The motivation of the study of normal graphs comes from perfect graphs. A graph $G$ is perfect if $\chi(H) = \omega(H)$ for every induced subgraph $H$ of $G$. Claude Berge first introduced perfect graphs in 1960. His motivation came, in part, from determining the zero-error capacity of a discrete memoryless channel in information theory. This can be formulated as finding the Shannon capacity $C(G)$ of a graph $G$ as follows:

$$C(G) = \lim_{n \to \infty} \frac{1}{n} \log \omega(G^n)$$

where $G^n$ is the $n^{th}$ co-normal power of $G$. The co-normal product $G_1 \ast G_2$ of two graphs $G_1$ and $G_2$ is the graph with vertex set $V(G_1) \times V(G_2)$, where vertices $(v_1, v_2)$ and $(u_1, u_2)$ are adjacent if $u_1$ is adjacent to $v_1$ or $u_2$ is adjacent to $v_2$. Shannon noticed that $\omega(G^n) = (\omega(G))^n$ whenever $\omega(G) = \chi(G)$. Since $\omega(G^n) = (\omega(G))^n$ holds for all graphs $G$ with $\omega(G) = \chi(G)$, one might have expected that perfect graphs are closed under co-normal products. Körner and Longo proved this to be false. This motivated Janos Körner to study graphs which are closed under co-normal products.

A graph $G = (V, E)$ is normal if there exist two coverings, $C$ and $S$, of its vertex set $V$ such that every member of $C$ induces a clique in $G$, every member of $S$ induces an independent set in $G$ and $C \cap S \neq \emptyset$ for every $C \in \mathcal{C}$ and $S \in \mathcal{S}$.

Körner showed that all co-normal products of normal graphs are normal. In the same paper, he also showed that all perfect graphs are normal.

By definition it follows that a graph is normal if and only if its complement is normal. The simplest graphs that are known to be normal but not perfect are the odd cycles of length at least 9. In fact, $C_5$, $C_7$ and $\overline{C_7}$ are the only minimally known graphs which are not normal. This led De Simone and Körner to conjecture the following conjecture, which has the same flavor as the Strong Perfect Graph Theorem.

**Conjecture 0.1** (The Normal Graph Conjecture (1999)). *A graph with no $C_5$, $C_7$ and $\overline{C_7}$ as induced subgraph is normal.*

We disproved the above conjecture. Our proof is probabilistic, i.e., we construct a random graph without short cycles that is not normal. In fact our proof shows that almost all graphs of arbitrary girth and with a certain edge-density are non-normal. Despite this, we cannot display a single counterexample.

**References**

Disproving the normal graph conjecture

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\textbf{Abstract}
A graph $G$ is called normal if there exist two coverings, $\mathcal{C}$ and $\mathcal{S}$ of its vertex set such that every member of $\mathcal{C}$ induces a clique in $G$, every member of $\mathcal{S}$ induces an independent set in $G$ and $\mathcal{C} \cap \mathcal{S} \neq \emptyset$ for every $C \in \mathcal{C}$ and $S \in \mathcal{S}$. It has been conjectured by De Simone and Körner in 1999 that a graph $G$ is normal if $G$ does not contain $C_5$, $C_7$ and $\overline{C_7}$ as an induced subgraph. We disprove this conjecture.

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1. Introduction

The motivation of the study of normal graphs comes from perfect graphs. A graph $G$ is \textit{perfect} if $\chi(H) = \omega(H)$ for every induced subgraph $H$ of $G$. Claude Berge first introduced perfect graphs in 1960. His motivation came, in part, from determining the
zero-error capacity of a discrete memoryless channel. This can be formulated as finding the Shannon capacity \( C(G) \) of a graph \( G \) as follows:

\[
C(G) = \lim_{n \to \infty} \frac{1}{n} \log \omega(G^n)
\]

where \( G^n \) is the \( n \)th co-normal power of \( G \). The co-normal product (also called the OR product) \( G_1 \ast G_2 \) of two graphs \( G_1 \) and \( G_2 \) is the graph with vertex set \( V(G_1) \times V(G_2) \), where vertices \((v_1, v_2)\) and \((u_1, u_2)\) are adjacent if \( u_1 \) is adjacent to \( v_1 \) or \( u_2 \) is adjacent to \( v_2 \). Shannon noticed that \( \omega(G^n) = (\omega(G))^n \) whenever \( \omega(G) = \chi(G) \). Since \( \omega(G^n) = (\omega(G))^n \) holds for all graphs \( G \) with \( \omega(G) = \chi(G) \), one might have expected that perfect graphs are closed under co-normal products. Körner and Longo in [11] proved this to be false. This motivated Körner [9] to study graphs which are closed under co-normal products. We note that a covering of \( G \) is a set of subsets of \( V(G) \) whose union is \( V(G) \).

**Definition.** A graph \( G \) is normal if there exist two coverings, \( \mathcal{C} \) and \( S \) of its vertex set such that every member of \( \mathcal{C} \) induces a clique in \( G \), every member of \( S \) induces an independent set in \( G \) and \( C \cap S \neq \emptyset \) for every \( C \in \mathcal{C} \) and \( S \in S \).

Körner showed that all co-normal products of normal graphs are normal [9]. In the same paper, he also showed that all perfect graphs are normal. It turns out that normal graphs, like perfect graphs, also have a close relationship with graph entropy. The entropy of a graph \( G \) with respect to a probability distribution \( P \) on \( V(G) \) is defined as:

\[
H(G, P) = \lim_{t \to \infty} \min_{U \subseteq V(G), P(U) > 1 - \epsilon} \frac{1}{t} \log \chi(G[U])
\]

where \( P(U) = \sum_{x \in U} \prod_{i=1}^{t} P(x_i) \) and \( \epsilon \in (0, 1) \) (we note that the limit is independent of \( \epsilon \) as shown by Körner [10]). The graph entropy is sub-additive [16] with respect to complementary graphs:

\[
H(P) \leq H(G, P) + H(\overline{G}, P)
\]

for all \( G \) and all \( P \), where \( H(P) = \sum_{i=1}^{n} p_i \log \frac{1}{p_i} \). In fact, the value

\[
\max_{P} \{ H(G, P) + H(\overline{G}, P) - H(P) \}
\]

is also a measure of how imperfect a graph \( G \) is, relating to a parameter introduced in [13] by McDiarmid called the imperfection ratio of graphs (see also [5] and [6]), which itself derives its motivation from the radio channel assignment problems (see [13] for details). In [2] Csiszár et al. showed that:

\[
H(P) = H(G, P) + H(\overline{G}, P)
\]

for all \( P \) if and only if \( G \) is perfect.
The relaxed version, i.e., equality holds for at least one $P$, is true whenever $G$ is normal, as shown in [12]:

$$H(P) = H(G, P) + H(G, P)$$ for at least one $P$ if and only if $G$ is normal

It has been proved that line-graphs of cubic graphs [15], circulants [17] and a few classes of sparse graphs [1] are normal. Normal graphs have also been studied for regular and random regular graphs. Hosseini et al. [8,7] have shown that all subcubic triangle-free graphs are normal as well as that almost all $d$-regular graphs are normal when $d$ is fixed.

By definition it follows that a graph is normal if and only if its complement is normal. The simplest graphs that are known to be normal but not perfect are the odd cycles of length at least 9 (see [9]). In fact, $C_5$, $C_7$ and $\overline{C_7}$ are the only minimally known graphs which are not normal. To this end, De Simone and Körner [3] conjectured the following.

**Conjecture 1.1** (*The normal graph conjecture*). A graph with no $C_5$, $C_7$ and $\overline{C_7}$ as induced subgraph is normal.

By analogy with perfect graphs, one can ask whether a graph $G$ is *strongly normal*, i.e., every induced subgraph of $G$ is normal. As for perfect graphs, it is natural to try to characterize strongly normal graphs by excluding forbidden induced subgraphs. This leads to a restatement of Conjecture 1.1.

**Conjecture 1.2** ([3]). A graph $G$ is strongly normal if and only if neither $G$ nor its complement contain a $C_5$ or a $C_7$ as an induced subgraph.

In this paper, we disprove the Normal Graph Conjecture. In fact, we prove the following stronger result.

**Theorem 1.3.** There exists a graph $G$ of girth at least 8 that is not normal.

Our proof is probabilistic, i.e., we construct a random graph of girth 8 which is not normal. In fact, our proof method can easily be mimicked to show something stronger: there exist graphs of arbitrary girth $g$ which are not normal.

The paper is organized as follows. In the next subsection, we introduce the well-known probabilistic tools that are heavily used in the paper. In Section 2, we state and prove some standard properties of the random graph $G_{n,p}$ most of which are folklore. In Section 3, using the results of Section 2 and additional arguments we prove our main result, except a key lemma which is proved in Section 4.

### 1.1. Probabilistic tools

To prove our main theorem, we need two basic and well-known probabilistic tools.
Theorem 1.4 (Chernoff’s inequality, see [14]). Let $X_1, \ldots, X_n$ be independent Bernoulli (that is, 0/1 valued) random variables where $P[X_i = 1] = p_i$. Let $X = \sum_{i=1}^{n} X_i$ and let $\mu = \sum_{i=1}^{n} p_i$ be the expectation of $X$. Then, for all $0 < \delta < 1$ we have:

\[
P[X \leq (1 - \delta)\mu] \leq e^{-\mu \delta^2 / 2}
\]

\[
P[X \geq (1 + \delta)\mu] \leq e^{-\mu \delta^2 / 3}
\]

Theorem 1.5 (Markov’s inequality). If $X$ is any non-negative discrete random variable and $a > 0$, then

\[
P[X \geq a] \leq \frac{\mathbb{E}[X]}{a}
\]

2. Random graph properties

Let $G_{n,p}$ denote the random graph on $n$ vertices in which every edge is randomly and independently chosen with probability $p$.

Consider the random graph $G := G_{n,p}$ with $p = n^{-9/10}$. Denote by $d := np = n^{1/10}$ and let $X_7$ be the number of cycles in $G$ of length at most 7. By $\alpha(G)$ we denote the size of the largest independent set in $G$. In the sequel, we always assume that $n$ is sufficiently large.

Lemma 2.1. The following properties hold for the graph $G$.

(a) $P[X_7 > 4n^{7/10}] < 1/2$.

(b) Let $c \geq 10$ be a fixed constant. Then $P[\alpha(G) \geq cn^{9/10} \log n] \leq n^{2e^{-3\log n}}$.

(c) Let $D$ be the event that $G$ has a vertex of degree greater than $2d$. Then $P[D] \leq e^{-n^{0.1}/10}$.

Proof. (a) Note that by linearity of expectation,

\[
\mathbb{E}[X_7] \leq \sum_{l=3}^{7} \binom{n}{l} (l-1)! p^l \leq \sum_{l=3}^{7} (np)^l \leq 2n^{7/10}.
\]

The result now follows by Markov’s inequality.

(b) is well-known and can be deduced from, for example, Frieze [4]. We include the proof for completeness. By the union bound, we have

\[
P[\alpha(G(n,p)) \geq x] \leq \binom{n}{x} (1-p)^{\binom{x}{2}} \leq n^x (e^{-p(x-1)/2})^x \leq (n e^{-n^{-0.9}(x-1)/2})^x
\]
Now, setting $x := cn^{0.9} \log n$ yields the result.

(c) Clearly, $\Pr[D] \leq n\Pr[\deg(v) > 2d]$, where $v$ is some fixed vertex. By Chernoff’s inequality $\Pr[\deg(v) > 2d] \leq e^{-n^{0.1}/3}$. The claim now follows. □

Let $G$ be a bipartite graph with $m$ edges on vertex bipartition $(A, B)$. We denote by $d$ its average degree in $A$, that is $d = m/|A|$ and by $e(X, Y)$ the number of edges between the set $X$ and $Y$ for any $X \subseteq A, Y \subseteq B$. A partial cover of $G$ is a set of pairs $(x_i, Y_i)$ where the $x_i$’s are distinct vertices of $A$, the $Y_i$’s are disjoint sets of $B$, $x_i$ is a neighbor of all vertices of $Y_i$, the size of each $Y_i$ is $\lceil d/3 \rceil$ and finally the union of $Y_i$’s has size at least $|B|/3$.

Lemma 2.2. Let $G$ be a random bipartite graph on vertex bipartition $(A, B)$, where each possible edge appears with some probability $p$, independently. If $\min\{|A|, |B|\} > 10^{100}p^{-1}$, then $G$ has $e(A, B) \in [0.99p|A||B|, 1.01p|A||B|]$ and a partial cover with probability at least $1 - e^{-cp|A||B|}$, where $c > 0$ is an absolute constant.

Proof. Let $A'$ be the set of vertices of $A$ with degree in $[0.99p|B|, 1.01p|B|]$ in $B$ and $B'$ be the set of vertices of $B$ with degree in $[0.99p|A|, 1.01p|A|]$ in $A$. By Chernoff’s inequality, there exists a constant $c > 0$, such that the probability that (i) $|A'| < 0.99|A|$ or, (ii) $|B'| < 0.99|B|$, or (iii) $m := e(A, B) \notin [0.99p|A||B|, 1.01p|A||B|]$ is at most $e^{-cp|A||B|}$. Indeed, note that probability of (i) is at most

$$\left(\frac{|A|}{0.01|A|}\right)(2e^{-\left(0.01\right)^2p|B|/3\cdot 0.01|A|} < 2|A|e^{-\left(0.01\right)^2p|A||B|} < e^{-c_1p|A||B|}$$

for some constant $c_1 > 0$ (here we used the fact that $10^{100}p^{-1} < \min\{|A|, |B|\}$). Similarly the probability of (ii) is at most $2|B|e^{-\left(0.01\right)^2p|A||B|} < e^{-c_2p|A||B|}$ for some constant $c_2 > 0$ (here again we used the fact that $\min\{|A|, |B|\} > 10^{100}p^{-1}$). The probability of (iii) is clear.

Now, we claim that if $G$ satisfies $|A'| \geq 0.99|A|$, $|B'| \geq 0.99|B|$ and $m \in [0.99p|A||B|, 1.01p|A||B|]$, then it has a partial cover. Observe first that at least $3m/4$ edges of $G$ must be between $A'$ and $B'$ (call these good edges). Now greedily pick pairs $(x_i, Y_i)$ where $x_i \in A'$ and $Y_i \subseteq B' \cap N(x_i)$ has size exactly $\lceil m/3|A| \rceil$ in order to construct a partial cover. If the process stops with $Y := Y_1 \cup \cdots \cup Y_k$ of size at least $|B|/3$, we have our partial cover. If not, denote by $X$ the set $\{x_1, \ldots, x_k\}$, and note that this implies that every vertex in $A' \setminus X$ has degree less than $\lceil m/(3|A|) \rceil$ in $B' \setminus Y$. Note that the size of $X$ is negligible compared to the size of $A'$. Indeed, $|X| < |B|/\lceil m/(3|A|) \rceil < 4p^{-1} < |A'|/10^{10}$. Hence the number of good edges incident to $X$ is negligible compared to the number of good edges. In particular, at least $2.99m/4$ good edges are incident to $A' \setminus X$. However, since every vertex in $A' \setminus X$ has degree at most $\lceil m/(3|A|) \rceil$ in $B' \setminus Y$, $e(A' \setminus X, Y) > 2.99m/4 - \lceil m/(3|A|) \rceil (|A' - X|) > 2.99m/4 - m/3$. Now, since $|Y| < |B|/3$, and every vertex in $Y$ has degree at most $1.01p|A|$, it follows
that \( e(A' \setminus X, Y) < 1.01p|A||B|/3 < 1.01m/(3 \cdot 0.99) \). This implies that 
\( 2.99m/4 - m/3 < 1.01m/(3 \cdot 0.99) \), a contradiction. \( \square \)

3. Proof of Theorem 1.3

In this section we prove our main result. We say that a graph \( G \) admits a star covering if there exist two coverings, \( C \) and \( S \), of \( V(G) \) such that:

(a) every member of \( C \) induces a clique \( K_2 \) or \( K_1 \) in \( G \), where no \( K_1 \) is included in some \( K_2 \).
(b) the graph on \( V(G) \) consisting of the edges of \( C \), denoted by \( E[C] \), is a spanning vertex-disjoint union of stars.
(c) every member of \( S \) induces an independent set in \( G \).
(d) \( C \cap S \neq \emptyset \) for every \( C \in C \) and \( S \in S \).

Every graph \( G \) admitting a star covering is normal, and the converse holds for triangle-free graphs:

Claim 3.1. If \( G \) is a normal triangle-free graph, then \( G \) admits a star covering \( (C, S) \) where \( E[C] \) contains at most \( \alpha(G) \) stars.

Proof. Let \((C', S')\) be a normal covering of \( G \). Since \( G \) is triangle-free, all cliques in \( C' \) are \( K_2 \)'s or \( K_1 \)'s. The cliques \( K_1 \) included in some \( K_2 \) can be deleted from \( C' \). All that remains to show is that we can reduce to cliques inducing vertex-disjoint stars. Indeed, suppose that \( E[C'] \) contains two adjacent vertices \( u, v \) with \( d_{E[C']}(u) \geq 2 \) and \( d_{E[C']}(v) \geq 2 \). Deleting the edge \( uv \) from \( C' \) gives another covering (since \( u \) and \( v \) are also covered by other edges) that is still intersecting with \( S' \). Repeating this, we obtain a star covering \( (C, S) \) of \( G \).

Now, we show that the number of stars in \( E[C] \) is at most \( \alpha(G) \). Indeed, let \( x_1, \ldots, x_k \) be the centers of the stars (some centers \( x_i \) may be trivial stars) in \( E[C] \), and let \( S \in S \) be any independent set. Then for each \( x_i \), \( S \) must contain either \( x_i \) or an adjacent neighbor of \( x_i \) in \( C \). Since the stars are disjoint, it follows that \( k \leq |S| \leq \alpha(G) \). \( \square \)

Let \( G = (V, E) \) be a graph. A star system \((Q, S)\) of \( G \) is a spanning set of vertex disjoint stars where \( S \) is the set of stars, and \( Q \) is the set of centers of the stars of \( S \). Therefore every \( x_i \in Q \) is the center of some star \( S_i \) of \( S \). Moreover, the union of vertices of the \( S_i \)'s is equal to \( V \). Note that some stars can be trivial, i.e. simply consisting of their center. To every star system \((Q, S)\), we associate a directed graph \( Q^* \) on vertex set \( Q \) by letting \( x_i \rightarrow x_j \) whenever a leaf of \( S_i \) is adjacent to \( x_j \). Of particular interest here is the following notion of out-section: A subset \( X \) of \( Q \) is an out-section if there exists \( v \) in \( Q \) such that for each \( x \in X \), there exists a directed path in \( Q^* \) from \( v \) to \( x \).

Observe that to every star-covering we can associate the star-system \( E[C] \).
Lemma 3.2. Let $G$ be a normal triangle-free graph with a star covering $(C, S)$. We denote by $(Q, S)$ its associated star-system. Assume that $X$ is an out-section of $Q^*$. Then the set of leaves of the stars with centers in $X$ form an independent set of $G$.

Proof. To see this, consider a vertex $v$ in $Q$ which can reach every vertex $x$ of $X$ in $Q^*$ by an oriented path $v = x_0 \rightarrow x_1 \rightarrow \cdots \rightarrow x_k = x$. For all $i$, we denote by $S_i$ the star with center $x_i$ (observe that they all have leaves, apart possibly $S_k$). Consider an independent set $I$ of $S$ which contains any leaf of $S_0$. Since $I$ is an independent set, it does not contain $x_0$, and hence by definition of normal cover $I$ must contain all the leaves of $S_0$. Now since $x_0 \rightarrow x_1$, there is a leaf of $S_0$ adjacent to $x_1$. In particular, $x_1$ is not in $I$, implying that every leaf of $S_1$ belongs to $I$. Applying the same argument, all leaves of $S_i$ belong to $I$, for each $i$. Since this argument can be done for every oriented path starting at $v$, any star $S_j$ whose center is reachable from $v$ in $Q^*$ by a directed path has all its leaves contained in $I$. In particular, all the leaves with centers in $X$ form an independent set.  

This lemma provides a roadmap to a disproof of the normal graph conjecture. Namely, a normal high girth dense enough random graph will have a star covering with large out-sections, in particular, large independent sets. By tuning the density we can contradict the typical stability of such graphs. To achieve this, we need to introduce the following definitions:

Given a graph $G$ and a subset $Q$ of its vertices partitioned into $Q_1, \ldots, Q_{10}$, we say that $w \in V \setminus Q$ is a private neighbor of a vertex $v \in Q_i$ if $w$ is adjacent to $v$ but not to any other vertex in $Q_1, \ldots, Q_i$. For each vertex $v$ in some $Q_i$, let $S_v$ be the (possibly trivial) star centered at $v$ consisting of $v$ and its private neighbors. Note that by definition, for any distinct vertices $v, v'$ in $Q$, the stars $S_v$ and $S_{v'}$ are vertex-disjoint. Thus, $Q$ and the set of stars $S_v$ form a star system for the graph induced by the set of vertices in $Q$ and their private neighbors. We define as previously our oriented graph $Q^*$ based on the star system consisting of $Q$ and the set of stars $S_v$. Observe that by definition of private neighbors, any arc $u \rightarrow v$ of $Q^*$ with $u \in Q_i$ and $v \in Q_j$ satisfies $i < j$. Given $Q_1, \ldots, Q_{10}$ in some graph $G$, we refer to this star system as the private star system over $Q_1, \ldots, Q_{10}$. The directed graph $Q^*$ is called the private directed graph over $Q_1, \ldots, Q_{10}$.

Let us now turn to our fundamental property:

Property $JQ$:

We say that $G$ satisfies property $JQ$ if for every choice of pairwise disjoint subsets of vertices $J, Q_1, \ldots, Q_{10}$, with $|J| \leq n^{0.91}$ and $n^{0.9} \leq |Q_i| \leq n^{0.9}$ for all $i = 1, \ldots, 10$, the private directed graph $Q^*$ over $Q_1, \ldots, Q_{10}$ defined on the induced subgraph $G \setminus J$ contains an out-section $X$ such that the sum of the number of private neighbors corresponding to all the vertices of $X$ is at least $n^{0.95}$.
The crucial point is that a random graph \( G := G_{n,p} \) with \( p = n^{-9/10} \) will almost surely have property \( JQ \), as claimed by the lemma below.

**Lemma 3.3.** \( \Pr[G \in JQ] = 1 - o(1) \).

We postpone the proof of this lemma to the end of the paper. Now, we show that Lemmas 2.1, 3.3 and Claim 3.1 are sufficient to prove our main theorem.

**Proof of Theorem 1.3.** We consider a random graph \( G := G_{n,p} \) with \( p = n^{-9/10} \). Using Lemma 2.1 and Lemma 3.3 and the union bound, for \( n \) sufficiently large, there exists a \( n \)-vertex graph \( G \) satisfying: (a) \( G \) has less than \( 4n^{0.7} \) cycles of length at most seven, (b) \( \alpha(G) < 10n^{0.9} \log n \), (c) \( G \) has maximum degree at most \( 2n^{0.1} \), (d) \( G \) has property \( JQ \).

Consider a set \( S \) of at most \( 4n^{0.7} \) vertices in \( G \) intersecting all cycles of length at most 7. Note that \( G[V \setminus S] \) has girth at least 8. Assume now for contradiction that \( G[V \setminus S] \) is a normal graph. By Claim 3.1, there is a star covering \((C,S)\) of \( G[V \setminus S] \) with the number of stars at most \( 10n^{0.9} \log n \). Let \( S' \) be the set of those stars which have size at most \( 10^{10} \log n \). Let \( J = S \cup S' \). Observe that \(|J| \leq 10^{10} \log n \cdot 10n^{0.9} \log n + 4n^{0.7} < n^{0.91}\). Now, consider \( G[V \setminus J] \) and call \( Q \) the set of centers of the remaining stars. Observe that the set of stars centered at \( Q \) still form a star covering of \( G[V \setminus J] \). Indeed, \( C \) and \( S \) restricted to \( G[V \setminus J] \) is a star covering.

Note that since \(|Q| < 10n^{0.9} \log n\), \(|V \setminus (J \cup Q)| > n - n^{0.91} - 10n^{0.9} \log n\). Now, since \( Q \) is a dominating set in \( G[V \setminus J] \), and the degree of every vertex in \( G[V \setminus J] \) is at most \( 2n^{0.1} \), it follows that \(|Q| > n^{0.9}/3\).

We now define the directed graph \( Q^* \) on \( Q \) based on the star covering of \( G[V \setminus J] \).

**Claim 3.4.** Every strongly connected component \( C \) of \( Q^* \) has size at most \( n^{0.9}/1000 \).

**Proof.** Observe that \( C \) is an out-section of any of its vertices, hence by Lemma 3.2 the set of leaves of stars with centers in \( C \) is an independent set. Since each star in the star covering of \( G[V \setminus J] \) has size at least \( 10^{10} \log n \), it follows that \( G[V \setminus J] \) has an independent set of size \( 10^{10} \log n \cdot |C| \). The result follows now from the fact that \( \alpha(G) < 10n^{0.9} \log n \).

Let \( C_1, \ldots, C_k \) be the strongly connected components of \( Q^* \), enumerated in such a way that all arcs \( xx' \) of \( Q^* \) with \( x \in C_i \) and \( x' \in C_j \) satisfy \( i \leq j \).

We concatenate subsets of the components \( C_1, \ldots, C_k \) into blocks \( Q_1, Q_2, \ldots, Q_{10} \) with \( Q_1 = C_1 \cup C_2 \cup \cdots \cup C_{i_1}, Q_2 = C_{i_1+1} \cup \cdots \cup C_{i_2}, \ldots, Q_{10} = C_{i_{9}+1} \cup \cdots \cup C_{i_{10}} \) for some \( i_1, \ldots, i_{10} \) such that for each \( Q_i \), \( 1 \leq i \leq 10 \), \( n^{0.9}/1000 \leq |Q_i| \leq n^{0.9}/500 \). This is clearly possible since for each \( i \leq k \), \( |C_i| < n^{0.9}/1000 \) and \(|Q| > n^{0.9}/3\).

The crucial remark now is that if a vertex \( v \) of \( G \setminus (J \cup Q) \) is a private neighbor of a vertex \( x_i \) in \( Q_i \), then the edge \( x_i v \) must be an edge of the star covering. Indeed, \( v \) has a unique neighbor in \( Q_1 \cup \cdots \cup Q_i \) by definition, and any edge \( vx_j \) where \( x_j \) is in \( Q \setminus (Q_1 \cup \cdots \cup Q_i) \) cannot belong to \( C \) since this would imply \( x_j \rightarrow x_i \). Now, by property
we know that the private directed graph $Q'*$ defined on the stars formed by the private neighbors of the $Q_i$’s has an out-section $O$ of size at least $n^{0.95}$. Since $Q'*$ is a subdigraph of $Q^*$, the set $O$ is also an out-section of $Q^*$. Hence the set of leaves with centers in $O$ forms an independent set of size $n^{0.95}$ by Lemma 3.2, contradicting the fact that $\alpha(G) < 10n^{0.9}\log n$. □

4. Proof of Lemma 3.3

In this section, we prove Lemma 3.3 to conclude the proof of Theorem 1.3.

Proof of Lemma 3.3. We will prove that $\mathbb{P}[\mathcal{J}Q^*] = o(1)$. We first fix the sets $J, Q_1, \ldots, Q_{10}$.

Note that there is at most $\sum_{i=1}^{n^{0.91}} \binom{n}{i} \leq 2n^{n^{0.91}}$ possible sets for $J$ and at most $(\sum_{i=n^{0.9}/1000}^{n^{0.9}/500} \binom{n}{i})^{10} \leq 2^{10n^{0.9}/50}$ sets for the $Q_1, \ldots, Q_{10}$. Thus, there are at most $2^{10n^{0.91}}$ ways to fix the sets $J, Q_1, \ldots, Q_{10}$. We will recall this fact later; in the sequel, the sets $J, Q_1, \ldots, Q_{10}$ are fixed.

Denote by $B := G \setminus \{\cup_{i=1}^{10} Q_i \cup \{J\}\}$.

Note that $|B| \geq n - n^{0.91} - \frac{n^{0.9}}{50} \geq n - 2n^{0.91}$. For a vertex $v \in Q_1$, let $D_v$ be the number of neighbors of $v$ in $B$. Let $D_{Q_1}$ be the event that at least $0.01|Q_1|$ vertices $v$ in $Q_1$ have $D_v \not\in (0.95d, 1.01d)$. We recall that $n^{0.9}/1000 \leq |Q_1| \leq n^{0.9}/500$.

Note that, for some sufficiently small $\delta, \epsilon > 0$

$$
\mathbb{P}[D_{Q_1}] \leq \left(\frac{|Q_1|}{0.01|Q_1|}\right)(\mathbb{P}[D_v \not\in (0.95d, 1.01d)])^{0.01|Q_1|} \\
\leq \left(\frac{n/500d}{n/50000d}\right)(\mathbb{P}[D_v \not\in (0.95d, 1.01d)])^{n/100000d} \\
\leq (n/500d)^{n/50000d}(e^{-\epsilon d})^{n/10^6d} < e^{-\delta n},
$$

where we used the fact that $D_v$ is a binomial random variable with mean $p|B| \in (0.96d, d)$ and thus Chernoff’s inequality applies.

For a vertex $v \in B$, let $X_v$ be the random variable counting the number of vertices in $Q_1$ adjacent to $v$, and $X$ be the number of vertices in $B$ that have degree equal to 1 in $Q_1$. Then $X$ is a binomial random variable. Now,

$$
\mathbb{E}[X] = |B| \times \mathbb{P}[X_v = 1] \\
\geq 0.96n\mathbb{P}[X_v = 1] \\
\geq 0.96n|Q_1|\frac{d}{n}(1 - d/n)^{|Q_1| - 1} \\
\geq 0.96|Q_1|d e^{-1/250} \\
\geq 0.95|Q_1|d.
$$
By Chernoff’s inequality, since $\mathbb{E}[X] \geq 0.95n/1000$, for some $\delta > 0$ sufficiently small,

$$\mathbb{P}[\{X < 0.9|Q_1|d]\} \leq e^{-\delta n}.$$ 

Next, let $Z_E$ be the number of edges from $Q_1$ to $B$. Note that $Z_E$ is a binomial random variable with mean $\mu = |Q_1||B|\frac{d}{n}$. Note that $\mu \in (0.96|Q_1|d, |Q_1|d)$. Then, for some $\delta > 0$ sufficiently small,

$$\mathbb{P}[\{Z_E \notin (0.95|Q_1|d, 1.01|Q_1|d)\}] \leq e^{-\delta n},$$

by Chernoff’s inequality. Now, let $M$ be the event

$$M := \{Z_E \in (0.95|Q_1|d, 1.01|Q_1|d)\} \cap D_{Q_1}^c \cap \{X > 0.9|Q_1|d\}.$$

Clearly,

$$\mathbb{P}[M^c] \leq 3e^{-\delta n}.$$

Thus,

$$\mathbb{P}[M] \geq 1 - 3e^{-\delta n}.$$ 

Let $N_{Q_1}$ be the event that at least $|Q_1|/2$ vertices in $Q_1$ have at least $d/2$ private neighbors. We claim that if the event $M$ holds then so does $N_{Q_1}$.

Assume that $M$ holds. Let us call an edge $e$ a good edge if its endpoint in $Q_1$, say $v$, has $D_v \in (0.95d, 1.01d)$ and its endpoint in $B$ has degree exactly 1 in $Q_1$. We compute the number of non-good edges. First, let us count the number of edges whose endpoint in $B$ has degree greater than 1.

Note that the number of vertices in $B$ that have degree 1 in $Q_1$ is at least $0.9|Q_1|d$. These vertices contribute at least $0.9|Q_1|d$ edges. Thus, the number of edges between $Q_1$ and $B$ whose endpoint in $B$ is not of degree 1 is at most $1.01|Q_1|d - 0.9|Q_1|d \leq 0.11|Q_1|d$.

Next, we count the number of edges between $Q_1$ and $B$ whose endpoint in $Q_1$, say $v$, satisfies $D_v \notin (.95d, 1.01d)$. Since at least $0.99|Q_1|$ vertices in $Q_1$ have degree in the interval $(.95d, 1.01d)$, they contribute at least $.99 \cdot 0.95|Q_1|d$ edges. The remaining number of edges is at most $1.01|Q_1|d - 0.99 \cdot 0.95|Q_1|d \leq 0.07|Q_1|d$.

Thus, the number of edges which are not good is at most $0.18|Q_1|d$.

Now, we prove our claim that if $M$ holds then $N_{Q_1}$ holds as well. We recall again that at least $0.99|Q_1|$ vertices in $Q_1$ have degree at least $0.95d$ in $B$. Let us compute the number of vertices in $Q_1$ (called bad vertices) which do not have at least $d/2$ private neighbors. By the remark above, the number of bad vertices which have degree at most $0.95d$ in $B$ is at most $0.01|Q_1|$. Thus, it suffices to bound the number of bad vertices which have degree at least $0.95d$ in $B$. Such a vertex is adjacent to at least $0.45d$ non-good edges since its degree is at least $0.95d$. Since the total number of non-good edges
is at most 0.18|Q_1|d it follows that the number of all bad vertices is easily at most 
\( \frac{0.18|Q_1|}{0.49} + 0.01|Q_1| < |Q_1|/2. \) Therefore, at least \( |Q_1|/2 \) vertices in \( Q_1 \) have at least \( d/2 \) private neighbors, proving the claim. Summarizing,

\[
\mathbb{P}[M] = \mathbb{P}[N_{Q_1} \cap M] + \mathbb{P}[N_{Q_1}^c \cap M] \\
= \mathbb{P}[N_{Q_1} \cap M].
\]

Thus,

\[
\mathbb{P}[N_{Q_1}] \geq 1 - 3e^{-\delta n}.
\]

Now, define \( B_2 = B \setminus \Gamma(Q_1), \) where \( \Gamma(Q_1) \) is the set of neighbors of \( Q_1 \) in \( B \). Define \( N_{Q_2} \) to be the event that at least \( |Q_2|/2 \) vertices in \( Q_2 \) have at least \( d/2 \) private neighbors in \( B_2 \). We would like to show that \( \mathbb{P}[N_{Q_2}] \) holds with high probability. First, note that

\[
\mathbb{P}[|\Gamma(Q_1)| > n/400] \leq \mathbb{P}[Z_E > n/400] < e^{-\delta n}.
\]

Thus, it suffices to bound \( \mathbb{P}[N_{Q_2} \mid \{|B_2| > |B| - n/400\}] \). By an identical argument as for \( N_{Q_1} \), we know that the probability of this event is at least \( 1 - O(\epsilon^{-\delta_2 n}) \), for some \( \delta_2 > 0 \). Indeed, the only assumption that we need that was used before is that

\[
|B_2| \geq 0.96n, \quad \text{which holds as } |B| \geq n - 2n^{0.91}.
\]

Thus, \( \mathbb{P}[N_{Q_2}] \geq (1 - e^{-\delta n})(1 - O(\epsilon^{-\delta_2 n})) \geq 1 - e^{-\beta n} \), for some \( \beta > 0 \).

For each \( i, 2 \leq i \leq 10 \), we define the sets \( B_i \) by \( B_{i+1} := B_i \setminus \Gamma(Q_i) \) and \( N_{Q_i} \), as the event that at least \( |Q_i|/2 \) vertices in \( Q_i \) have at least \( d/2 \) private neighbors in \( B_i \). By repeating the same argument as before we obtain that with probability at least \( 1 - O(e^{-\epsilon n}) \) the event \( N_{Q_i} \) holds, for some \( \epsilon > 0 \). Indeed, the size of the \( B_i \)’s almost surely never decreases by more than \( n/400 \) at a time and thus for each \( i, |B_i| > |B| - n/40 > 0.97n \), allowing us to guarantee that the event \( M \) holds with high probability in each iteration.

It follows that

\[
\mathbb{P}[(\cap_{i=1}^{10} N_{Q_i})^c] = O(e^{-\epsilon' n}),
\]

for some \( \epsilon' > 0 \).

Armed with the fact that the event \( \cap_{i=1}^{10} N_{Q_i} \) holds with very high probability, we will finish the proof. We will say that a vertex \( v \) in some \( Q_i \) is rich if \( v \) has at least \( d/2 \) private neighbors; similarly, a set \( S \) of vertices is called rich if every vertex of \( S \) is rich. Let \( \tau \) be an ordering of the vertices of \( G \setminus J \); we will use this ordering a bit later.

Consider the following set of events. We remark that for our purposes we are only interested in the case \( i = 10 \).

*There exist positive constants \( \epsilon_i \) and \( C_i \) such that in each \( Q_i, 2 \leq i \leq 10 \), there exist at least \( \frac{\epsilon n}{d} \) rich, disjoint out-sections of \( Q^* \), each of size at least \( \frac{\epsilon n - 1}{C_i} \).*

Let \( J_i \) be the \( i \)th event in the above statement.
We inductively prove the following claim (*): for appropriate values of $\epsilon_i$ and $C_i$, there exist $\epsilon'_i > 0$ such that $\mathbb{P}[J_i] \geq 1 - e^{-\epsilon'_i n}$.

We first show that $\mathbb{P}[J_2] \geq 1 - e^{-\epsilon'_2 n}$ for some values of $\epsilon_2, C_2$ and $\epsilon'_2$. Note that

$$\mathbb{P}[J_2] \geq \mathbb{P}[J_2 \mid N_{Q_1} \cap N_{Q_2}] \mathbb{P}[N_{Q_1} \cap N_{Q_2}] \geq \mathbb{P}[J_2 \mid N_{Q_1} \cap N_{Q_2}](1 - O(e^{-\epsilon'_n})).$$

Thus, it is sufficient to show that $\mathbb{P}[J_2 \mid N_{Q_1} \cap N_{Q_2}] \geq 1 - e^{-c_1 n}$, for some constant $c_1 > 0$.

We apply Lemma 2.2. We construct the following auxiliary bipartite graph. Consider the bipartite graph $H_1 = (A_1, A_2)$, where the partite sets $A_1$ and $A_2$ are the set of rich vertices of $Q_1$ and $Q_2$, respectively. Note that conditional on $N_{Q_1} \cap N_{Q_2}$, min$\{|A_1|, |A_2|\} \geq \frac{n}{20000}$. We put an edge between $v_1 \in A_1$ and $v_2 \in A_2$ in $H_1$ if at least one of the first $\frac{d}{2}$ private neighbors of $v_1$ under the ordering $\tau$ is adjacent to $v_2$.

Claim 4.1. $H_1$ is a random bipartite graph where the probability of any edge is $p_1 = 1 - (1 - p)^{d/2}$ with the edges appearing independently.

Proof of Claim. We note the following: let $v_1, v'_1$ be any elements in $Q_1$ (not necessarily distinct) which have distinct private neighbors $w_1$ and $w'_1$, respectively. Then, conditional on $N_{Q_1} \cap N_{Q_2}$, it is still the case that $\mathbb{P}[w_1 v_2 \in E(G)] = \mathbb{P}[w'_1 v_2 \in E(G)] = p$, and furthermore these two events are still independent. It follows that the probability of any edge in $H_1$ is $1 - (1 - p)^{d/2}$ and that these edges appear independently. □

It is easily seen that $\frac{d^2}{n} \leq p_1 \leq \frac{d^2}{n}$. We apply Lemma 2.2.

Indeed, $10^{100} p_1^{-1} < 4 \cdot 10^{100} n/d^2 < n/2000d \leq \min\{|A_1|, |A_2|\}$, if $n$ is sufficiently large. Thus, $H_1$ has a partial cover and $e(A_1, A_2) \in [0.99 p_1 |A_1||A_2|, 1.01 p_1 |A_1||A_2|]$ with probability at least $1 - e^{-c_0 |A_1||A_2|} > 1 - e^{-c_1 n}$, for some constant $c_1 > 0$.

Let $(x_1, Y_1), ..., (x_k, Y_k)$ be the set of pairs in the partial cover. It follows that $|Y_i| = e(A_1, A_2)/3 |A_1| > d/C_2$ for some $C_2 > 0$ and at least $|A_2|/3$ of the vertices of $A_2$ are covered by the $Y_i$’s. Since $e(A_1, A_2) < 1.01 p_1 |A_1||A_2|$, it follows that $k > \frac{\epsilon n}{d}$ for some $\epsilon > 0$. Finally, note that each $Y_i$ is a rich out-section. Indeed, each element of $Y_i$ is a vertex of $Q_2$ that is adjacent to at least one of the private neighbors of $x_i$ and all vertices of $A_2$ are rich. Now, the facts $|Y_i| > d/C_2$ and $k > \frac{\epsilon n}{d}$ are sufficient to establish that $\mathbb{P}[J_2 \mid N_{Q_1} \cap N_{Q_2}] \geq 1 - e^{-c_1 n}$. Thus, $\mathbb{P}[J_2] \geq 1 - e^{-c_1 n}$, for some $c_1 > 0$.

The general case is similar. Suppose that we know that $\mathbb{P}[J_i] \geq 1 - e^{-\epsilon'_i n}$ with the corresponding constants $C_i$ and $\epsilon_i$. We will prove that $\mathbb{P}[J_{i+1}] \geq 1 - e^{-\epsilon'_{i+1} n}$, for some constant $\epsilon'_{i+1}$.

Note that

$$\mathbb{P}[J_{i+1}] \geq \mathbb{P}[J_{i+1} \mid J_i \cap N_{Q_{i+1}}] \mathbb{P}[J_i \cap N_{Q_{i+1}}] \geq \mathbb{P}[J_{i+1} \mid J_i \cap N_{Q_{i+1}}](1 - e^{-\epsilon'_i n} - e^{-\epsilon'_n}).$$

Therefore, it suffices to show that $\mathbb{P}[J_{i+1} \mid J_i \cap N_{Q_{i+1}}] \geq 1 - e^{-c_i n}$, for some constant $c_i > 0$. 

Suppose that $I_i \cap N_{i+1}$ holds. We argue similarly as for the case $i = 1$. In the set $Q_i$ we will have at least $\epsilon_i n / d^i$ disjoint out-sections each of which is rich and has size at least $d^{-i}/C_i$ for some constants $C_i, \epsilon_i > 0$.

Consider the following bipartite graph $H_i = (A_i, A_{i+1})$. For each of the disjoint, rich out-sections guaranteed by $J_i$ we will have a vertex $v$ in $A_i$, and $A_{i+1}$ will consist of the rich vertices of $Q_{i+1}$. Note that conditional on $J_i \cap N_{Q_{i+1}}$, $\min\{|A_i|, |A_{i+1}|\} \geq \frac{\epsilon_{i+1} n}{d^i}$. Let $v_i \in A_i$ and $v_{i+1} \in A_{i+1}$, and let $O_i$ be the out-section associated with $v_i$. We put an edge between $v_i$ and $v_{i+1}$ in $H_i$ if at least one of the first $\lceil d^{-i}/C_i \rceil$ vertices of $O_i$ (under $\tau$) has one of its first $\lceil \frac{d}{2} \rceil$ private neighbors (under $\tau$) adjacent to $v_{i+1}$.

Claim 4.2. $H_i$ is a random bipartite graph where the probability of any edge is $p_i = 1 - (1 - p_1)^{\lceil d^{-i}/C_i \rceil}$ with the edges appearing independently.

Proof of Claim. We note the following: let $v_i, v'_{i}$ be any elements in $Q_i$ (not necessarily distinct) which have distinct private neighbors $w_i$ and $w'_i$, respectively, and let $v_{i+1}$ be a vertex in $Q_{i+1}$. Then, conditional on $J_i \cap N_{Q_{i+1}}$, it is still the case that $\mathbb{P}[w_i v_{i+1} \in E(G)] = \mathbb{P}[w'_i v_{i+1} \in E(G)] = p$, and furthermore these two events are still independent. It follows that the probability of any edge in $H_i$ is $1 - (1 - p_1)^{\lceil d^{-i}/C_i \rceil}$ and that these edges appear independently. \hfill \Box

It is easily seen that $\frac{d^{i+1}}{4C_i n} < p_i < \frac{2d^i}{C_i n}$. We again apply Lemma 2.2. Indeed, $10^{100} p_i^{-1} < 10^{100} \frac{4C_i n}{d^{i+1}} = \lfloor \epsilon_i n / d^i \rfloor = \min\{|A_i|, |A_{i+1}|\}$, if $n$ is sufficiently large. Thus, $H_i$ has a partial cover and $e(A_i, A_{i+1}) \in [0.99p_i |A_i| |A_{i+1}|, 1.01p_i |A_i| |A_{i+1}|]$ with probability at least $1 - e^{-c_i n}$, for some constant $c_i > 0$. Let $(x_1, Y_1), ..., (x_k, Y_k)$ be the set of pairs in the partial cover. It follows that the size of each $Y_i$ is $|e(A_i, A_{i+1})| / 3 |A_i| > d^i / C_{i+1}$ for some $C_{i+1} > 0$ and at least $|A_{i+1}| / 3$ of the vertices of $A_{i+1}$ are covered by the $Y_i$’s. Since $e(A_i, A_{i+1}) < 1.01p_i |A_i| |A_{i+1}|$, it follows that $k > \frac{\epsilon_{i+1} n}{d^{i+1}}$ for some $\epsilon_{i+1} > 0$.

Finally, note that each $Y_i$ is a rich out-section. Indeed, each element of $Y_i$ is a vertex of $Q_{i+1}$ that is adjacent to at least one of the private neighbors of an out-section $O_i \subset Q_i$ associated with the vertex $x_i$. Moreover, all the vertices of $A_{i+1}$ are rich. Now, the facts $|Y_i| > d^i / C_{i+1}$ and $k > \frac{\epsilon_{i+1} n}{d^{i+1}}$ are sufficient to establish that $\mathbb{P}[J_{i+1} \mid J_i \cap N_{Q_{i+1}}] \geq 1 - e^{-c_i n}$. Thus, $\mathbb{P}[J_{i+1}] \geq 1 - e^{-\epsilon_{i+1} n}$, for some $\epsilon_{i+1} > 0$.

This proves the claim (*).

Recall that $d = n^{0.1}$. Now, if $J_{i0}$ holds, then we have that there exist at least $\epsilon_{i0} n / d^{10} = \epsilon_{i0} > 0$ rich out-sections of size at least $d^0 / C_{10}$. Therefore, there is at least one rich out-section of size at least $\frac{n}{d^{10}}$. Now, since this out-section is rich, each of its vertices has at least $[\frac{d}{2}]$ private neighbors, giving at least $n / 2 C_{10} > n^{0.95}$ total private neighbors.

Now, suppose that the event $JQ$ does not hold. Then, by the argument above, for some fixed sets $J, Q_1, ..., Q_{10}$, we must have $J_{10}'$. As argued at beginning of this proof, the num-
ber of ways to fix such sets is at most $2^{11} n^{2n^{0.91}}$ and thus, $\Pr [JQ^c] \leq 2^{11} n^{2n^{0.91}} \Pr [J_{10}] < 2^{11} n^{2n^{0.91}} e^{-\varepsilon_{10} n} = o(1)$.

This completes the proof of the lemma. □

5. Concluding remarks

Our intent in this paper was to disprove the normal graph conjecture. In fact, by setting $p := n^{-1+1/10g}$ and identically mimicking the argument one can prove that for every $g$, there exist graphs of girth $g$ which are not normal.

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