PhD title: "Box-perfect graphs"

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Scope. The characterization of box-perfect graphs is an open question posed by Cameron and Edmonds in 1982 [1]. These graphs are a subclass of perfect graphs for which the maximum stable set problem satisfies properties akin to a max-flow min-cut theorem [2]. More precisely, they are the graphs for which the clique inequalities form a box-totally dual integral system. The past few years, several new results appeared concerning this intriguing class of graphs: the exhibition of several new subclasses in [6], a weak box-perfect graph theorem in [4], and a new polyhedral characterization in [3].

Despite these recent progress, no good conjecture is still known that would characterize these graphs by forbidding induced minors. The goal of this PhD is to push even further the investigation of these graphs, hopefully obtaining a purely graphical characterization.

This topic fits perfectly within the research directions of LAMSADE's pole 2, as its mixes combinatorial optimization considerations with matricial and graph theoretic aspects.

Objectives. For a start, we would like to build infinite families of box-perfect and non box-perfect graphs. The goal there is to try to obtain a lead towards a suitable conjecture of forbidden structures. A first step will be the following:

- Question 1: Given a minimally non-totally unimodular matrix, what is the minimum number of unit vectors to be added as new columns to obtain a matrix describing a face of the stable set polytope of some perfect graph?

A complementary approach is the study of operations preserving box-perfection. On some examples, we noticed the existence of "bad" even pairs: pairs of vertices whose contraction preserve non box-perfection.

- Question 2: Characterize/generalize the existence of "bad" even pairs.

As a consequence of the weak box-perfect graph theorem [4], it is known when the complete join of two graphs is box-perfect. Several other graph operations appear in the decomposition of perfect graphs, and the next steps are:

- Question 3: Under which condition do operations that preserve perfection also preserve box-perfection?

Underlying all these question also lies an algorithmic one:

- Question 4: Can box-perfect graphs be recognized in polynomial time?

These steps will be a good starting point for exploring the general case, which is the aim of this PhD Thesis.

Planning for the first year.

- September-December 2025: Learn the various new results about box-perfect graphs [6, 4, 3], together with the matricial and geometrical structure of box-totally dual integral polyhedra [5].
- January-March 2026: Via the study of Question 1, provide new families of non box-perfect graphs.
- April-November 2026: Investigate Questions 2 and 3 for the new families.
- End of 2026: Express a suitable conjecture for the structures to be excluded.

Impact and consequences. Progress on this more than 40 years old problem would help understanding a fundamental question: what are the structures in which stable sets satisfy some kind of max-flow min-cut theorem? The more general topic of box-total dual integrality has been especially lively in the combinatorial optimization community the past few years, and this PhD would also offer progress about: how does box-total dual integrality interact with graph theoretic frameworks? Answering even partially some of the questions above has the potential to identify parameters that drive combinatorial min-max theorems and may lead to practical applications. Additionally, the research may extend recent results, solving previously unsolved special cases and contributing to both theoretical foundations and algorithmic efficiency in optimization problems.

Planning for 3 years. The planning for the first year can be found above. For the next two years, the planning will heavily rely on the progress of the first year. Yet, we can imagin the following: at the beginning of 2027, a suitable conjecture will have been expressed, and the tools needed to tackle it will have been narrowed. In particular, the candidate will be able then to focus on specific unresolved subcases, before dealing with the general case.

Required skills. The Ph.D. student should have a master's degree in optimization or mathematical programming. A solid background in mathematics, especially in linear algebra, graph theory, and polyhedra, will be highly valued. Familiarity with integer programming concepts and techniques is expected, given the optimization focus.

References

[1] K. Cameron. Polyhedral and algorithmic ramifications of antichains. *Ph.D. Thesis, University of Waterloo. (Supervisor: J. Edmonds)*, 1982.

[2] Kathie Cameron. A min-max relation for the partial q-colourings of a graph. Part II: Box perfection. *Discrete Mathematics*, 74(1):15–27, 1989.

[3] Patrick Chervet and Roland Grappe. The essence of a box-totally dual integral polytope. *In preparation.*

[4] Patrick Chervet and Roland Grappe. A weak box-perfect graph theorem. *Journal of Combinatorial Theory, Series B*, 169:367–372, 2024.

[5] Patrick Chervet, Roland Grappe, and Louis-Hadrien Robert. Box-total dual integrality, box-integrality, and equimodular matrices. *Mathematical Programming*, 188(1):319–349, 2021.

[6] Guoli Ding, Wenan Zang, and Qiulan Zhao. On box-perfect graphs. *Journal of Combinatorial Theory, Series B*, 128:17–46, 2018.