PhD title: "Algorithmic and Structural Properties of box-TDI Polytopes"

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Scope. A large part of optimization problems amounts to optimizing a linear program, where the feasible region is a polyhedron defined by linear inequalities. The complexity of solving such problems is heavily influenced by the structure of the polyhedron. In particular, when a polyhedron is integer, it is well known that we can solve such problems in polynomial time on the size of the problem [7]. In practice, one of the most efficient algorithms is still the Simplex Method developed by Dantzig. Even if this method is known for having bad theoretical performances [8, 9], it has seen a renewed interest and several theoretical advances [5], in particular some recent developments connect the structure of a polyhedron and the efficiency of this algorithm [1]. One other point of interest for this algorithm is the strong connection with the polyhedral structure of the problem itself. In particular, one key factor influencing the performance of the simplex algorithm is the polyhedral diameter, which bounds the number of pivots required in the worst case. In this context, a weak form of the Hirsch Conjecture has been proven valid for polytopes defined by Totally Unimodular Matrices [2, 6].

Box-TDI polyhedra are the polyhedra that can be described by a box-TDI system. These polyhedra directly generalize polyhedra described by Totally Unimodular Matrices [3]. Moreover, even if the integer linear programming has recently proved to be NP-Hard on box-TDI polyhedra [4], the topic has not yet been explored when such polyhedra are integer.

The main goal of this project is to study whether box-TDI polyhedra admit improved diameter bounds and whether this has implications for the efficiency of linear programming algorithms.

Objectives. This PhD project will focus on the following key objectives:

- Investigate whether the diameter of box-TDI polyhedra admits tighter bounds than general polyhedra, particularly in relation to the Hirsch conjecture.
- Study whether classic optimization algorithms, such as the simplex and interior point methods, perform efficiently on box-TDI polyhedra.
- Analyze whether, when a box-TDI polyhedron is integer, its structural properties (e.g., diameter and number of facets) allow the simplex algorithm to solve optimization problems in polynomial time.
- Explore the combinatorial and geometric characteristics of box-TDI polyhedra that influence algorithmic performance.

This topic fits perfectly within the research directions of LAMSADE's Pole 2, as it mixes combinatorial optimization considerations with matricial and geometrical aspects.

Planning for the first year.

- September-December 2025: compose a solid foundation on the topic of box-TDI polyhedra, as well as a knowledge of the recent results on the Simplex Method.
- January-August 2026: apply the recent techniques, proposed for instance in [2], to Box-TDI polyhedra, and deduce a lower bound on the diameter of such polyhedra.
- September-December 2026: redaction of the paper.

Impact and consequences. Progress in this area could lead to new insights into the relationship between polyhedral structure and optimization complexity. If it is shown that integer box-TDI polyhedra admit polynomial-time solutions via the simplex algorithm, this would represent a significant breakthrough in understanding the property of box-TDI polyhedra, as well as giving an applicative interest for the resolution of applied problems. Additionally, improvements in diameter bounds could refine our theoretical understanding of polyhedral graphs and pivoting methods.

Planning for 3 years. The research direction will be adjusted based on progress in the first year. However, a tentative plan is:

- (2027) - generalize recent advances on the Simplex Method [5, 1] to deduce its theoretical complexity on box-TDI polyhedra.
- (2028) - Finalize theoretical contributions, refine algorithmic analysis, and prepare the PhD dissertation.

Required skills. The candidate should have a Master's degree in Mathematics, Computer Science, Operations Research, or a strictly related field. A solid background in mathematics, especially in linear algebra and polyhedra, will be highly valued. Familiarity with integer programming concepts and techniques is expected, given the optimization focus.

References

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