20	0	601	10	414	83	5	2359	8	3639	48.61	36.86	18000.00	10268.2
geant	22	72	10	0	124	63	97	93	0	3075	2	1942	48.03	37.06	18000.00	9418.52
geant	22	72	12	0	451	208	133	237	13	2185	2	1344	52.28	46.65	18000.00	10065.5
geant	22	72	14	0	333	95	220	168	7	2368	9	1276	68.24	64.93	18000.00	11173.1
geant	22	72	16	0	143	104	151	94	7	2761	0	961	67.08	65.07	18000.00	13713.3
geant	22	72	18	0	306	119	148	129	20	2422	0	1119	65.30	60.57	18000.00	12257.4
geant	22	72	20	0	173	92	157	174	10	2942	3	707	62.98	61.01	18000.00	13572.2
Avg.												2706.7	52.05	29.96	14116.89	8299.4

Table 6: Branch-and-Cut results for real instances

Topology	$\|V_2\|$	$\|A_2\|$	$\|K\|$	C (Gbit/s)	NcI	nCII	NMSI	NCCS	NC	NFCS	NMSII	Nodes	Root Gap (%)	Final Gap (%)	TT	TTsep
bretagne9	9	20	15	10	67	47	133	31	181	288	0	94	29.43	0.00	97.77	48.49
bretagne9	9	20	15	12	109	47	81	47	333	267	0	208	23.08	0.00	163.40	67.53
bretagne9	9	20	15	25	270	114	148	97	83142	0	0	3601	56.25	29.29	18000.00	5312.34
bretagne9	9	20	20	10	107	66	318	58	645	753	2	379	7.11	0.00	483.27	297.86
bretagne9	9	20	20	12	175	86	223	66	5514	859	1	3082	17.62	0.00	2949.18	1664.58
bretagne9	9	20	20	25	352	78	173	156	74608	0	11	1615	48.72	39.55	18000.00	8416.61
bretagne9	9	20	30	10	293	210	968	72	3916	2437	7	4192	8.8	0.00	7573.20	4655.07
bretagne9	9	20	30	12	438	280	1060	264	22684	4407	50	7097	20.90	12.64	18000.00	7367.87
bretagne9	9	20	30	25	353	166	472	183	33698	0	34	3451	45.12	36.22	18000.00	11502.80
bretagne22	22	52	15	10	213	707	192	35	1003	894	2	1102	27.79	24.10	18000.00	8760.96
bretagne22	22	52	15	12	107	294	84	54	411	411	2	379	28.40	0.00	6312.77	3666.88
bretagne22	22	52	15	25	140	575	95	60	451	0	5	1130	65.63	62.07	18000.00	14012.60
bretagne22	22	52	20	10	144	213	159	131	738	108	0	323	22.98	22.49	18000.00	12571.80
bretagne22	22	52	20	12	209	504	94	147	230	950	0	90	25.02	18.61	18000.00	9852.16
bretagne22	22	52	20	25	129	193	53	167	675	0	0	372	65.02	60.87	18000.00	10070.25
bretagne22	22	52	30	10	120	502	149	156	535	145	1	148	21.74	13.94	18000.00	12887.50
bretagne22	22	52	30	12	218	486	135	176	712	130	0	112	44.65	25.78	18000.00	11593.70
bretagne22	22	52	30	25	131	111	85	217	436	0	2	267	67.76	67.30	18000.00	15329.30
Avg.												1535.7	34.78	22.94	12976.64	7671.02

We have proposed a cut-based ILP formulation for the problem and studied its basic polyhedral properties. Exploiting the underlying sub-problems, we have proposed several families of valid inequalities and investigated their facial structure. These inequalities have been embedded within a branch-and-cut algorithm to solve the problem.

The proposed valid inequalities have helped a lot in reducing the number of nodes in the branch-and-cut enumeration tree, leading to optimal or provably good solution for a significant number of instances. Yet, some of the associated separation procedures are still time-consuming and could be enhanced. In addition, deriving good upper bounds from the fractional solutions is a non-trivial task but would hopefully allow to further reduce the size of the branch-and-cut tree. Finally, we expect that finding other valid inequalities involving the demand structure would be interesting for increasing the efficiency of the algorithm, especially on instances where many demands have small traffic value compared to the subband capacity.

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Appendices

Definition 3. *A solution S of the CMLND-U problem can be represented by two subsets of arcs F_1, F_2 of A_1 (with F_2 eventually empty), $|K|$ subsets of arcs $\mathcal{C}_1, \ldots, \mathcal{C}_k$, of A_1, a subset of subbands $\overline{W}$ of W installed on the arcs of $F_1 \cup F_2$, a subset of arcs Δ of A_2, and $|A_1| \times |W|$ subsets of arcs Δ_{ew}, $e \in A_1$, $w \in W$, of A_2 in such a way that*

(i) *at least one subband is installed on each arc of $F_1 \cup F_2$,*

(ii) $F_1 = \bigcup_{k \in K} \mathcal{C}_k$,

(iii) *$\mathcal{C}_k$, $k \in K$, consists in a path between o_k and d_k,*

(iv) $\Delta = \bigcup_{e \in F_1 \cup F_2, w \in \overline{W}} \Delta_{ew}$,

(v) *with every arc $e = (u, v) \in F_1 \cup F_2$ and $w \in \overline{W}$, one can associate an arc subset Δ_{ew} (which may be empty), in such a way that if w is installed on e, then Δ_{ew} contains a path, say $P_{ew} \subseteq \Delta_{ew}$ between u' and v',*

(vi) *for every $w \in \overline{W}$, any arc of Δ belongs to at most one path P_{ew}, for $e \in F_1 \cup F_2$.*

We then denote the solution S by $S = (F_1, F_2, \Delta, \overline{W})$. The incidence vector of S, $(x^S, y^S, z^S) \in \mathbb{R}^{K \times A_1 \times W} \times \mathbb{R}^{A_1 \times W} \times \mathbb{R}^{A_1 \times W \times A_2}$, will be given by:

$$x^S_{kew} = \begin{cases} 1, & \text{if } e \in \mathcal{C}_k \text{ and } w \text{ is installed on } e, \\ 0, & \text{otherwise.} \end{cases}$$

$$y^S_{ew} = \begin{cases} 1, & \text{if } w \in \overline{W}, e \in F_1 \cup F_2 \text{ and } w \text{ is installed on } e, \\ 0, & \text{otherwise.} \end{cases}$$

$$z^S_{ew}(a) = \begin{cases} 1, & \text{if } a \in \Delta_{ew}, \\ 0, & \text{otherwise.} \end{cases}$$

We will consider throughout the polyhedral analysis that each commoditiy uses a different subband for its routing, unless stated explicitely.

Appendix A. Proof for Theorem 1

Assume that $P(G_1, G_2, K, C)$ is contained in the hyperplane defined by the linear equation

$$\alpha x + \beta y + \gamma z = \delta \tag{A.1}$$

where $\alpha = (\alpha^k_{ew}, k \in K, e \in A_1, w \in W) \in \mathbb{R}^{K \times A_1 \times W}$, $\beta = (\beta^{ew}, e \in A_1, w \in W) \in \mathbb{R}^{A_1 \times W}$, $\gamma = (\gamma^{ew}_a, e \in A_1, w \in W, a \in A_2) \in \mathbb{R}^{A_1 \times W \times A_2}$ and $\delta \in \mathbb{R}$. We will show that α=0, β=0, γ=0 implying that $P(G_1, G_2, K, C)$ can not be included in the hyperplane (A.1), since it is not empty. To this end, let us first construct a solution $S^0 = (F_1^0, F_2^0, \Delta^0, W^0)$ of the problem.

For each commodity $k \in K$, we consider a path in G_1 between its origin and destination nodes, consisting of arc (o_k, d_k). This is possible since G_1 is complete. We install over this arc one subband. In other words, every subband is assigned at most to one commodity. Note that every arc (u, v) receives as much subbands as there are demands going from u to v. All the installed subbands are supposed to be different. After that, we associate with each subband, installed over (o_k, d_k), $k \in K$, a path in G_2 consisting in the arc (o'_k, d'_k). Again, this is possible since G_2 is also a complete graph.

Let $S^0 = (F_1^0, F_2^0, \Delta^0, W^0)$, be the solution given by $F_1^0 = \{(o_k, d_k), k \in K\}$, $F_2^0 = \emptyset$, $\Delta^0 = \{(o'_k, d'_k), k \in K\}$ and W^0 the subset of $|K|$ different subbands installed on the arcs of F_1^0.

Note that, as all the set up subbands are different, every considered path between o'_k and d'_k is associated with different subbands, and therefore, disjunction constraints (6) are satisfied. Moreover, since the capacities of the subbands are all greater than or equal to the commodity values, and a different subband is associated with each commodity, we have that capacity constraints (4) are also satisfied. Furthermore, by construction, the solution given above also satisfies the cut constraints (2) and (5). Thus the solution S^0 is feasible.

Consider a pair $(e, w) \in A_1 \times W$. Let $S^1 = (F_1^1, F_2^1, \Delta^1, W^1)$ be the solution obtained from S^0 by adding an arc $f \in A_2 \setminus \Delta^0$ to Δ^0_{ew}, while the other elements of S^0 remain the same. In other words, S^1 is such that $F_1^1 = F_1^0$, $F_2^1 = F_2^0$, $\Delta^1 = \Delta^0 \cup \{f\}$, and $W^1 = W^0$.

Obviously, S^1 is also feasible for the problem. As S^0 and S^1 are both feasible, their incidence vectors $(x^{S^0}, y^{S^0}, z^{S^0})$ and $(x^{S^1}, y^{S^1}, z^{S^1})$ both satisfy equality (A.1). Hence,

$$\alpha x^{S^0} + \beta y^{S^0} + \gamma z^{S^0} = \alpha x^{S^1} + \beta y^{S^1} + \gamma z^{S^1} = \alpha x^{S^0} + \beta y^{S^0} + \gamma z^{S^0} + \gamma_f^{ew}$$

This implies that $\gamma_f^{ew} = 0$. As f, e and w are chosen arbitrarily in $A_2 \setminus \Delta^0$, A_1 and W, respectively, we obtain that

$$\gamma_f^{ew} = 0, \text{ for all } f \text{ in } A_2 \setminus \Delta^0, e \in A_1 \text{ and } w \in W. \tag{A.2}$$

Now let $f = (u', v') \in \Delta^0$, $e = (u, v) \in A_1$ and $w \in W$. Suppose first that $f \in \Delta^0_{ew}$. Consider the solution $S^2 = (F_1^2, F_2^2, \Delta^2, W^2)$ such that $F_1^2 = F_2^2$, $F_2^2 = \emptyset$, $\Delta^2 = (\Delta^0 \cup \{f_1, f_2\}) \setminus \{f\}$, $W^2 = W^0$, where $f_1 = (u', s)$, $f_2 = (s, v')$ with $s \in V_2 \setminus \{u', v'\}$. In particular, $\Delta^2_{e'w'} = \Delta^0_{e'w'}$ if $(e', w') \neq (e, w)$ and $\Delta^2_{ew} = (\Delta^0_{ew} \cup \{f_1, f_2\})\{f\}$. As both solutions S^0 and S^2 are feasible, their incidence vectors satisfy (A.1). It follows that $\gamma_f^{ew} = \gamma_{f_1}^{ew} + \gamma_{f_2}^{ew}$. As by A.2, $\gamma_{f_1}^{ew} = \gamma_{f_2}^{ew} = 0$, we get $\gamma_f^{ew} = 0$.

If $f \notin \Delta^0_{ew}$, by considering the same solution S^0 where we add f to Δ^0_{ew}, we obtain that $\gamma_f^{ew} = 0$. We thus have, $\gamma_f^{ew} = 0$ for all $f \in \Delta^0$, $e \in A_1$ and $w \in W$. Hence,

$$\gamma_a^{ew} = 0, \text{ for all } a \in A_2, e \in A_1, \text{ and } w \in W. \tag{A.3}$$

Next, we will show that $\beta^{ew} = 0$, for all $(e, w) \in A_1 \times W$.

Consider an arc $g = (u, v) \in A_1 \setminus F_1^0$. Let us install a subband $\omega \in W$ over g. Let $S^3 = (F_1^3, F_2^3, \Delta^3, W^3)$, such that $F_1^3 = F_1^0$, $F_2^3 = F_2^0 \cup \{g\}$, $\Delta^3 = \Delta^0 \cup \{(u', v')\}$ and $W^3 = W^0 \cup \{\omega\}$.

Solution S^3 is clearly feasible and its incidence vector satisfies (A.1). In consequence, we get β^{gw} = 0 and

$$\beta^{g\omega} = 0, \text{ for all } g \in A_1 \setminus F_1^0 \text{ and } \omega \in W. \tag{A.4}$$

Now suppose that $g = (u, v) \in F_1^0$ (that is g is used in the solution S^0). Let w be a subband installed on g and k be a commodity of K using the pair (g, w). Let $S^4 = (F_1^4, F_2^4, \Delta^4, W^4)$ be a solution obtained from S^0 as follows. We consider two additional arcs $g_1 = (u, s)$ and $g_2 = (s, v)$ of $A_1 \setminus F_1^0$, where $s \in V_1 \setminus \{u, v\}$. And both g_1 and g_2 are added to the solution S^0 by receiving the subband w. In this solution, commodity k is moved from g to path $\{g_1, g_2\}$. In other words, the routing of k uses g_1, g_2 instead of g. Then, S^4 is such that $F_1^4 = F_1^0 \cup \{g_1, g_2\}$, $F_2^4 = F_2^0$, $\Delta^4 = \Delta^0 \cup \{(u', s'), (s', v')\}$, where s' is the node of V_2 associated with s. In addition, note that $W^4 = W^0$ and $\mathcal{C}_k^4 = (\mathcal{C}_k^0 \setminus \{g\}) \cup \{g_1, g_2\}$, while the remaining elements of $\mathcal{C}^0$ do not change in $\mathcal{C}^4$. The solution S^4 is clearly feasible for CMLND-U problem.

Now we will introduce the solution S^5 which is obtained from S^4 by removing the pair (g, w). Recall that, in S^4, (g, w) is not involved anymore in the routing of k. In consequence, the removal of (g, w) does not affect the feasibility of this solution, which is actually ensured since all the constraints of the problem are satisfied. As both S^4 and S^5 are feasible, $(x^{S^4}, y^{S^4}, z^{S^4})$ and $(x^{S^5}, y^{S^5}, z^{S^5})$ verify (A.1). Hence, we get β^{gw} = 0. As g and w are arbitrary in F_1^0 and W, we obtain that

$$\beta^{ew} = 0, \text{ for all } e \in F_1^0 \text{ and for all } w \in W. \tag{A.5}$$

And, by (A.4) and (A.5), we have

$$\beta^{ew} = 0, \text{ for all } e \in A_1 \text{ and for all } w \in W. \tag{A.6}$$

Now let us show that α_{ew}^k = 0, for all $k \in k$, $e \in A_1$, and $w \in W$.

Consider a commodity $\overline{k} \in K$, an arc $g = (u, v) \in A_1 \setminus F_1^0$, and a subband $\omega \in W$. We will install ω over g. Let $S^6 = (F_1^6, F_2^6, \Delta^6, W^6)$ be the solution defined as follows. $F_1^6 = F_1^0 \cup \{g\}$, $F_2^6 = F_2^0$, $\Delta^6 = \Delta^0 \cup \{(u', v')\}$ and $W^6 = W^0 \cup \{\omega\}$. Moreover, $\Delta_{g\omega}^6 = \Delta_{g\omega}^0 \cup \{(u', v')\}$ and $\mathcal{C}_k^6 = \mathcal{C}_k^0$, for all $k \in K \setminus \{\overline{k}\}$ and $\mathcal{C}_{\overline{k}}^6 = \mathcal{C}_{\overline{k}}^0 \cup \{g\}$, while $\Delta_{ew}^6 = \Delta_{ew}^0$, if $(e, w) \neq (g, \omega)$ and $\Delta_{ew}^6 = \Delta_{ew}^0 \cup \{(u', v')\}$ if $(e, w) = (g, \omega)$. S^6 is obviously a feasible solution. Hence, both incidence vectors of S^0 and S^6 verify (A.1), and consequently, we have,

$$\alpha_{g\omega}^{\overline{k}} + \beta^{g\omega} + \gamma_{(u',v')}^{g\omega} = 0,$$

As by (A.3) and (A.6), $\beta^{g\omega} = \gamma_{(u',v')}^{g\omega}$ = 0, we get $\alpha_{g\omega}^{\overline{k}}$ = 0. Since $g \in A_1 \setminus F_1^0$, $\omega \in W$ and $\overline{k} \in K$ are chosen arbitrarily and all the subbands play the same role, we obtain that

$$\alpha_{ew}^k = 0, \text{ for all } k \in K, e \in A_1 \setminus F_1^0 \text{ and } w \in W. \tag{A.7}$$

Suppose now that $g = (o_{\underline{k}}, d_{\underline{k}}) \in F_1^0$. Consider the subband $w_0 \in W^0$, such that (g, w_0) is involved in the routing of some commodity, say $\underline{k}$. Let S^7 be a solution obtained from S^0 as follows. We pick two arcs $g_1 = (o_{\underline{k}}, s)$ and $g_2 = (s, d_{\underline{k}})$ of $A_1 \setminus F_1^0$, with $s \in V_1 \setminus \{o_{\underline{k}}, d_{\underline{k}}\}$. We install w_0 on both g_1 and g_2, and we associate with pairs (g_1, w_0) and (g_2, w_0) paths $\{(o'_{\underline{k}}, s')\}$ and $\{(s', d'_{\underline{k}})\}$, respectively, with $s' \in V_2 \setminus \{o'_{\underline{k}}, d'_{\underline{k}}\}$ the corresponding node of s in V_2. Then $S^7 = (F_1^7, F_2^7, \Delta^7, W^7)$, where

$F_1^7 = (F_1^0 \cup \{g_1, g_2\}) \setminus \{g\}$, $F_2^7 = F_2^0$, $\Delta^7 = \Delta^0 \cup \{(o'_{\underline{k}}, s'), (s', d'_{\underline{k}})\}$ and $W^7 = W^0$. Consider here $\mathcal{C}_k^7 = \mathcal{C}_k^0$, for all $k \in K \setminus \{\underline{k}\}$ and $\mathcal{C}_{\underline{k}}^7 = (\mathcal{C}_{\underline{k}}^0 \cup \{g_1, g_2\}) \setminus \{g\}$. Furthermore, $\Delta_{ew}^7 = \Delta_{ew}^0$ if $(e, w) \notin \{(g_1, w_0), (g_2, w_0)\}$, while $\Delta_{g_1 w_0}^7 = \Delta_{g_1 w_0}^0 \cup \{(o'_{\underline{k}}, s')\}$ and $\Delta_{(s', d'_{\underline{k}}) w_0}^7 = \Delta_{g_2 w_0}^0 \cup \{(s', d'_{\underline{k}})\}$. Solution S^7 is also feasible, and its incidence vector as the one of S^0, verifies equality (A.1). Thus we obtain that

$$\alpha_{gw_0}^{\underline{k}} + \beta^{gw_0} + \gamma_{(o'_{\underline{k}}, d'_{\underline{k}})}^{gw_0} = \alpha_{g_1 w_0}^{\underline{k}} + \alpha_{g_2 w_0}^{\underline{k}} + \beta^{g_1 w_0} + \beta^{g_2 w_0} + \gamma_{(o'_{\underline{k}}, s')}^{g_1 w_0} + \gamma_{(s', d'_{\underline{k}})}^{g_2 w_0},$$

By (A.3), $\gamma_{(o'_{\underline{k}}, d'_{\underline{k}})}^{gw_0} = \gamma_{(o'_{\underline{k}}, s')}^{g_1 w_0} = \gamma_{(s', d'_{\underline{k}})}^{g_2 w_0} = 0$. By (A.6) and (A.7) we also have $\beta^{gw_0} = \beta^{g_1 w_0} = \beta^{g_2 w_0} = 0$ and $\alpha_{g_1 w_0}^{\underline{k}} = \alpha_{g_2 w_0}^{\underline{k}} = 0$. This yields $\alpha_{gw_0}^{\underline{k}} = 0$. As $\underline{k}$, g and w_0 are chosen arbitrarily in K, F_1^0 and W, we get

$$\alpha_{ew}^k = 0, \text{ for all } k \in K, e \in F_1^0, \text{ and } w \in W. \tag{A.8}$$

Hence, by (A.7) and (A.8), we obtain

$$\alpha_{ew}^k = 0, \text{ for all } k \in K, e \in A_1, \text{ and } w \in W. \tag{A.9}$$

By (A.3), (A.5) and (A.9), $\alpha = \beta = \gamma = 0$. Thus, $P(G_1, G_2, K, C)$ can not be included in hyperplane (A.1). Consequently, it is full dimensional. □

Appendix B. Proof of Theorem 3

Suppose that conditions (i) to (iv) of Theorem 3 are fulfilled. Let us denote by $\alpha x + \beta y + \gamma z \geq \delta$ the capacitated cutset inequality induced by T, and let $\widetilde{\mathcal{F}}$ denote the face induced by (10). Then,

$$\tilde{\mathcal{F}} = \{(x, y, z) \in P(G_1, G_2, K, C) : \sum_{e \in \delta_{G_1}^+(T)} \sum_{w \in W} y_{ew} = BP(K^+(T))\}.$$

Lemma 1. *$\tilde{\mathcal{F}}$ is a proper face of $P(G_1, G_2, K, C)$.*

Proof. We first show that $\tilde{\mathcal{F}}$ is a proper face of $P(G_1, G_2, K, C)$. To this end, let us construct a feasible solution $S^0 = (F_1^0, F_2^0, \Delta^0, W^0)$ that satisfies (10) with equality.

We will denote by $K(s, t)$ all the demands of $K(T)$ (respectively in $K(V_1 \setminus T)$) with origin s and destination t and we install $BP(K(s, t))$ different subbands on the arc (s, t) of A_1. Moreover, each commodity k in $K(T)$ (respectively in $K(V_1 \setminus T)$) is associated with the path $\{(s, t)\} = \{(o_k, d_k)\}$ and a subband w_k. Note that, in this solution, a subband w_k may be associated with more than one commodity.

Now, let $u, v \in V_1$ such that $u \in T$ and $v \in V_1 \setminus T$. For every demand (o_k, d_k) of $K^+(T)$, we install on the arcs (o_k, u) and (v, d_k) $BP(K(o_o, d_k))$ new subbands of W, while (u, v) receives $BP(K^+(T))$ new subbands to route the demands of $K^+(T)$. Note that (u, v) is the only arc of the cut $\delta_{G_1}^+(T)$ that is used in this solution. We do the same operation for the commodities of $K^-(T)$. Furthemore, we associate with each pair (e, w) such that w is installed on $e = (i, j)$ the path $\{(i', j')\}$ in G_2. This is possible since G_2 is a complete graph. Notice that, in this solution, each commodity $k \in K(T)$, (respectively $k \in K(V \setminus T)$) uses the subband w_k on path $\{e_k\}$,

e_k = (o_k, d_k) for its routing while the commodities of $K^+(T)$ have a path of length at most three $\{(o_i, u), (u, v), (v, d_i), i \in K^+(T)\}$. The node u (respectively v) can obviously be equal to some o_i (respectively d_i), $i \in K^+(T)$. Moreover, in this solution, every commodity of K uses at least one subband for its routing, and we assume that all the set up subbands are different so that the disjunction constraints (6) are satisfied. Also note that many commodities may share the same subband, however, as $BP(K(s,t))$ subbands are installed for each demand $(s,t) \in T$, we ensure that the capacity constraints (4) are satisfied.

In this solution S^0, a path in G_1 is assigned to each commodity of K. Moreover, a path is also associated with every pair (e, w) such that w is installed on e. Furthermore, both capacity constraints (4) and disjunction constraints (6) are satisfied, as enough different subbands are installed on each arc used in the solution. As by condition (ii), $\lceil \frac{D(K^+(T))}{C} \rceil$ = $BP(K^+(T))$, we have that S^0 induces a feasible solution of $P(G_1, G_2, K, C)$ whose incidence vector belongs to $\widetilde{\mathcal{F}}$. Hence, $\widetilde{\mathcal{F}} \neq \emptyset$. Moreover, as we have supposed that a large enough set of subbands is available, one can install more than $BP(K^+(T))$ different subbands on the arc (u, v) and still have a feasible solution. Clearly, this new solution is not in $\widetilde{\mathcal{F}}$. Hence, $P(G_1, G_2, K, C) \setminus \widetilde{\mathcal{F}} \neq \emptyset$. Therefore, $\widetilde{\mathcal{F}}$ is a proper face of $P(G_1, G_2, K, C)$. □

Note that

- F_1^0 is the set of arcs (o_k, d_k) for $k \in K$ such that $\{o_k, d_k\} \subseteq T \cup (V \setminus T)$ and the paths (o_k, u, v, d_k) (resp. (d_k, v, u, o_k)) for the demands of $K^+(T)$ (resp. $K^-(T)$),
- $F_2^0 = \emptyset$,
- Δ^0 is the set of all the arcs of A_2 that correspond to the paths associated with all the pairs (e, w) such that $e \in F_1^0$ ans w is installed on e.

Now suppose that there exists a facet defining inequality $\lambda x + \mu y + \nu z \geq \xi$ such that

$$\tilde{\mathcal{F}} \subseteq \mathcal{F} = \{(x, y, z) \in P(G_1, G_2, K, C) : \lambda x + \mu y + \nu z = \xi\}.$$

We will show that there exists a scalar $\rho \in \mathbb{R}$ such that $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$.

Lemma 2. *ν_a^{ew} = 0, for all $e \in A_1$, $w \in W$ and $a \in A_2$.*

Proof. Consider an arc $a \in A_2 \setminus \Delta^0$, and a pair $(e^*, w^*) \in A_1 \times W$. Clearly, the solution S^1 = $(F_1^0, F_2^0, \Delta^1, W^0)$, where $\Delta^1_{e_i w_i}$ = $\Delta^0_{e_i w_i}$ if $(e_i, w_i) \neq (e^*, w^*)$ and $\Delta^1_{e^* w^*}$ = $\Delta^0_{e^* w^*} \cup \{a\}$ is a solution of $P(G_1, G_2, K, C)$, and its incidence vector satisfies $\alpha x + \beta y + \gamma z \geq \delta$ with equality. It then follows that

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^1} + \mu y^{S^1} + \nu z^{S^1} = \lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} + \nu_a^{e^* w^*},$$

which implies that $\nu_a^{e^* w^*}$ = 0. Since, a, e^* and w^* are arbitrary in $A_2 \setminus \Delta^0$, A_1 and W, we obtain

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, a \in A_2 \setminus \Delta^0. \tag{B.1}$$

Now let $a = (s', t') \in \Delta^0$, such that $a \in \Delta^0_{e^*w^*}$ for some (e^*, w^*) such that w^* is installed on e^*. Then consider the solution S^2 obtained from S^0 by replacing a by (s', r') and (r', t') in $\Delta^0_{e^*w^*}$, with $(s', r'), (r', t') \in A_2 \setminus \Delta^0$ and $r' \in V_2 \setminus \{s', t'\}$. $S^2 = (F_1^0, F_2^0, \Delta^2, W^0)$, with $\Delta^2_{ew} = \Delta^0_{ew}$ if $(e, w) \neq (e^*, w^*)$, and $\Delta^2_{e^*w^*} = (\Delta^0_{e^*w^*} \setminus \{a\}) \cup \{(s', r'), (r', t')\}$ is obviously feasible for $P(G_1, G_2, K, C)$. As its incidence vector belongs to $\tilde{\mathcal{F}}$ and thus to $\mathcal{F}$, we have

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^2} + \mu y^{S^2} + \nu z^{S^2} =$$

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} - \nu_a^{e^*w^*} + \nu^{e^*w^*}_{(s',r')} + \nu^{e^*w^*}_{(r',t')},$$

which gives that $\nu_a^{e^*w^*} = \nu^{e^*w^*}_{(s',r')} + \nu^{e^*w^*}_{(r',t')}$. As by (B.1), $\nu^{e^*w^*}_{(s',r')} = \nu^{e^*w^*}_{(r',t')} = 0$, it follows that $\nu_a^{e^*w^*} = 0$, which yields

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, a \in \Delta^0. \tag{B.2}$$

By (B.1) and (B.2), we obtain that

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, a \in A_2. \tag{B.3}$$

□

Lemma 3. *$\mu^{ew} = 0$, for all $e \in A_1 \setminus \delta^+_{G_1}(T)$ and $w \in W$.*

Proof. Let $e^* = (s, t) \in A_1 \setminus (F_1^0 \cup F_2^0)$ such that $e^* \notin \delta^+_{G_1}(T)$. Let w^* be a subband of W. We introduce the solution S^3 obtained from S^0 by adding e^* to the subset F_2^0. Thus $S^3 = (F_1^0, F_2^0 \cup \{e^*\}, \Delta^0 \cup \{(s', t')\}, W^0 \cup \{w^*\})$, where $(s', t') \in A_2 \setminus \Delta^0$, induces a feasible solution of $P(G_1, G_2, K, C)$. In addition, $(x^{S^3}, y^{S^3}, z^{S^3}) \in \tilde{\mathcal{F}}$, and then $(x^{S^3}, y^{S^3}, z^{S^3}) \in \mathcal{F}$. Hence

$$\lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} = \lambda x^{S^3} + \mu y^{S^3} + \nu z^{S^3} =$$

$$\lambda x^{S^0} + \mu y^{S^0} + \mu^{e^*w^*} + \nu z^{S^0} + \nu^{e^*w^*}_{(s',t')},$$

implying that $\mu^{e^*w^*} = -\nu^{e^*w^*}_{(s',t')}$. In consequence, by (B.3), we obtain that $\mu^{e^*w^*} = 0$. Since e^* and w^* are arbitrarily selected in $A_1 \setminus (F_1^0 \cup F_2^0)$ and W, we get

$$\mu^{ew} = 0, \text{ for all } e \in A_1 \setminus (F_1^0 \cup F_2^0), e \notin \delta^+_{G_1}(T), w \in W. \tag{B.4}$$

Now consider an arc $e^* = (s, t) \in F_1^0 \cup F_2^0 = F_1^0$ such that $e^* \notin \delta^+_{G_1}(T)$. Let w^* be a subband installed on e^* and assume that $e^* \in \mathcal{C}^0_{k^*}$ for some commodity $k^* \in K$ (that is to say e^* is used in the routing of k^*). Consider the solution S^4 defined as follows. $S^4 = (F_1^0 \cup \{(s, r), (r, t)\}, F_2^0, \Delta^0 \cup \{(s', r'), (r', t')\}, W^0)$, with (s, r), $(r, t) \in A_1 \setminus F_1^0 \cup F_2^0$, $r \in V_1 \setminus \{s, t\}$ and $(s', r'), (r', t') \in A_2 \setminus \Delta^0$, $r' \in V_2 \setminus \{s', t'\}$. Notice that $\mathcal{C}^4_k = \mathcal{C}^0_k$ if $k \neq k^*$ and $\mathcal{C}^4_{k^*} = (\mathcal{C}^0_{k^*} \setminus \{e^*\}) \cup \{(s, r), (r, t)\}$, while $\Delta^4_{(s,r)w^*} = \Delta^0_{(s,r)w^*} \cup \{(s', r')\}$ and $\Delta^4_{(r,t)w^*} = \Delta^0_{(r,t)w^*} \cup \{(r', t')\}$. We will construct a further solution S^5, obtained from S^4 by removing the pair (e^*, w^*) from the solution S^4. More formally, S^5 is such that $S^5 = (F_1^4 \setminus \{e^*\}, F_2^4, \Delta^4, W^4 \setminus \{w^*\})$. Both solutions S^4 and S^5 are feasible for $P(G_1, G_2, K, C)$ and their incidence vectors belong to $\tilde{\mathcal{F}}$ and then, to $\mathcal{F}$. In consequence, it follows that

$$\lambda x^{S^4} + \mu y^{S^4} + \nu z^{S^4} = \lambda x^{S^5} + \mu y^{S^5} + \nu z^{S^5} = \lambda x^{S^4} + \mu y^{S^4} - \mu^{e^*w^*} + \nu z^{S^4}.$$

Hence, we get that $\mu^{e^*w^*} = 0$. Since e^* is arbitrary in $(F_1^0 \cup F_2^0) \setminus \delta^+_{G_1}(T)$, we conclude that

$$\mu^{ew} = 0, \text{ for all } e \in (F_1^0 \cup F_2^0) \setminus \delta^+_{G_1}(T), (e,w) \in \Gamma^0. \tag{B.5}$$

By (B.4) and (B.5) we obtain

$$\mu^{ew} = 0, \text{ for all } e \in A_1 \setminus \delta^+_{G_1}(T), w \in W. \tag{B.6}$$

□

In what follows, we will show that $\lambda^k_{ew} = 0$, for all $k \in K$, $e \in A_1$ and $w \in W$.

Lemma 4. *$\lambda^k_{ew} = 0$, for all $k \in K$, $e \in A_1$ and $w \in W$.*

Proof. Let $e^* = (s,t)$ be an arc $A_1 \setminus (F_1^0 \cup F_2^0)$ that does not belong to $\delta^+_{G_1}(T)$ and let w^* be a subband of W. Consider the solution S^6 obtained from S^0 by installing w^* on e^*, and adding e^* to any subset $\mathcal{C}^0_{k^*}$, $k^* \in K$. This means setting $x^{S^0}_{k^*e^*w^*}$ to 1. Then $S^6 = (F_1^0 \cup \{e^*\}, F_2^0, \Delta^0 \cup \{(s',t')\}, W^0 \cup \{w^*\})$, where $(s',t') \in A_2 \setminus \Delta^0$. Observe that $\mathcal{C}^6_k = \mathcal{C}^0_k$ if $k \neq k^*$ and $\mathcal{C}^6_{k^*} = \mathcal{C}^0_{k^*} \cup \{e^*\}$, while $\Delta^6_{e^*w^*} = \Delta^0_{e^*w^*} \cup \{(s',t')\}$ and $\Delta^6_{ew} = \Delta^0_{ew}$ if $(e,w) \neq (e^*,w^*)$. It is easy to see that S^6 induces a feasible solution of $P(G_1, G_2, K, C)$ whose incidence vector verifies $\lambda x + \mu y + \nu z \geq \xi$ with equality. Hence, we have that

$$\lambda^{k^*}_{e^*w^*} + \mu^{e^*w^*} + \nu^{e^*w^*}_{(s',t')} = 0.$$

Since $\mu^{e^*w^*} = \nu^{e^*w^*}_{(s',t')} = 0$, by (B.3) and (B.6), we obtain that $\lambda^{k^*}_{e^*w^*} = 0$. As e^*, w^* and k^* are arbitrary, we get

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, e \in A_2 \setminus (F_1^0 \cup F_2^0).e \notin \delta^+_{G_1}(T), w \in W, \tag{B.7}$$

Now consider an arc $e^* = (s,t)$ of $(F_1^0 \cup F_2^0)$ and let w^* be a subband of W installed on e^*. Assume without loss of generality that e^* is different from (u,v), and that the pair (e^*,w^*) is associated with the routing of some commodity, say k^*. Let us introduce the solution S^7, obtained from S^0 by replacing e^* in $\mathcal{C}^0_{k^*}$ by two arcs (s,r) and (r,t) of $A_1 \setminus (F_1^0 \cup F_2^0)$. Then, $S^7 = (F_1^0 \cup \{(s,r),(r,t)\}, F_2^0, \Delta^0 \cup \{(s',r'),(r',t')\}, W^0)$, where (s',r'), $(r',t') \in A_2 \setminus \Delta^0$ and $\mathcal{C}^7_{k^*} = (\mathcal{C}^0_{k^*} \setminus \{e^*\}) \cup \{(s,r),(r,t)\}$. Also remark that $\Delta^7_{(s,r)w^*} = \Delta^0_{(s,r)w^*} \cup \{(s',r')\}$ while $\Delta^7_{(r,t)w^*} = \Delta^0_{(r,t)w^*} \cup \{(r',t')\}$. It is clear that S^7 is a feasible solution whose incidence vector is in $\tilde{\mathcal{F}}$ and $\mathcal{F}$. Hence, we have

$$\lambda^{k^*}_{e^*w^*} = \lambda^{k^*}_{(s,r)w^*} + \lambda^{k^*}_{(r,t)w^*},$$

which implies that $\lambda^{k^*}_{e^*w^*} = 0$, as $\lambda^{k^*}_{(s,r)w^*} = \lambda^{k^*}_{(r,t)w^*} = 0$ by (B.7). Furthermore, since (e^*,w^*) is an arbitrary pair in the solution S^0, $e^* \notin \delta^+_{G_1}(T)$, we get

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, (e,w) \in \Gamma^0, e \notin \delta^+_{G_1}(T). \tag{B.8}$$

Now consider a commodity $k^* \in K$. We will show that coefficient λ related to commodities of K and arcs of $\delta^+_{G_1}(T)$ are equal to zero. Two cases may hold here.

Case 1.

Suppose that $k^* \in K \setminus K^+(T)$. Consider an arc e^* of $\delta^+_{G_1}(T)$ and a subband w^* of W. We will assume that $e^* = (u, v)$, since the arcs of the cut $\delta^+_{G_1}(T)$ are interchangeable. Also suppose that w^* is installed on e^*. Consider the solution S^8, obtained from S^0 by associating a new routing path to k^* that uses e^*. This operation means that we install two more new subbands on the arcs (o_{k^*}, u), (v, d_{k^*}) so as to put enough capacity on the path $\{(o_{k^*}, u), e^*, (v, d_{k^*})\}$. As in previous solutions, we also include the corresponding path of G_2 in this solution ($\Delta^8 = \Delta^0 \cup \{(o'_{k^*}, u'), (v', d'_{k^*})\}$). In other words, $S^8 = (F_1^0 \cup \{(o_{k^*}, u), (v, d_{k^*})\}, F_2^0, \Delta^8, W^0 \cup \{w_{k^*, w'_{k^*}}\})$ and $\mathcal{C}^8_{k^*} = \{(o_{k^*}, u), e^*, (v, d_{k^*})\}$. Condition (iii) makes the solution feasible for the problem, as it allows capacity constraints to be satisfied. Thus, S^8 as well as S^0 belong to $\mathcal{F}$, and consequently to $\mathcal{F}$. Hence, both incidence vectors $(x^{S^0}, y^{S^0}, z^{S^0})$ and $(x^{S^0}, y^{S^0}, z^{S^0})$ satisfy the following

$$\lambda x^{S^8} + \mu y^{S^8} + \nu z^{S^8} = \lambda x^{S^0} + \mu y^{S^0} + \nu z^{S^0} + \lambda^{k^*}_{e^* w^*},$$

which yields $\lambda^{k^*}_{e^* w^*} = 0$. Since, k^*, e^* and w^* are arbitrary in $K \setminus P^+$, $\delta^+_{G_1}(T)$ and W, we obtain that

$$\lambda^k_{ew} = 0, \text{ for all } k \in K \setminus P^+, e \in \delta^+_{G_1}(T), w \in W, \tag{B.9}$$

Note that, at this step, we have shown that coefficients $\lambda^k_{ew} = 0$ for all the arcs $e \notin \delta^+_{G_1}$. In consequence, the coefficients corresponding to the previous (resp. new) routing path of k^* are equal to 0.

Case 2.

Now consider the case where $k^* \in K^+(T)$, and let k be a commodity of $K^+(T)$ such that $D^{k^*} + D^k \leq C$. Such a commodity exists because of condition (iv). Let $e^* = (s, t)$ be an arc of $\delta^+_{G_1}(T)$ and let w^* be one of the subbands installed on (u, v). We will construct a solution S^9 obtained from S^0 by moving w^* from arc (u, v) to arc (s, t), and associating with $((s, t), w^*)$ the path $\{(s', t')\}$ in G_2. In this solution, we will also replace (u, v) in the routing path of k^* by $\{(u, s), e^*, (t, v)\}$, where (u, s) and (t, v) are two arcs of $A_1 \setminus \delta^+_{G_1}(T)$. (u, s) and (t, v) also receive the subband w^* and are assigned the paths $\{(u', s')\}$ and $\{(t', v')\}$ in G_2, respectively. S^9 is feasible, since we know, by condition (iv) that capacity constraints (4) are satisfied. Now let us derive a solution S^{10} which slightly differs from S^9 in that we associate (s, t) to k instead of (u, v) in its routing. Again, this is possible thanks to condition (iv). This variation in the solution induces $x^{S^{10}}_{k(s,t)w^*} = 1$ while $x^{S^9}_{k(s,t)w^*} = 0$. Solution S^{10} is clearly feasible, and both incidence vectors $(x^{S^9}, y^{S^9}, z^{S^9})$ and $(x^{S^{10}}, y^{S^{10}}, z^{S^{10}})$ are in $\tilde{\mathcal{F}}$, and then also in $\mathcal{F}$. Thus, we obtain that $\lambda^k_{(s,t)w^*} = 0$. By the interchangeability argument on the elements of $K^+(T)$, $\delta^+_{G_1}(T)$ and W, we get

$$\lambda^k_{ew} = 0, \text{ for all } k \in K^+(T), e \in \delta^+_{G_1}(T), w \in W, \tag{B.10}$$

And, by (B.7), (B.8), (B.9) and (B.10), we finally obtain

$$\lambda^k_{ew} = 0, \text{ for all } k \in K, e \in A_1, w \in W, \tag{B.11}$$

□

Lemma 5. *The μ^{ew} are the same for the arcs of the cut $\delta^+_{G_1}(T)$.*

Proof. Indeed, let $e^* = (s,t)$ be an arc of $\delta^+_{G_1}(T)$, different from (u,v). Recall that $BP(K^+(T))$ different subbands are installed over the arc (u,v). Let $\overline{w}$ be one of these subbands. Consider the solution S^9 where we replace the pair $((u,v),\overline{w})$ in the solution by $((u,s),\overline{w})$, $((s,t),\overline{w})$ and $((t,v),\overline{w})$, with (u,s), $(t,v) \in A_1 \setminus (F_1^0 \cup F_2^0)$. In other words, all the demands of $K^+(T)$ use (s,t) instead of (u,v) to cross the cut $\delta^+_{G_1}(T)$. Comparing solutions S^0 and S^9 gives

$$\mu^{(u,v)\overline{w}} = \mu^{(u,s)\overline{w}} + \mu^{(s,t)\overline{w}} + \mu^{(t,v)\overline{w}} + \nu^{(u,s)\overline{w}}_{(u',s')} + \nu^{(s,t)\overline{w}}_{(s',t')} + \nu^{(t,v)\overline{w}}_{(t',v')}.$$

By (B.3) and (B.6), we obtain that

$$\mu^{(u,v)\overline{w}} = \mu^{(s,t)\overline{w}}.$$

Since (s,t) is arbitrary in $\delta^+_{G_1}(T)$, we get

$$\mu^{ew} = \begin{cases} \rho, & \text{for some } \rho \in \mathbb{R}^*, \text{ for all } e \in \delta^+_{G_1}(T), w \in W, \\ 0, & \text{otherwise.} \end{cases} \tag{B.12}$$

□

By Lemmas 1, 2, 3, 4, 5 and when replacing $(x^{S^0}, y^{S^0}, z^{S^0})$ in hyperplane $\lambda x + \mu y + \nu z \geq \xi$ we obtain

$$\rho \sum_{w\in W} \sum_{e\in\delta^+_{G_1}(T)} y_{ew} = \xi,$$

Note that $\rho \neq 0$, since $\mathcal{F} \neq \emptyset$. Consequently, $\sum_{w\in W}\sum_{e\in\delta^+_{G_1}(T)} y_{ew} = \xi/\rho = \lceil \frac{D(P^+)}{C} \rceil$. Thus, $(\alpha,\beta,\gamma) = \rho(\lambda,\mu,\nu)$, and the proof is complete.

Appendix C. Proof of Theorem 4

Let T be a subset of nodes of V_1 and $\overline{T} = V_1 \setminus T$. Consider the cut $\delta^+_{G_1}(T)$ induced by T, and let F, $\overline{F}$ be a partition of $\delta^+_{G_1}(T)$. Now consider the flow-cutset inequality induced by T and F

$$\sum_{e\in F}\sum_{w\in W} y_{ew} + \sum_{e\in\overline{F}}\sum_{w\in W}\sum_{k\in K^+(T)} x_{kew} \geqslant \lceil \frac{D(K^+(T))}{C} \rceil.$$

Let $\mathcal{F}$ be the hyperplane defined as follows

$$\mathcal{F} = \{(x,y,z) \in P(G_1,G_2,K,C) : \sum_{e\in F}\sum_{w\in W} y_{ew} + \sum_{e\in\overline{F}}\sum_{w\in W}\sum_{k\in K^+(T)} x_{kew} = \lceil \frac{D(K^+(T))}{C} \rceil\}.$$

(i) If $\overline{F} = \emptyset$, then $F = \delta^+_{G_1}(T)$ and (11) is equivalent to

$$\sum_{w\in W}\sum_{e\in\delta^+_{G_1}(T)} y_{ew} \geqslant \lceil \frac{D(K^+(T))}{C} \rceil,$$

which reduces to the cutset inequality (10) and (11) cannot be a facet of $P(G_1, G_2, K, C)$ different from (10). If $F = \emptyset$, then $\overline{F} = \delta^+_{G_1}(T)$ and (11) can be written as

$$\sum_{e\in\delta^+_{G_1}(T)} \sum_{w\in W} \sum_{k\in K^+(T)} x_{kew} \geqslant \lceil \frac{D(K^+(T))}{C} \rceil, \tag{C.1}$$

which implies that the number of commodities allowed to use the cut $\delta^+_{G_1}(T)$ is greater than or equal to $\lceil \frac{D(K^+(T))}{C} \rceil$. Note that $\lceil \frac{D(K^+(T))}{C} \rceil \leqslant |K^+(T)|$, as $D^k \leqslant C$, for all $k \in K^+(T)$. Thus, inequality (C.1) is dominated by inequality

$$\sum_{k\in K^+(T)} \sum_{e\in\delta^+_{G_1}(T)} \sum_{w\in W} x_{kew} \geqslant |K^+(T)|,$$

which is nothing but the sum of the connectivity constraints (2) over the commodities of $K^+(T)$. Thus, (11) cannot define a facet for $P(G_1, G_2, K, C)$.

(ii) Now if $D(K^+(T)) < C$, then (11) is equivalent to

$$\sum_{e\in F} \sum_{w\in W} y_{ew} + \sum_{k\in K^+(T)} \sum_{e\in\overline{F}} \sum_{w\in W} x^k_{ew} \geqslant 1. \tag{C.2}$$

First, suppose that $|K^+(T)| = 1$, that is to say there is only one demand, say k_0, crossing the cut $\delta^+_{G_1}(T)$. By inequalities (4) (capacity constraints), it follows that $x^k_{ew} \leqslant y_{ew}$ holds for all $k \in K$, $e \in A_1$ and $w \in W$. Therefore

$$\sum_{e\in F} \sum_{w\in W} y_{ew} + \sum_{e\in\overline{F}} \sum_{w\in W} x^{k_0}_{ew} \geqslant \sum_{e\in F} \sum_{w\in W} x^{k_0}_{ew} + \sum_{e\in\overline{F}} \sum_{w\in W} x^{k_0}_{ew} \geqslant 1,$$

the last inequality comes from inequality (2) (connectivity constraints). In consequence, inequality (11) cannot define a facet.

Now suppose that $|K^+(T)| \geqslant 2$. We claim that no solution of the face defined by (11) can use $\overline{F}$. Indeed, if such a solution S uses $\overline{F}$, then it cannot use F. For otherwise, we would have the left hand side greater than or equal to 2. In consequence, all the demands have to use $\overline{F}$. However, as $|K^+(T)| \geqslant 2$, at least two variables x^k_{ew} would have value 1, and hence $\mathcal{X}^S$ would not be in the face of (11), a contradiction. As a consequence, we have that the face induced by (11)in contained in $\subseteq \{(x, y, z) \in P(G_1, G_2, K, C) : x^k_{ew} = 0,$ for all $e \in \overline{F}, k \in K, w \in W\}$. Thus (11) cannot be a facet.

(iii) If $D(K^+(T))/C$ is integer, then (11) can be obtained from inequalities (2), (4) and the trivial constraints $x^k_{ew} \geqslant 0$. Thus, it cannot be facet defining.

(iv) Suppose the condition does not hold. Then $\frac{D(K^+(T))}{C} < BP(K^+(T))$. We claim that no solution S of CMLND-U may have its incidence vector satisfying inequality (11) with equality. In fact, to be routed, S needs at least $BP(K^+(T))$ subbands. This implies that at least

$BP(K^+(T))$ subbands should be installed on the arcs of $\delta^+_{G_1}(T)$. In consequence, at least $BP(K^+(T))$ variables among y_{ew}, $e \in F$, $w \in W$ and x^k_{ew}, $k \in K^+(T)$, $e \in \overline{F}$, $w \in W$ must have value 1. Hence, $\mathcal{X}^S$ does not satisfy (11) with equality. In consequence, (11) does not define a proper face and cannot then be facet defining.

(v) Suppose that $BP(K^+(T)) = |K^+(T)|$. We may assume that condition (iv) holds. For otherwise, as previously shown, (11) cannot define a facet. Hence (11) can be written as

$$\sum_{e\in F}\sum_{w\in W} y_{ew} + \sum_{k\in K^+(T)}\sum_{e\in\overline{F}}\sum_{w\in W} x^k_{ew} \geqslant BP(K^+(T)) = |K^+(T)| \tag{C.3}$$

as $BP(K^+(T))$ is a lower bound on the number of subbands used to route the demands of $K^+(T)$. Since $D^k \leqslant C$ for all $k \in K$, $|K^+(T)|$ is an upper bound on the the installed subbands on $\delta^+(T)$. As a consequence, every solution whose incidence vector satisfies inequality (C.3) also verifies the equation

$$\sum_{k\in K^+(T)}\sum_{e\in\delta^+(T)}\sum_{w\in W} x^k_{ew} = |K^+(T)|. \tag{C.4}$$

As (C.3) is not a multiple of (C.4) and the polytope $P(G_1, G_2, K, C)$ is full dimensional, the inequality cannot define a facet.

(vi) Suppose on the contrary that for all $Q \subseteq K^+(T)$ we have $BP(K^+(T) \setminus Q) + |Q| > BP(K^+(T))$. If (11) defines a facet, there must exist a solution S which uses arcs of $\overline{F}$. For otherwise, one would have $\mathcal{F} \subseteq \{(x, y, z) \in P(G_1, G_2, K, C) : x^k_{ew} = 0, \text{ for all } k \in K, e \in \overline{F}, w \in W\}$ implying that (11) cannot define a facet. Therefore, let S be a solution such that $\mathcal{X}^S \in \mathcal{F}$ and some demands, say $Q \subseteq K^+(T)$, use $\overline{F}$. The minimum number of subbands needed for routing the demands of $K^+(T) \setminus Q$ (on F) is $BP(K^+(T) \setminus Q)$. Therefore, $\sum_{e\in F}\sum_{w\in W} y_{ew} \geqslant BP(K^+(T) \setminus Q)$. Since $|Q|$ demands cross $\overline{F}$, we must have $\sum_{k\in Q}\sum_{e\in\overline{F}}\sum_{w\in W} x^k_{ew} = |Q|$. In consequence, $\mathcal{X}^S$ cannot satisfy (11) with equality, contradicting the fact that $\mathcal{X}^S \in \mathcal{F}$. Hence, $\mathcal{F} = \emptyset$, and thus $\mathcal{F}$ cannot define a facet.

(vii) Finally, suppose that there exists a commodity $k \in K \setminus K^+(T)$ such that $BP(K^+(T) \cup \{k\}) \geqslant BP(K^+(T)) + 1$. Then there would be no solution of $P(G_1, G_2, K, C)$ with $x^k_{ew} = 1$ for $e \in F \cup \overline{F}$, $w \in W$, meaning that k cannot cross the cut $\delta^+_{G_1}(T)$. In this case, $\mathcal{F}$ is included in

$$\overline{\mathcal{F}} = \{(x, y, z) \in P(G_1, G_2, K, C) : x^k_{ew} = 0, \text{ for } e \in F \cup \overline{F}, w \in W\},$$

and thus, (11) cannot define a facet.

Appendix D. Proof of Theorem 5

Suppose that conditions (i) to (vii) are satisfied. Let $\alpha x + \beta y + \gamma z \geq \delta$ denote the flow-cutset inequality produced by T and F, and let

$$\tilde{\mathcal{F}} = \{(x, y, z) \in P(G_1, G_2, K, C) : \sum_{e\in F}\sum_{w\in W} y_{ew} + \sum_{k\in P^+}\sum_{e\in\overline{F}}\sum_{w\in W} x_{kew} = \lceil\frac{D(K^+(T))}{C}\rceil\},$$

Let us first show that $\tilde{\mathcal{F}} \neq \emptyset$. To this end, we will construct a solution S^0 whose incidence vector belongs to $\tilde{\mathcal{F}}$.

Lemma 6. *$\tilde{\mathcal{F}}$ is a proper face.*

Proof. We install, for each $k \in K_{in}(T)$ (resp. $k \in K_{in}(V \setminus T)$), a subband w_k on the arc (o_k, d_k). This is to associate with every commodity of $K_{in}(T)$ (resp. $K_{in}(V \setminus T)$) a path linking o_k and d_k composed by one arc, and entirely contained in T (resp. in $V \setminus T$). This solution is such that each arc (i, j) of A_1 with $i, j \in T$ (resp. $i, j \in V \setminus T$), receives as many subbands as there exist commodities with $(o_k, d_k) = (i, j)$, $k \in K_{in}(T)$ (resp. $k \in K_{in}(V \setminus T)$). In other words, every commodity k of $K_{in}(T) \cup K_{in}(V \setminus T)$ is associated with the pair (e_k, w_k) for its routing, where $e_k = (o_k, d_k)$.

Recall that $K^+(T)$ (resp. $K^-(T)$) are the sets of commodities of K having their origin in T (resp. $V \setminus T$) and their destination in $V \setminus T$ (resp. T). Consider two nodes u, s in T and two nodes v, t in $V \setminus T$. Note that u, s (resp. v, t) may be the same, and both arcs (u, v) and (s, t) belong to the directed cut $\delta^+_{G_1}(T)$. We can assume, without loss of generality, that $(u, v) \in F$ and $(s, t) \in \overline{F}$. Now, for every commodity $k \in K^+(T)$ (resp. $K^-(T)$), we install a subband w_k (resp. w'_k) over the arc (o_k, u) (resp. on (o_k, v)). Similarly, we install a subband w_k over (v, d_k) (resp. a subband w'_k over (u, d_k)), for every $k \in K^+(T)$ (resp. $K^-(T)$). We then set up $\lceil \frac{D(K^+(T))}{C} \rceil$ different subbands on the arc (u, v) so that all the commodities of $K^+(T)$ may be routed across $\delta^+_{G_1}(T)$ using (u, v). Note that we exactly need $BP(K^+(T))$ subbands to pack the commodities of $K^+(T)$. The same is done on the arc (v, u) so as the commodities of $K^-(T)$ may be routed as well from their origins in $V \setminus T$ to their destinations in T using the cut $\delta^-_{G_1}(T)$. Remark that nodes o_k and u (resp. d_k and v) may coincide. Observe that (u, v) is the unique arc of $\delta^+_{G_1}(T)$ used in this solution. Now we assign to each pair (e, w) with $w \in W$ installed on $e = (i, j) \in A_1$ the path $\{(i', j')\}$ in G_2. This is possible since G_2 is a complete graph, and all the subbands used in this solution are supposed to be different. So both constraints (5) and (6) are satisfied. The solution S^0 is clearly feasible for the CMLND-U problem. Moreover, its incidence vector is such that

$$\sum_{e \in F} \sum_{w \in W} y^{S^0}_{ew} = \sum_{w \in W} y^{S^0}_{(u,v)w} = \lceil \frac{D(K^+(T))}{C} \rceil,$$
$$\sum_{k \in K^+(T)} \sum_{e \in \overline{F}} \sum_{w \in W} x^{S^0}_{kew} = 0.$$

Thus, by condition (iv), solution S^0 satisfies $\alpha x + \beta y + \gamma z \geq \delta$ with equality. Hence, $\widetilde{\mathcal{F}} \neq \emptyset$ and $\widetilde{\mathcal{F}} \neq P(G_1, G_2, K, C)$ is a proper face of $P(G_1, G_2, K, C)$. □

More formally, solution S^0 described above is such that $S^0 = (F^0_1, F^0_2, \Delta^0, W^0)$ where F^0_1 is the set of all arcs of A_1 used by the commodities and $F^0_2 = \emptyset$. Δ^0 contains the set of paths assigned to the installed subbands and W^0 is the set of subbands used in S^0. Observe that, in this solution, a path is assigned to each commodity of K. Indeed, $\mathcal{C}^0_k = \{(o_k, d_k)\}$ if $k \in K_{in}(T) \cup K_{in}(V \setminus T)$, $\mathcal{C}^0_k = \{(o_k, u), (u, v), (v, d_k)\}$ if $k \in K^+(T)$, and $\mathcal{C}^0_k = \{(o_k, v), (v, u), (u, d_k)\}$ if $k \in K^-(T)$. Moreover, for each pair (e, w) such that w is installed on $e = (i, j)$ $\Delta^0_{ew} = \{(i', j')\}$, with $(i', j') \in A_2$.

Now let $\lambda x + \mu y + \nu z \geq \xi$ be a constraint that defines a facet of $P(G_1, G_2, K, C)$ and suppose that

$$\widetilde{\mathcal{F}} \subseteq \mathcal{F} = \{(x, y, z) \in P(G_1, G_2, K, C) : \lambda x + \mu y + \nu z = \xi\}.$$

We will show that there exists a scalar $\rho \in \mathbb{R}$ such that $(\alpha, \beta, \gamma) = \rho(\lambda, \mu, \nu)$.

Lemma 7. *ν_a^{ew} = 0, for all $e \in A_1$, $w \in W$ and $a \in A_2$*

Proof. Similar to proof of Lemma 2. □

Lemma 8. *μ^{ew} = 0, for all $(e, w) \in (A_1 \setminus F) \times W$.*

Proof. Similar to proof of Lemma 3. □

The case where $e^* \in F$ will be treated further in the proof. In what follows, we will examine the λ coefficients related to commodities not in $K^+(T)$.

Lemma 9. *λ_{ew}^k = 0, for all $k \in K \setminus K^+(T)$, $e \in A_1$, $w \in W$.*

Proof. Consider a commodity k^* of $K \setminus K^+(T)$. Let e = (i, j) and w be an arc of A_1 and a subband of W, respectively, such that w is not already installed on e. The case where $e \in A_1 \setminus F$ is treated similarly as we did in proof of Lemma 4. Let us consider an arc e of F. Note that if $|F|$ = 1, then e = (u, v). Let w be some subband installed on e, such that w still has enough residual capacity to carry k^* in addition to the commodities already using it. Because of condition (vii), we know that such subband exists. Consider the solution S^1, obtained from S^0 by associating a path using e to the commodity k^* instead of its initial routing, that is to say k^* is routed along $\{(o_{k^*}, i), e, (j, d_{k^*})\}$. A subband is installed on both arcs (o_{k^*}, i) and (j, d_{k^*}) and those arcs are associated the paths $\{(o'_{k^*}, i')\}$ and $\{(j', d'_{k^*})\}$ in G_2, respectively. It is clear that S^1 is feasible and its incidence vector belongs to $\widetilde{\mathcal{F}}$. Thus comparing S^1 and S^0 yields

$$\lambda_{o_{k^*} d_{k^*} w_{k^*}}^{k^*} = \lambda_{o_{k^*} i w_{k^*}}^{k^*} + \lambda_{ew}^{k^*} + \lambda_{j d_{k^*} w'_{k^*}}^{k^*}$$

which is equivalent to $\lambda_{ew}^{k^*}$ = 0 (since $\lambda_{fw}^{k^*}$ = 0, for all $f \in A_1 \setminus F$, $w \in W$). As k^*, e and w are arbitrary in $K \setminus K^+(T)$, F and W, we get $\lambda_{ew}^k = 0$, for all $k \in K \setminus K^+(T)$, $e \in F$, $w \in W$. □

Now, let us look at the λ coefficients related to the commodities of $K^+(T)$.

Lemma 10. *λ_{ew}^k = 0, for all $k \in K^+(T)$, $e \in A_1 \setminus \overline{F}$, $w \in W$.*

Proof. Let k^* be a commodity of $K^+(T)$ and e = (i, j) an arc of A_1. Again, the case of $e \in A_1 \setminus (F \cup \overline{F})$ is discussed in the proof of Lemma 4, and we will assume that $e \in \delta_{G_1}^+(T)$ (that is to say $e \in F \cup \overline{F}$). Two case may then be considered.

Suppose that $e \in F$, and let us show that λ_{ew}^k = 0, for $k \in K^+(T)$, $e \in F$ and $w \in W$.

(a) First, suppose that $|F|$ = 1, that is to say F = e = $\{(u, v)\}$. Let k^* be a commodity of $K^+(T)$ and let w be some subband of W installed on e in the solution S^0. Consider the solution S^2 obtained from S^0 as follows. The subband w is involved in the routing of k^* while the remaining $BP(K^+(1))$ - 1 subbands are re-assigned for the routing of the left $K^+(T) \setminus \{k^*\}$ commodities using (u, v). Condition (ii) ensures that this induces a feasible solution.

Now let k be a commodity of $K^+ \setminus \{k^*\}$ such that $D^{k^*} + D^k \leq C$. This is possible since condition (iii) guarantees that such a commodity exists. Consider the solution S^3, which slightly differs from S^2 as follows. We move the commodity k from its initial subband to w, that is k will use the pair (e, w) for its routing while the remaining elements of the solution S^2 are the same. In other words, $x^{S^3}_{kew}$ is set to 1, while $x^{S^2}_{kew}$ = 0. Clearly, S^3 is feasible for the problem, and both incidence vectors of S^2 and S^3 belong to $\widetilde{\mathcal{F}}$, and then to $\mathcal{F}$. Hence, we obtain that λ^k_{ew} = 0. As k^* is arbitrary in $K^+(T)$, we get λ^k_{ew} = 0, for all $k \in K^+(T)$, $e \in F$, $w \in W$.

In what follows, we will show that the λ coefficients related to commodities of $K^+(T)$ on arcs of $\overline{F}$ and μ coefficients for F are equal.

Let k^* be some commodity of $K^+(T)$ and let w the subband used for its routing along the arc (u, v). Consider the solution S^4 obtained from S^0 as follows. We move the subband w from (u, v) to (s, t) and install two subbands w' and w'' on the arcs (o_{k^*}, s) and (t, d_{k^*}). Recall that $(u, v) \in F$ and $(s, t) \in \overline{F}$. We then replace the initial routing of k^*, that uses (u, v), by $\{(o_{k^*}, s), (s, t), (t, d_{k^*})\}$. This solution is feasible as condition (ii) ensures that enough capacity is available on (u, v) to carry the commodities of $K^+(T) \setminus \{k^*\}$. Clearly, S^4 is feasible for the problem, and comparing $(x^{S^4}, y^{S^4}, z^{S^4})$ and $(x^{S^0}, y^{S^0}, z^{S^0})$ gives

$$\lambda^{k^*}_{(u,v)w} + \mu^{(u,v)w} = \lambda^{k^*}_{(s,t)w} + \mu^{(s,t)w},$$

which, as shown above, implies that $\mu^{(u,v)w} = \lambda^{k^*}_{(s,t)w}$. As k^*, w and (s, t) are arbitrary in $K^+(T)$, W and $\overline{F}$, we obtain that

$$\mu^{(u,v)w} = \lambda^k_{(s,t)w} = \rho, \text{ for all } k \in K^+(T), (s, t) \in \overline{F}, w \in W,$$

for some $\rho \in \mathbb{R}$.

(b) Suppose now that $|F| \geq 2$. Let e = (i, j) be an arc of F different from (u, v). Consider two commodities k', k'' of $K^+(T)$, such that $D^{k'} + D^{k''} \leq C$. Condition (iii) guarantees that such two commodities exist. Recall that in S^0, all the commodities of $K^+(T)$ are routed along (u, v). Let w be the subband installed on (u, v) and involved in the routing of k'. We will consider a new solution S^5 obtained from S^0 as follows. We install w on (i, j) and replace the routing path $\{(o_{k'}, u), (u, v), (v, d_{k'})\}$ of k' by $\{(o_{k'}, i), (i, j), (j, d_{k'})\}$, after installing two new subbands on the arcs $(o_{k'}, i)$, $(j, d_{k'})$. By condition (ii), the remaining commodities can be routed along (u, v) using the left $BP(K^+(T)) - 1$ subbands. It is clear that S^5 is a feasible solution for the problem. Now consider the solution S^6, obtained from S^5 by associating with the commodity k'' one more arc, namely (i, j). Note that k'' is still routed through (u, v). Arc (i, j) is just added to the solution. As $D^{k'} + D^{k''} \leq C$, the capacity constraint (4) related to $((i, j), w)$ is satisfied. Hence S^6 is feasible. Moreover, as the incidence vectors of S^5 and S^6 both belong to $\widetilde{\mathcal{F}}$, and hence to $\mathcal{F}$, we have that $\lambda^{k''}_{(i,j)w}$ = 0. Consequently, λ^k_{ew} = 0, for all $k \in K^+(T)$, $e \in F$, $w \in W$.

□

We will go over the coefficients related to the demands in $K^+(T)$ and the arcs of $\overline{F}$ at the end of the proof. Let us first get back to the coefficients μ related to the arcs of F.

Simply compare solutions S^5 and S^0, by Lemma 10 we that

$$\mu^{(i,j)w} = \mu^{(u,v)w}.$$

Since the arc (i,j) is arbitrary in F, we get the equality of coefficients μ for the arcs of F. Hence, we conclude that there exists a positive scalar $\rho \in \mathbb{R}$, such that

$$\mu^{ew} = \rho, \text{ for all } e \in F, w \in W.$$

The last case of our proof involves the coefficients of the commodities of $K^+(T)$ and the arcs of $\overline{F}$.

Consider a commodity $k^* \in K^+(T)$, and let w be a subband installed on an arc (u,v), such that the pair $((u,v),w)$ is involved in the routing of k^*. Assume that w is moved from (u,v) to the arc (s,t) (which belongs to $\overline{F}$). This allows to introduce the latter arc in the solution S^0. Let us install subbands w' and w'' on the arcs (o_{k^*}, s) and (t, d_{k^*}), respectively. In this way, k^* is assigned the path $\{(o_{k^*}, s), (s,t), (t, d_{k^*})\}$ instead of the initial routing path $\{(o_{k^*}, u), (u,v), (v, d_{k^*})\}$. And the sections of this path are themselves assigned the paths $\{(o'_{k^*}, s')\}$, $\{(s',t')\}$ and $\{(t', d'_{k^*})\}$ in G_2, respectively.

Let us denote by S^8 the solution described above, and give in what follows its different subsets. $S^8 = (F_1^0 \cup \{(o_{k^*}, s), (s,t), (t, d_{k^*})\}, F_2^0, \Delta^0 \cup \{(o'_{k^*}, s'), (s',t'), (t', d'_{k^*})\}, W^0 \cup \{w', w''\})$. S^8 is obviously feasible, and $(x^{S^8}, y^{S^8}, z^{S^8})$ together with $(x^{S^0}, y^{S^0}, z^{S^0})$ belong to $\widetilde{\mathcal{F}}$ and then to $\mathcal{F}$. In addition, S^8 is such that

$$\begin{aligned}
&\sum_{e\in F}\sum_{w\in W} y^{S^8}_{ew} = \sum_{w\in W} y^{S^8}_{(u,v)w} = BP(K^+(T)) - 1,\\
&\sum_{e\in \overline{F}}\sum_{w\in W} y^{S^8}_{ew} = y^{S^8}_{(s,t)w} = 1,\\
&\sum_{k\in K^+(T)}\sum_{e\in \overline{F}}\sum_{w\in W} x^{S^8}_{kew} = x^{S^8}_{k^*(s,t)w} = 1.
\end{aligned}$$

Comparing both incidence vectors $(x^{S^8}, y^{S^8}, z^{S^8})$ and $(x^{S^0}, y^{S^0}, z^{S^0})$ implies the following

$$\lambda x^{S^8} + \mu y^{S^8} + \nu z^{S^8} = \lambda x^{S^0} - \lambda^{k^*}_{(u,v)w} + \lambda^{k^*}_{(o_{k^*},s)w'} + \lambda^{k^*}_{(s,t)w} + \lambda^{k^*}_{(t,d_{k^*})w''}$$

$$+\mu y^{S^0} - \mu^{(u,v)w} + \mu^{(o_{k^*},s)w'} + \mu^{(s,t)w} + \mu^{(t,d_{k^*})w''} + \nu z^{S^0} + \nu^{(o_{k^*},s)w'}_{(o'_{k^*},s')} + \nu^{(s,t)w}_{(s',t')} + \nu^{(t,d_{k^*})w''}_{(t',d'_{k^*})}.$$

By Lemmas 7, 8, 9, 10, it follows from the previous equality that $\lambda^{k^*}_{(s,t)w} = \mu^{(u,v)w}$. As k^* is arbitrary in $K^+(T)$, we conclude that all the coefficients λ of $K^+(T)$ and $\overline{F}$ are equal up to the scalar ρ. That is

$$\lambda^k_{ew} = \rho, \text{ for } k \in K^+(T), e \in \overline{F}, w \in W.$$

To summarize, all together we get

$$\nu_a^{ew} = 0, \text{ for all } e \in A_1, w \in W, a \in A_2,$$
$$\mu^{ew} = \begin{cases} \rho, & \text{for some scalar } \rho \in \mathbb{R}_+^*, \text{ for all } e \in F, w \in W, \\ 0, & \text{otherwise.} \end{cases}$$
$$\lambda_{ew}^k = \begin{cases} \rho, & \text{for } k \in K^+(T), e \in \overline{F}, w \in W, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\rho \neq 0$, since $\mathcal{F} \neq \emptyset$. Thus, replacing the values of our coefficients in $\lambda x + \mu y + \nu z \geq \xi$, yields

$$\sum_{e \in F} \sum_{w \in W} \rho y^{ew} + \sum_{k \in K^+(T)} \sum_{e \in \overline{F}} \sum_{w \in W} \rho x_{ew}^k \geq \xi$$

And, as $(x^{S^0}, y^{S^0}, z^{S^0}) \in \mathcal{F}$, it follows that $\rho BP(K^+(T)) = \xi$ and hence $\frac{\xi}{\rho} = BP(K^+(T))$, which completes the proof of Theorem 3.

Appendix E. Proof of Theorem 6

Necessity

(i) The validity condition for inequalities (17) states that $p \geqslant |S| - BP(S)$. It is clear that any value of $p > |S| - BP(S)$ would induce a valid inequality which is redundant with respect to

$$\sum_{w \in W} \sum_{k \in S} x_{ew}^k \leq \sum_{w \in W} y_{ew} + (|S| - BP(S)), \text{ for all } e \in A_1,$$

and thus, that could not define a facet of $P(G_1, G_2, K, C)$.

(ii) Now let us denote by s the largest element of $K \setminus S$ such that $BP(S \cup \{s\}) = BP(S) + 1$. Then, inequality (17) with respect to $S \cup \{s\}$ can be written as

$$\sum_{w \in W} \sum_{k \in S \cup \{s\}} x_{ew}^k \leq \sum_{w \in W} y_{ew} + |S| + 1 - BP(S \cup \{s\}) = \sum_{w \in W} y_{ew} + p. \quad \text{(E.1)}$$

However, (E.1) dominates (17). Thus the latter cannot define a facet of $P(G_1, G_2, K, C)$.

(iii) Now let s' denote the smallest element of S, and suppose that $BP(S \setminus \{s'\}) = BP(S) = |S| - p$. Then inequality (17) with respect to $S \setminus \{s'\}$ can be written as

$$\sum_{w \in W} \sum_{k \in S \setminus \{s'\}} x_{ew}^k \leqslant \sum_{w \in W} y_{ew} + |S| - 1 - BP(S \setminus \{s'\}) = \sum_{w \in W} y_{ew} + p - 1. \quad \text{(E.2)}$$

Note that inequalities (17) can be obtained as a linear combination of (E.2) and $\sum_{w \in W} x_{ew}^{s'} \leqslant 1$ (constraints (3) in the formulation of the problem).

Sufficiency

Suppose that conditions (i) to (iii) of Thorem 6 are fulfilled. Let us denote by $\alpha x + \beta y + \gamma z \leqslant \delta$ the inequality (17) induced by $\tilde{S}$ and $\tilde{e}$ = (u, v), and let $widetilde\mathcal{F}$ be the face defined as follows

$$\widetilde{\mathcal{F}} = \{(x, y, z) \in P(G_1, G_2, K, C) : \sum_{k \in \tilde{S}} \sum_{w \in W} x^k_{\tilde{e}w} = \sum_{w \in W} y_{\tilde{e}w} + p\}.$$

We first show that $\widetilde{\mathcal{F}}$ is a proper face of $P(G_1, G_2, K, C)$. To this end, we construct a feasible solution S^1, whose incidence vector belongs to $\widetilde{\mathcal{F}}$.

The solution S^1 is obtained from the solution S^0 introduced in the proof of Theorem 1 as follows. We install a set of $BP(\tilde{S})$ non previously used and different subbands $\tilde{W} \subseteq W$ over the arc $\tilde{e}$ and we add the pairs $(\tilde{e}, w)$, $w \in \tilde{W}$ to S^0. In other words, we let $y^{S^1}_{\tilde{e}w}$ = $y^{S^0}_{\tilde{e}w}$ + 1, for all $w \in \tilde{W}$. We then associate to every pair $(\tilde{e}, w)$, $w \in \tilde{W}$ of the solution a path in G_2 that is the arc $\tilde{a} = (u', v') \in A_2$. This is possible since the subbands of $\tilde{W}$ are newly installed and are not assigned physical paths in the solution S^0. The disjunction constraints are therefore satisfied. Now let us associate with the commodities of $\tilde{S}$ the pairs $(\tilde{e}, w)$, $w \in \tilde{W}$, in addition to their initial routing paths. Such assignment is possible since there are enough subbands installed on $\tilde{e}$ so that the capacity constraint (4) for every pair $(\tilde{e}, w)$ such that $w \in W$ is installed on $\tilde{e}$ is satisfied. It is clear that the solution S^1 is feasible and $(x^{S^1}, y^{S^1}, z^{S^1})$ belongs to $\tilde{\mathcal{F}}$. Hence $\widetilde{\mathcal{F}} \neq \emptyset$, and also different from $P(G_1, G_2, K, C)$. Therefore, it is a proper face of $P(G_1, G_2, K, C)$.

Now suppose that there exists a facet-defining inequality λx + μy + $\nu z \geqslant \xi$, such that

$$\tilde{\mathcal{F}} \subseteq \mathcal{F} = \{(x, y, z) \in P(G_1, G_2, K, C) : \lambda x + \mu y + \nu z = \xi\}.$$

We will show that there exists a scalar $\rho \in \mathbb{R}$ such that (α, β, γ) = $\rho(\lambda, \mu, \nu)$.

Lemma 11. *ν^a_{ew} = 0, for all $e \in A_1$, $w \in W$, $a \in A_2$.*

Proof. Similar to proof of Lemma 2. □

Lemma 12. *μ_{ew} = 0, for all $e \in A_1 \setminus \{\tilde{e}\}$, $w \in W$.*

Proof. Similar to proof of Lemma 3. □

Lemma 13. *λ^k_{ew} = 0, for all $k \in K \setminus \tilde{S}$, $e \in A_1$ and $w \in W$.*

Proof. Similar to proof of Lemma 4. □

It is easy to show that λ^k_{ew} = 0, for any commodity $k \in \widetilde{S}$, $e \in A_1 \setminus \{\tilde{e}\}$, $w \in W$, by constructing further feasible solutions where k is shifted to a routing path that avoids the use of arc $\tilde{e}$. Besides, if we pick a commodity k^* in $K \setminus \tilde{S}$ and a subband installed on $\tilde{e}$, say $\tilde{w}$, then we can easily see that a solution obtained by associating the pair $(\tilde{e}, \tilde{w})$ to $\tilde{k}$ in addition to its routing path remains feasible and its incidence vector belongs to both $\tilde{\mathcal{F}}$ and $\mathcal{F}$ (by condition (ii) of Theorem 6). This observation implies that $\lambda^k_{\tilde{e}w}$ = 0, for all $k \in K \setminus \widetilde{S}$, $w \in W$.

Now let k^* be a commodity arbitrarily choosen in $\widetilde{S}$, and let $\tilde{w}$ be the subband of W such that ($\tilde{e}$, $\tilde{w}$) is associated with the routing of k^*. We will construct a solution S^2 from S^1 such that all the commodities of $\widetilde{S} \setminus \{k^*\}$ use $\widetilde{W} \setminus \{\tilde{w}\}$ for their routing and $\tilde{w}$ is completely devoted to the commodity k^*. Such a solution is feasible and its incidence vector belongs to $\widetilde{\mathcal{F}}$ as well as $\mathcal{F}$ thanks to condition (iii).

Now if we remove some commodity, say k^*, of $\widetilde{S}$ from $\tilde{e}$ and move it to a routing path that does no longer use $\tilde{e}$ (that is to say only commodities of $\widetilde{S} \setminus \{k^*\}$ use $\tilde{e}$), then by condition (iii), setting $\sum_{w \in W} y_{\tilde{e}w}$ to $BP(\widetilde{S})$ - 1 keeps the new solution feasible. Moreover, its incidence vector belongs to $\widetilde{\mathcal{F}}$ and then to $\mathcal{F}$. As a consequence, we obtain, by comparing both solutions, that $\lambda^{k^*}_{\tilde{e}\tilde{w}}$ = - $\mu_{\tilde{e}\tilde{w}}$. Since k^* and $\tilde{w}$ are arbitrarily chosen in $\widetilde{S}$ and W, respectively, we obtain that

$$\mu^{\tilde{e}w} = -\rho, \qquad \text{for all } w \in W, \tag{E.3}$$

$$\lambda^k_{\tilde{e}w} = \rho, \quad \text{for all } k \in \tilde{S}, w \in W, \tag{E.4}$$

where $\rho \in \mathbb{R}^+$. Finally, by replacing $(x^{S^0}, y^{S^0}, z^{S^0})$ in the hyperplane defined by $\lambda x + \mu y + \nu z \leqslant \xi$, we obtain $\lambda|S| - \mu BP(S) = \xi$. By condition (i), we have that $|S| - BP(S) = p$, yielding λ = 1, μ = 1 and ξ = p and the proof is complete.

Appendix F. Proof of Theorem 7

Suppose that conditions (i) and (ii) hold. Denote inequality (6) by $\alpha x + \beta y + \gamma z \leqslant \delta$ and suppose there is a facet defining inequality $\lambda x + \mu y + \nu z \leqslant \xi$ such that

$$\widetilde{\mathcal{F}} = \{(x, y, z) \in P(G_1, G_2, K, C) : \alpha x + \beta y + \gamma z = \delta\} \subset$$

$$\mathcal{F} = \{(x, y, z) \in P(G_1, G_2, K, C) : \lambda x + \mu y + \nu z = \xi\}.$$

It suffises to show that there is a scalar h such that $h(\alpha, \beta, \gamma) = (\lambda, \mu, \nu)$.

To this end, consider the solution $S^0 = (F^0_1, F^0_2, \Delta^0, W^0)$ defined as follows. for each commodity $k \in K \setminus S'$, we consider a path in G_1 consisting of the arc (o_k, d_k). We install over this arc a subband. **All the subbands installed over an arc are supposed to be different**. Now, for a commodity in S' consider the path $\{(o_k, u), (u, v), (v, d_k)\}$ (recall that $e = (u, v)$). We install on arc e $BP(S')$ different subbands (denoted by W_e) and on each arc (o_k, u) (resp. (v, d_k)) a subband. Then we associate with each subband installed on an arc (u_1, v_1) a path in G_2 consisting of the arc (u'_1, v'_1). Let $(F^0_1, F^0_2, \Delta^0, W^0)$ be the solution given by $F^0_1 = \{(o_k, d_k), k \in K \setminus S'\} \cup \{(o_k, u), (v, d_k), k \in S'\} \cup \{e\}$, $F^0_2 = \emptyset$, $\Delta^0 = \{(u'_1, v'_1) | (u_1, v_1) \in F^0_1\}$ and W^0 the set of subbands used to route the demands (observe that $|W^0| = |K \setminus S'| + BP(S') + 2|S'|$). Clearly, S^0 is feasible for the problem and its incidence vector belongs to $\mathcal{F}$.

We show that $\nu = 0$ exactly as we did in the previous proofs. Similary, we claim that $\mu_{fw} = 0$, for all $f \in A_1 \setminus \{e\}$, $w \in W$. Indeed, let $f = (v_1, v_2)$ be some arc of $A_1 \setminus \{e\}$. Then, either 1) f does not belong to the set $F^0_1 \cup F^0_2$ (f is not used in the solution) or 2) $f \in F^0_1 \cup F^0_2$. In the first case, it is easy to see that a solution S^1 created from S^0 by adding f to F^0_2 and installing some unused subband $w \in W \setminus W^0$, then associating the path $\{(v'_1, v'_2)\}$ (that is to set Δ^1 = $\Delta^0 \cup \{(v'_1, v'_2)\}$) to the pair (f, w), is feasible and its incidence vector belongs to $\mathcal{F}$. Since f and w are arbitrarily chosen in $A_1 \setminus F^0_1 \cup F^0_2 \cup \{e\}$ and $W \setminus W^0$, respectively, the coefficients μ_{fw} = 0 for all $f \in A_1 \setminus F^0_1 \cup F^0_2 \cup \{e\}$, $w \in W \setminus W^0$. Now if f is used in S^0 (that is $f \in F^0_1 \cup F^0_2$), we can introduce two further solutions S^2, S^3, respectively obtained from S^0 as follows. Denote K_f and W_f the demands using f and the set of subbands installed on f in the solution S^0, respectively. We first add to F_2^0 two arcs $f_1 = (v_1, v_3)$, $f_2 = (v_3, v_2)$ and install as much new subbands on f_1, f_2 as there is subbands carried by f in the solution S^0 ($2 \times |W_f|$ new subbands). The paths $\{(v_1', v_3')\}$ and $\{(v_3', v_2')\}$ are associated to f_1 and f_2, respectively. Finaly, we shift the demands of K_f from their initial routing so as to let them use f_1, f_2 instead of f. The solution S^3 is obtained from S^2 by removing f (and the subbands installed on it). Both S^2 and S^3 ate clearly feasible and their incidence vectors belong to $\mathcal{F}$, and we can conclude that $\mu_{fw} = 0$ also for all $f \in F_1^0 \cup F_2^0 \cup \{e\}$, $w \in W \setminus W^0$. Similarly, we show that $\lambda_{fw}^k = 0$, for all $k \in K \setminus S'$, $f \in A_1 \setminus \{e\}$, $w \in W$ as follows. Let k be some demand of $K \setminus S'$ and denote by $f_k = (o_k, d_k)$ defining its initial routing path. We introduce the solution S^4 obtained from S^0 by installing two new subbands w_1, w_2 on any pair of arcs (o_k, l), (l, d_k) and associating to $\{(o_k, l), (l, d_k)\}$ the path $\{(o_k', d_k')\}$ in G_2. Now further associate the routing path $\{(o_k, l), (l, d_k)\}$ to k in addition to its initial path (that is f_k). Now, consider the solution S^5 obtained from S^4 by deleting the arc f_k. We can easily observe that both S^4 and S^5 are feasible and their incidence vectors are both in $\mathcal{F}$. Then, we get that $\lambda_{f_k w}^k = 0$, and $\lambda_{fw}^k = 0$ for all $k \in K \setminus S'$, $f \in A_1 \setminus \{e\}$, $w \in W$.

Now let s be some commodity of $K \setminus S$ and consider the solution S^6 obtained from S^0 by associating a routing path to s that uses e and some subband already installed on e, say w. Two non used subbands w_1, w_2, are also installed on the arcs (o_s, u), (v, d_s) so as to route s from its origin to its destination. Note that, by condition (ii) we know that such solution exists since $BP(S') = BP(S' \cup \{s\})$. Comparing solutions S^0 and S^6 yields

$$\lambda_{o_s d_s w_s}^s = \lambda_{o_s u w_1}^s + \lambda_{ew}^s + \lambda_{v d_s, w_2}^s + \mu_{o_s u w_1} + \mu_{v d_s, w_2},$$

implying that $\lambda_{ew}^k = 0$, for all $k \in K \setminus S$.

We still have to show that the coefficients λ_{ew}^k are the same for $k \in S$ and $w \in W$. Consider a subset of commodities $S'' \subseteq S \setminus S'$ such that $|S'| = |S''| = r$, where $r \in \mathbb{Z}^+$ and let S^7 be the solution obtained from S^0 by setting $S'' = S' \setminus \{k_1\} \cup \{k_2\}$, that is to say we replace some commodity k_1 using e by a commodity $k_2 \in S \setminus S'$. By condition (i), we know that S^7 is still feasible and its incidence vector belongs to $\mathcal{F}$. Comparing S^0 and S^7 yields $\lambda_{ew}^{k_1} = \lambda_{ew}^{k_2}$ and since k_1, k_2 are arbitrarily chosen in S, we can conclude that the coefficients λ_{ew}^k are equal for $k \in S$, $w \in W$. A similar reasoning allows to show that the coefficients μ_{ew} are the same for the subbands installed over e as all the subbands of W are exchangeable (they have the same capacity).

Finaly, by replacing $(x^{S^0}, y^{S^0}, z^{S^0})$ in the hyperplane defined by $\lambda x + \mu y + \nu z \leqslant \xi$, we obtain

$$\lambda \sum_{k \in S} \sum_{w \in W} x_{ew}^k + \mu \sum_{w \in W} y_{ew} = \xi,$$

anbd hence

$$\lambda |S'| + \mu BP(S') = \xi.$$

By condition (i), we have that $|S'| - qBP(S') = p$, yielding $\lambda = 1$, $\mu = -q$, $\xi = p$ and the proof is complete.