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St. Moretti, H. Norde

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Stefano Moretti[1,2] and Henk Norde[3]

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Abstract

In this note we consider the class of weighted multi-glove games. We will show that these games are totally balanced and we will characterize the weighted multi-glove games that are supermodular and pmas-admissible. Moreover, we will provide an explicit expression for the Shapley value of the supermodular and a large part of the pmas-admissible weighted multi-glove games.

Keywords: Multi-glove games, (totally) balancedness, population monotonic allocation schemes (pmas), supermodularity, Shapley value.

JEL code: C 71.

1 Introduction

Glove games have been introduced in [8]: players are divided in two groups L and R, players in L own one left glove and players in R one right glove. The worth of a coalition S is the number of pairs of gloves that can be formed by the members of S:

$$v(S) = \min\{|L \cap S|, |R \cap S|\}.$$

Players of different type (i.e. owning different types of gloves) are perfectly complementary, whereas players of the same type (i.e. owning the same type of gloves) are perfectly interchangeable and in competition for doing business with players of the other type. Glove games are examples of market games ([9]) where players own nonnegative bundles of goods and the value of a coalition is computed by redistributing the initial endowments of their members such that the sum of the individual (continuous and concave) utilities is maximal. To be more precise, glove games are glove-market games: market games where all utility functions are the same and given by $u(a) = \min\{a_1, \ldots, a_m\}$ for every $a \in \mathbb{R}_+^m$. In other words, combinations of 1 unit of every good are valuable and can be sold at a selling price that is normalized to 1. The classes of glove-market games, market games, flow games ([4]) and linear production games ([5]) all coincide with the class of totally balanced games as argued in [1]. Glove-market games where each player owns one unit of precisely one good (like in glove games) are called unitary glove-market games in [1], T(ype-based)-market games in [2] and ocean games in [6]. We will refer to these games as multi-glove games here.

Multi-glove games are totally balanced, so core elements exist for the game and all its subgames. However, it is not always possible to choose such core elements in a monotonic way: allocations to players do not decrease if the coalitions to which they belong become larger. In other words multi-glove games do not have to admit a population monotonic allocation scheme (pmas). In [11] it is shown that the four-player glove game with two players owning a left glove and two players a right glove lacks a pmas. In this note we generalize multi-glove games in the sense that players are allowed to own more than one glove but only gloves of the same type. We will refer to these games as weighted multi-glove games. We will show that these games are totally balanced (a result that by the way also directly follows from the fact that the class of weighted multi-glove games fall within the class of glove-market games). We will characterize the class of pmas-admissible weighted multi-glove games and the class of convex or supermodular weighted multi-glove games. Finally, we will provide an explicit expression for the Shapley value of the supermodular and a large part of the pmas-admissible weighted multi-glove games.

This note is organized as follows. Some preliminaries and notation is introduced in Section 2. In Section 3 we present our main results.

[1]Université Paris-Dauphine, PSL Research University, CNRS, UMR [7243], LAMSADE, 75016 Paris, France. Email: stefano.moretti@dauphine.fr

[3]Corresponding author, CentER and Department of Econometrics and Operations Research, Tilburg University, P.O. Box 90153,5000 LE Tilburg, The Netherlands, Email: h.norde@tilburguniversity.edu

2 Preliminaries and notation

A *Transferable Utility (TU) game* (or, simply, a *game*) is a pair (N, v), where $N = \{1, \ldots, n\}$ is the set of *players* and $v : 2^N \to \mathbb{R}$ the *characteristic function*. By convention $v(\emptyset) = 0$. For every *coalition* $S \subseteq N$ the number $v(S)$ represents the (monetary) *worth* or *profit* S can achieve by cooperation.

An *allocation* is a vector $x \in \mathbb{R}^N$. An allocation x is in the *core* $C(N, v)$ ([3]) of game (N, v) if it is *efficient* (i.e., $\sum_{i \in N} x_i = v(N)$) and *stable* (i.e., $\sum_{i \in S} x_i \geq v(S)$ for every coalition $S \in 2^N \backslash \{\emptyset\}$). So,

$$C(N, v) = \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N), \sum_{i \in S} x_i \geq v(S), \text{ for every } S \subseteq N, S \neq \emptyset\}.$$

A game (N, v) with a nonempty core is called *balanced*. For a game (N, v) and a coalition $S \in 2^N \backslash \{\emptyset\}$ the *subgame* $(S, v_{|S})$ of (N, v) restricted to coalition S is defined by $v_{|S}(R) = v(R)$ for every $R \in 2^S$. A game (N, v) with all subgames (i.e. including (N, v) itself) having a nonempty core is called *totally balanced*.

A *population monotonic allocation scheme* or *pmas* ([11]) of the game (N, v) is a *scheme* $(x_{S,i})_{S \in 2^N \backslash \{\emptyset\}, i \in S}$ with the properties

i) $\sum_{i \in S} x_{S,i} = v(S)$ for every $S \in 2^N \backslash \{\emptyset\}$ (*efficiency*);

ii) $x_{S,i} \leq x_{T,i}$ for every $S, T \in 2^N \backslash \{\emptyset\}$ and $i \in N$ with $i \in S \subset T$ (*monotonicity*).

A pmas provides an allocation vector for every coalition in a monotonic way: the value allocated to some player increases if the coalition to which he belongs becomes larger. If a game (N, v) has a pmas, then it is said to be *pmas-admissible*. It is easy to check that a pmas $(x_{S,i})_{S \in 2^N \backslash \{\emptyset\}, i \in S}$ provides a core element for the game and all its subgames, i.e. the allocation $(x_{S,i})_{i \in S} \in C(S, v_{|S})$ for every $S \in 2^N \backslash \{\emptyset\}$. Therefore, a pmas-admissible game is also totally balanced.

A game (N, v) is *supermodular* or *convex* ([10]) if for every $i \in N$ and for every $S, T \subseteq N \backslash \{i\}$ with $S \subset T$ we have $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$. Supermodular games are pmas-admissible ([11]) and hence totally balanced as well.

The *Shapley value* $\Phi(N, v)$ ([7]) is defined by

$$\Phi_i(N, v) = \sum_{S \in 2^{N \backslash \{i\}}} \frac{(|S|)!(n - |S| - 1)!}{n!}(v(S \cup \{i\}) - v(S))$$

for every $i \in N$. An alternative formula for the Shapley value is obtained by writing a game (N, v) as linear combination of unanimity games. For $T \in 2^N \backslash \{\emptyset\}$ the *unanimity game* u_T is defined by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise. Now $v = \sum_{T \in 2^N \backslash \{\emptyset\}} \alpha_T(v) u_T$, where $\alpha_T(v) = \sum_{S \in 2^T \backslash \{\emptyset\}} (-1)^{|T| - |S|} v(S)$ for every $T \in 2^N \backslash \{\emptyset\}$, and the Shapley value is given by

$$\Phi_i(N, v) = \sum_{T \in 2^N \backslash \{\emptyset\} : i \in T} \frac{\alpha_T(v)}{|T|}$$

for every $i \in N$.

3 Weighted multi-glove games

We first provide the definition of weighted multi-glove games.

Definition 3.1 *Let* $P = \{P_1, \ldots, P_k\}$ $(k \geq 2)$ *be a partition of the player set* N *and* $w \in \mathbb{N}^N$. *The* weighted multi-glove game $(N, v^{P,w})$ *is defined by*

$$v^{P,w}(S) = \min\{\sum_{i \in S \cap P_j} w_i : j \in \{1, \ldots, k\}\}$$

for every $S \in 2^N$ *(with the usual convention that* 0 *is the result of an empty summation).*

The context of weighted multi-glove games is the following: there are k different complementary goods, and the player set N is partitioned into k groups $P_1, \ldots, P_k$, each group containing players owning units of the same good. Every player $i \in N$ owns w_i units of the good specified by the partition element to which he belongs. Combinations of 1 unit of every good are valuable and can be sold at a selling price that is normalized to 1. Standard glove games ([8]) refer to the situation of 2 complementary goods (left gloves and right gloves), where each player owns one unit of one of the two goods. Weighted multi-glove games generalize these games in two directions: 1) there can be more than two goods (justifying the adjective 'multi'); 2) every player is allowed to have more than one unit of some good (justifying the adjective 'weighted'). In order to define weighted multi-glove games we assume that the player set N is partitioned in at least two sets. Of course, the game $(N, v^{P,w})$ can also be defined in case $k = 1$ (i.e. if $P = \{N\}$). In this case the game is an additive game ($v^{P,w}(S) = \sum_{i \in S} w_i$ for every $S \in 2^N$) which is totally balanced, supermodular and pmas-admissible anyhow. Also note that weighted multi-glove games are monotonic: if $S, T \in 2^N$ with $S \subset T$ then $v^{P,w}(S) \leq v^{P,w}(T)$.

Standard glove games are known to be totally balanced. This result can be generalized to weighted multi-glove games in an obvious way.

Theorem 3.2 *Weighted multi-glove games are totally balanced.*

Proof Let $(N, v^{P,w})$ be the weighted multi-glove game corresponding to partition $P = \{P_1, \ldots, P_k\}$ and weight vector $w \in \mathbb{N}^N$. Let $S \in 2^N \setminus \{\emptyset\}$. Determine $j^* \in \{1, \ldots, k\}$ such that $\sum_{i \in S \cap P_{j^*}} w_i = \min\{\sum_{i \in S \cap P_j} w_i : j \in \{1, \ldots, k\}\} = v^{P,w}(S)$. Define the vector $x \in \mathbb{R}^S$ by $x_i = w_i$ if $i \in S \cap P_{j^*}$ and $x_i = 0$ if $i \in S \setminus P_{j^*}$. Then $\sum_{i \in S} x_i = \sum_{i \in S \cap P_{j^*}} x_i = \sum_{i \in S \cap P_{j^*}} w_i = v^{P,w}(S) = (v^{P,w})_{|S}(S)$. Moreover, for every $R \subseteq S$, $R \neq \emptyset$, we have $\sum_{i \in R} x_i = \sum_{i \in R \cap P_{j^*}} x_i = \sum_{i \in R \cap P_{j^*}} w_i \geq \min\{\sum_{i \in R \cap P_j} w_i : j \in \{1, \ldots, k\}\} = v^{P,w}(R) = (v^{P,w})_{|S}(R)$. So $x \in C(S, (v^{P,w})_{|S})$. Since S was chosen in an arbitrary way we conclude that $(N, v^{P,w})$ is totally balanced. $\square$

Weighted multi-glove games do not have to be pmas-admissible: in [11] it is shown that the standard glove game with two players owning a left glove and two players owning a right glove does not have a pmas. So these games do not have to be supermodular as well. In this note we characterize the subclasses of weighted multi-glove games that are supermodular and pmas-admissible respectively. First we characterize the subclass of supermodular weighted multi-glove games.

Theorem 3.3 *Let $(N, v^{P,w})$ be a weighted multi-glove game with weight vector $w \in \mathbb{N}^N$ and partition $P = \{P_1, \ldots, P_k\}$. Then $(N, v^{P,w})$ is supermodular if and only if there is a $j^* \in \{1, \ldots, k\}$ such that for every $j \in \{1, \ldots, k\} \setminus \{j^*\}$ partition element P_j contains precisely one player i_j and $w_{i_j} \geq \sum_{i \in P_{j^*}} w_i$ for every $j \in \{1, \ldots, k\} \setminus \{j^*\}$.*

Proof "$\Leftarrow$". Suppose there is a $j^* \in \{1, \ldots, k\}$ such that for every $j \in \{1, \ldots, k\} \setminus \{j^*\}$ partition element P_j contains precisely one player i_j and $w_{i_j} \geq \sum_{i \in P_{j^*}} w_i$ for every $j \in \{1, \ldots, k\} \setminus \{j^*\}$. Let $l \in N$ and $S \subset T \subseteq N \setminus \{l\}$. If $l \notin P_{j^*}$ and $\cup_{j \in \{1, \ldots, k\} \setminus \{j^*\}} P_j \subseteq S \cup \{l\}$ then $v^{P,w}(S \cup \{l\}) - v^{P,w}(S) = \sum_{i \in (S \cup \{l\}) \cap P_{j^*}} w_i - 0 = \sum_{i \in (S \cup \{l\}) \cap P_{j^*}} w_i \leq \sum_{i \in (T \cup \{l\}) \cap P_{j^*}} w_i = \sum_{i \in (T \cup \{l\}) \cap P_{j^*}} w_i - 0 = v^{P,w}(T \cup \{i\}) - v^{P,w}(T)$. If $l \notin P_{j^*}$ and $\cup_{j \in \{1, \ldots, k\} \setminus \{j^*\}} P_j \not\subseteq S \cup \{l\}$ then $v^{P,w}(S \cup \{l\}) - v^{P,w}(S) = 0 - 0 = 0 \leq v^{P,w}(T \cup \{l\}) - v^{P,w}(T)$ by monotonicity of $v^{P,w}$. If $l \in P_{j^*}$ and $\cup_{j \in \{1, \ldots, k\} \setminus \{j^*\}} P_j \subseteq S$ then $v^{P,w}(S \cup \{l\}) - v^{P,w}(S) = w_l = v^{P,w}(T \cup \{l\}) - v^{P,w}(T)$. If $l \in P_{j^*}$ and $\cup_{j \in \{1, \ldots, k\} \setminus \{j^*\}} P_j \not\subseteq S$ then $v^{P,w}(S \cup \{l\}) - v^{P,w}(S) = 0 - 0 = 0 \leq v^{P,w}(T \cup \{l\}) - v^{P,w}(T)$, again by monotonicity of $v^{P,w}$.

"$\Rightarrow$". Suppose $(N, v^{P,w})$ is supermodular. First, assume that there are at least two partition elements with at least two players. Without loss of generality assume that $|P_1| \geq 2$ and $|P_2| \geq 2$. Choose $a, b \in P_1$, $a \neq b$, and $c, d \in P_2$, $c \neq d$. Choose $i_j \in P_j$ for every $j \in \{3, \ldots, k\}$. By supermodularity we have

$$v^{P,w}(\{a, c, d, i_3, \ldots, i_k\}) - v^{P,w}((\{a, d, i_3, \ldots, i_k\}) \geq v^{P,w}(\{a, c, i_3, \ldots, i_k\}) - v^{P,w}(\{a, i_3, \ldots, i_k\})$$

so

$$\min\{w_a, w_c + w_d, w_{i_3}, \ldots, w_{i_k}\} - \min\{w_a, w_d, w_{i_3}, \ldots, w_{i_k}\} \geq \min\{w_a, w_c, w_{i_3}, \ldots, w_{i_k}\} - 0,$$

so

$$\min\{w_a, w_c + w_d, w_{i_3}, \ldots, w_{i_k}\} \geq \min\{w_a, w_c, w_{i_3}, \ldots, w_{i_k}\} + \min\{w_a, w_d, w_{i_3}, \ldots, w_{i_k}\}. \tag{1}$$

3

From (1) we get

$$
\begin{aligned}
\min\{w_a, w_{i_3}, \ldots, w_{i_k}\} &\geq \min\{w_a, w_c + w_d, w_{i_3}, \ldots, w_{i_k}\} \\
&\geq \min\{w_a, w_c, w_{i_3}, \ldots, w_{i_k}\} + \min\{w_a, w_d, w_{i_3}, \ldots, w_{i_k}\} \\
&> \min\{w_a, w_c, w_{i_3}, \ldots, w_{i_k}\},
\end{aligned}
$$

so $\min\{w_a, w_c, w_{i_3}, \ldots, w_{i_k}\} = w_c$. In a similar way we get $\min\{w_a, w_d, w_{i_3}, \ldots, w_{i_k}\} = w_d$. Now, again using (1), we have $w_a \geq \min\{w_a, w_c + w_d, w_{i_3}, \ldots, w_{i_k}\} \geq \min\{w_a, w_c, w_{i_3}, \ldots, w_{i_k}\} + \min\{w_a, w_d, w_{i_3}, \ldots, w_{i_k}\} = w_c + w_d$. Using a symmetry argument, interchanging the roles of a and b, we get $w_b \geq w_c + w_d$. Again using a symmetry argument, now interchanging the roles of a and c and the roles of b and d, we get $w_c \geq w_a + w_b$ and $w_d \geq w_a + w_b$ as well. Hence $w_a \geq w_c + w_d > w_c \geq w_a + w_b > w_a$. A contradiction.

So there is at most one partition element which is not singleton. Without loss of generality assume that $|P_j| = 1$ for every $j \in \{1, \ldots, k-1\}$. Let i_j be the unique player in P_j for every $j \in \{1, \ldots, k-1\}$. If $|P_k| = 1$ we are done (taking $j^* \in \{1, \ldots, k\}$ such that P_{j^*} contains the player with minimal weight) so assume that $|P_k| \geq 2$. Let $S = \cup_{j=1}^{k-1} P_j = \{i_1, \ldots, i_{k-1}\}$ and let $i^* \in P_k$ be such that $w_{i^*} = \min_{i \in P_k} w_i$. Again, by supermodularity we have

$$
v^{P,w}(N) - v^{P,w}(N \setminus \{i^*\}) \geq v^{P,w}(S \cup \{i^*\}) - v^{P,w}(S),
$$

so

$$
\min\{w_{i_1}, \ldots, w_{i_{k-1}}, \sum_{i \in P_k} w_i\} - \min\{w_{i_1}, \ldots, w_{i_{k-1}}, \sum_{i \in P_k \setminus \{i^*\}} w_i\} \geq \min\{w_{i_1}, \ldots, w_{i_{k-1}}, w_{i^*}\} - 0,
$$

so

$$
\min\{w_{i_1}, \ldots, w_{i_{k-1}}, \sum_{i \in P_k} w_i\} \geq \min\{w_{i_1}, \ldots, w_{i_{k-1}}, \sum_{i \in P_k \setminus \{i^*\}} w_i\} + \min\{w_{i_1}, \ldots, w_{i_{k-1}}, w_{i^*}\}. \tag{2}
$$

Repeating the arguments above we get

$$
\min\{w_{i_1}, \ldots, w_{i_{k-1}}, \sum_{i \in P_k \setminus \{i^*\}} w_i\} = \sum_{i \in P_k \setminus \{i^*\}} w_i
$$

and

$$
\min\{w_{i_1}, \ldots, w_{i_{k-1}}, w_{i^*}\} = w_{i^*}.
$$

Now (2) reads $\min\{w_{i_1}, \ldots, w_{i_{k-1}}, \sum_{i \in P_k} w_i\} \geq \sum_{i \in P_k} w_i$ from which we infer that $w_{i_j} \geq \sum_{i \in P_k} w_i$ for every $j \in \{1, \ldots, k-1\}$. This finishes the proof. $\qquad \square$

So, the supermodular weighted multi-glove games are characterized by the fact that all but possibly one of the partition elements are singleton sets and the weight of the player in these sets exceeds the total weight of the players in the partition element which is not singleton (if any). In particular this implies that weighted multi-glove games with only singleton partition elements are supermodular anyhow.

The characterization of the pmas-admissible weighted multi-glove games has a similar flavour. If there is a singleton partition element the weighted multi-glove games are pmas-admissible anyhow and, in case all partition elements contain at least two players, they are precisely pmas-admissible if there is one 'dominated' partition element in the sense that every individual player in a non-dominated partition element has a weight that exceeds the total weight of the players in the dominated partition element.

Theorem 3.4 *Let $(N, v^{P,w})$ be a weighted multi-glove game with weight vector $w \in \mathbb{N}^N$ and partition $P = \{P_1, \ldots, P_k\}$. Then $(N, v^{P,w})$ is pmas-admissible if and only if either there is at least one singleton partition element or $|P_j| \geq 2$ for every $j \in \{1, \ldots, k\}$ and there is a $j^* \in \{1, \ldots, k\}$ such that for every $j \in \{1, \ldots, k\}, j \neq j^*$ and every $i \in P_j$ we have $w_i \geq \sum_{l \in P_{j^*}} w_l$.*

Proof "$\Leftarrow$". First, assume that there is at least one singleton partition element. Let $j \in \{1, \ldots, k\}$ be such that $|P_j| = 1$ and let i^* be the unique player in P_j. It is obvious that $v^{P,w}(S) = 0$ if $i^* \notin S$. Using monotonicity of $(N, v^{P,w})$ we find that the scheme $(x_{S,i})_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ defined by $x_{S,i} = v^{P,w}(S)$ if $i = i^*$ and $x_{S,i} = 0$ if $i \neq i^*$ is a

pmas of $(N, v^{P,w})$. Second, assume that $|P_j| \geq 2$ for every $j \in \{1, \ldots, k\}$ and there is a $j^* \in \{1, \ldots, k\}$ such that for every $j \in \{1, \ldots, k\}, j \neq j^*$ and every $i \in P_j$ we have $w_i \geq \sum_{l \in P_{j^*}} w_l$. It is obvious that for S with $S \cap P_j \neq \emptyset$ for every $j \in \{1, \ldots, k\}, j \neq j^*$ we have $v^{P,w}(S) = \sum_{i \in S \cap P_{j^*}} w_i$ and for other S we have $v^{P,w}(S) = 0$. Now the scheme $(x_{S,i})_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ defined by $x_{S,i} = w_i$ if $i \in P_{j^*}$ and $S \cap P_j \neq \emptyset$ for every $j \in \{1, \ldots, k\}, j \neq j^*$ and $x_{S,i} = 0$ otherwise is a pmas of $(N, v^{P,w})$.

"⇒". Assume that $(N, v^{P,w})$ is pmas-admissible, let $(x_{S,i})_{S \in 2^N \setminus \{\emptyset\}, i \in S}$ be a pmas for $(N, v^{P,w})$. Suppose that $|P_j| \geq 2$ for every $j \in \{1, \ldots, k\}$. Let $i_1 \in N$ be such that $w_{i_1} = \min_{i \in N} w_i$. Without loss of generality we may assume that $i_1 \in P_1$. Let $i_2 \in \cup_{j=2}^k P_j$ be such that $w_{i_2} = \min_{i \in \cup_{j=2}^k P_j} w_i$. We can also assume without loss of generality that $i_2 \in P_2$. Choose players $i_j \in P_j$ for every $j \in \{3, \ldots, k\}$ in an arbitrary way. Define $S = \{i_1, i_2, \ldots, i_k\}$. Let $j \in \{2, \ldots, k\}$. As $|P_j| \geq 2$ we can choose a player $i^* \in P_j$, $i^* \neq i_j$. Now $v^{P,w}(S \cup \{i^*\}) = v^{P,w}((S \cup \{i^*\}) \setminus \{i_j\}) = w_{i_1}$. Hence $x_{S \cup \{i^*\}, i_j} = \sum_{l \in S \cup \{i^*\}} x_{S \cup \{i^*\}, l} - \sum_{l \in (S \cup \{i^*\}) \setminus \{i_j\}} x_{S \cup \{i^*\}, l} = v^{P,w}(S \cup \{i^*\}) - \sum_{l \in (S \cup \{i^*\}) \setminus \{i_j\}} x_{S \cup \{i^*\}, l} \leq v^{P,w}(S \cup \{i^*\}) - \sum_{l \in (S \cup \{i^*\}) \setminus \{i_j\}} x_{(S \cup \{i^*\}) \setminus \{i_j\}, l} = v^{P,w}(S \cup \{i^*\}) - v^{P,w}((S \cup \{i^*\}) \setminus \{i_j\}) = w_{i_1} - w_{i_1} = 0$. Now from monotonicity of a pmas it follows that $x_{S,i_j} \leq x_{S \cup \{i^*\}, i_j} \leq 0$ as well. On the other hand we have $x_{S,i_j} \geq x_{\{i_j\}, i_j} = v^{P,w}(\{i_j\}) = 0$. We conclude that $x_{S,i_j} = 0$ for every $j \in \{2, \ldots, k\}$. As $v^{P,w}(S) = w_{i_1}$ we get $x_{S,i_1} = w_{i_1}$. Now consider coalition $T = P_1 \cup \{i_2, \ldots, i_k\}$. We have $v^{P,w}(T) = \min\{\sum_{l \in P_1} w_l, w_{i_2}\}$ and $v^{P,w}(T \setminus \{i_1\}) = \min\{\sum_{l \in P_1 \setminus \{i_1\}} w_l, w_{i_2}\}$. Now $x_{T,i_1} = \sum_{l \in T} x_{T,l} - \sum_{l \in T \setminus \{i_1\}} x_{T,l} = v^{P,w}(T) - \sum_{l \in T \setminus \{i_1\}} x_{T,l} \leq v^{P,w}(T) - \sum_{l \in T \setminus \{i_1\}} x_{T \setminus \{i_1\}, l} = v^{P,w}(T) - v^{P,w}(T \setminus \{i_1\})$. As $S \subset T$ we get $w_{i_1} = x_{S,i_1} \leq x_{T,i_1} \leq v^{P,w}(T) - v^{P,w}(T \setminus \{i_1\}) = \min\{\sum_{l \in P_1} w_l, w_{i_2}\} - \min\{\sum_{l \in P_1 \setminus \{i_1\}} w_l, w_{i_2}\}$, so $\min\{\sum_{l \in P_1} w_l, w_{i_2}\} \geq w_{i_1} + \min\{\sum_{l \in P_1 \setminus \{i_1\}} w_l, w_{i_2}\} = \min\{\sum_{l \in P_1} w_l, w_{i_1} + w_{i_2}\}$. From this last inequality we derive that $w_{i_2} \geq \sum_{l \in P_1} w_l$. So, for every $j \in \{2, \ldots, k\}$ and $i \in P_j$ we have $w_i \geq w_{i_2} \geq \sum_{l \in P_1} w_l$. This finishes the proof. □

The Shapley value of a weighted multi-glove game can be computed in an efficient way in case there is a 'dominated' partition element: every individual player in the other partition elements has a weight that exceeds the total weight of the players in the dominated partition element. Note that this covers the supermodular weighted multi-glove games and the pmas-admissible weighted multi-glove games without singleton partition elements.

Theorem 3.5 *Let $(N, v^{P,w})$ be a weighted multi-glove game with weight vector $w \in \mathbb{N}^N$ and partition $P = \{P_1, \ldots, P_k\}$. Assume there is a $j^* \in \{1, \ldots, k\}$ such that for every $j \in \{1, \ldots, k\}, j \neq j^*$ and every $i \in P_j$ we have $w_i \geq \sum_{l \in P_{j^*}} w_l$. Then for every $t \in N$ we have*

$$
\Phi_t(N, v^{P,w}) = \begin{cases} w_t \cdot \displaystyle\sum_{T \in 2^{N \setminus P_{j^*}} : T \cap P_j \neq \emptyset \forall j \neq j^*} (-1)^{|T|-(k-1)} \cdot \frac{1}{1+|T|} & \text{if } t \in P_{j^*} \\[2em] (\displaystyle\sum_{i \in P_{j^*}} w_i) \cdot \displaystyle\sum_{T \in 2^{N \setminus P_{j^*}} : T \cap P_j \neq \emptyset \forall j \neq j^*, t \in T} (-1)^{|T|-(k-1)} \cdot \frac{1}{1+|T|} & \text{if } t \notin P_{j^*}. \end{cases}
$$

Proof We will show that $v^{P,w}$ can be written as linear combination of unanimity games in the following way:

$$
v^{P,w} = \sum_{i \in P_{j^*}} (w_i \cdot \sum_{T \in 2^{N \setminus P_{j^*}} : T \cap P_j \neq \emptyset \forall j \neq j^*} (-1)^{|T|-(k-1)} \cdot u_{\{i\} \cup T}).
$$

The formula for the Shapley value then follows in a straightforward way. Let $S \in 2^N$. We have

$$
\sum_{i \in P_{j^*}} (w_i \cdot \sum_{T \in 2^{N \setminus P_{j^*}} : T \cap P_j \neq \emptyset \forall j \neq j^*} (-1)^{|T|-(k-1)} \cdot u_{\{i\} \cup T}(S)) =
$$

$$
\sum_{i \in P_{j^*} \cap S} (w_i \cdot \sum_{T \in 2^{S \setminus (P_{j^*} \cap S)} : T \cap P_j \neq \emptyset \forall j \neq j^*} (-1)^{|T|-(k-1)}) =
$$

$$
\sum_{i \in P_{j^*} \cap S} (w_i \cdot \sum_{(A_j)_{j \neq j^*} : \emptyset \neq A_j \subseteq P_j \cap S \forall j \neq j^*} (-1)^{\sum_{j \neq j^*} (|A_j|-1)}) =
$$

$$\sum_{i \in P_{j^*} \cap S} \left(w_i \cdot \sum_{(A_j)_{j \neq j^*}: \emptyset \neq A_j \subseteq P_j \cap S \forall j \neq j^*} \prod_{j \neq j^*} (-1)^{|A_j|-1} \right) =$$
$$\sum_{i \in P_{j^*} \cap S} \left(w_i \cdot \prod_{j \neq j^*} \sum_{A_j: \emptyset \neq A_j \subseteq P_j \cap S} (-1)^{|A_j|-1} \right).$$

For a finite nonempty set A the number of odd subsets equals the number of even subsets: $\sum_{B:B \subseteq A} (-1)^{|B|-1} = 0$. If we exclude the empty (even) subset from the summation we get $\sum_{B:\emptyset \neq B \subseteq A} (-1)^{|B|-1} = 1$. Therefore we get that if $P_j \cap S \neq \emptyset$ then $\sum_{A_j: \emptyset \neq A_j \subseteq P_j \cap S} (-1)^{|A_j|-1} = 1$. Of course, if $P_j \cap S = \emptyset$ then $\sum_{A_j: \emptyset \neq A_j \subseteq P_j \cap S} (-1)^{|A_j|-1} = 0$ (empty sum). So

$$\sum_{i \in P_{j^*} \cap S} \left(w_i \cdot \prod_{j \neq j^*} \sum_{A_j: \emptyset \neq A_j \subseteq P_j \cap S} (-1)^{|A_j|-1} \right) = \begin{cases} \sum_{i \in P_{j^*} \cap S} w_i & \text{if } P_j \cap S \neq \emptyset \forall j \neq j^* \\ 0 & \text{otherwise} \end{cases}$$
$$= v^{P,w}(S).$$

This finishes the proof. $\qquad\qquad\square$

Note that according to the Shapley value players in the dominated partition element receive a payoff proportional to their weight, whereas players in dominating partition elements receive a share of the total revenue $\sum_{i \in P_{j^*}} w_i$. In the supermodular case (so $|P_j| = 1$ for every $j \neq j^*$) it is easy to see that $\Phi_t(N, v^{P,w}) = \frac{1}{k} \cdot w_t$ if $t \in P_{j^*}$ and $\Phi_t(N, v^{P,w}) = \frac{1}{k} \cdot \sum_{i \in P_{j^*}} w_i$ if $t \notin P_{j^*}$, because the only set $T \in 2^{N \backslash P_{j^*}}$ such that $T \cap P_j \neq \emptyset$ for every $j \neq j^*$ is the set $N \backslash P_{j^*}$ with cardinality $k-1$. So all players in the dominating partition elements get an equal share of the total revenue. In cases where not all dominating partition elements are singleton players in different dominating partition elements can get a different share of the total revenue. The following example illustrates this.

Example 3.6 *Let $N = \{1, 2, 3, 4, 5, 6, 7\}$ be partitioned into $P_1 = \{1, 2\}$, $P_2 = \{3, 4, 5\}$ and $P_3 = \{6, 7\}$. Let $w \in \mathbb{N}^N$ be such that $w_i \geq w_1 + w_2$ for every $i \in P_2 \cup P_3$. So P_1 is the dominated partition element. Applying Theorem 3.5 we get $\Phi_t(N, v^{P,w}) = \frac{7}{12} w_t$ if $t \in P_1$, $\Phi_t(N, v^{P,w}) = \frac{1}{20}(w_1 + w_2)$ if $t \in P_2$ and $\Phi_t(N, v^{P,w}) = \frac{2}{15}(w_1 + w_2)$ if $t \in P_3$.*

In absence of a dominated partition element we cannot expect to be able to compute the Shapley value in an efficient way, as suggested by the following example.

Example 3.7 *Let $N = \{1, 2, 3, 4\}$ be partitioned into $P_1 = \{1, 2\}$ and $P_2 = \{3, 4\}$. Let $w \in \mathbb{N}^N$ be given by $w_1 = 2$, $w_2 = 3$, $w_3 = 4$ and $w_4 = 5$. Note that P_1 is not dominated by P_2 because $w_3 < w_1 + w_2$. One readily checks that $v^{P,w}$ can be written as linear combination of unanimity games in the following way:*

$$v^{P,w} = 2u_{\{1,3\}} + 2u_{\{1,4\}} + 3u_{\{2,3\}} + 3u_{\{2,4\}} - u_{\{1,2,3\}} - 2u_{\{1,3,4\}} - 3u_{\{2,3,4\}} + u_{\{1,2,3,4\}}.$$

From this we infer that $\Phi(N, v^{P,w}) = (\frac{15}{12}, \frac{23}{12}, \frac{9}{12}, \frac{13}{12})$. Note that in none of the partition elements the allocation is proportional to the individual weights.

Let us finish this note by making a remark about the weight vector. We assumed the weights to be positive and integer-valued, in order to stay close to the context of gloves. However, all results in this note are valid as well when the weights are assumed to be positive and real-valued. If we would like to generalize our results to nonnegative weights (i.e. we allow for zero weights as well) we have to be careful. Of course, a player with zero weight is a null player and adding or removing such a player does not affect properties like totally balancedness, pmas-admissibility and supermodularity. The statement in Theorem 3.3 however should be rephrased into "$(N, v^{P,w})$ is supermodular if and only if there is a $j^* \in \{1, \dots, k\}$ such that for every $j \in \{1, \dots, k\} \backslash \{j^*\}$ partition element P_j contains precisely one player i_j with positive weight and $w_{i_j} \geq \sum_{i \in P_{j^*}} w_i$ for every $j \in \{1, \dots, k\} \backslash \{j^*\}$."

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