JOURNAL OF MULTI-CRITERIA DECISION ANALYSIS, VOL 5, 145–159 (1996)

A Theoretical Framework for Analysing the Notion of Relative Importance of Criteria

BERNARD ROY and VINCENT MOUSSEAU
LAMSADE, Université Paris-Dauphine, Place du M[al] De Lattre de Tassigny, F-75775 Paris Cedex 16, France

ABSTRACT

Multiple-criteria decision aid almost always requires the use of weights, importance coefficients or even a hierarchy of criteria, veto thresholds, etc. These are importance parameters that are used to differentiate the role devoted to each criterion in the construction of comprehensive preferences. Many researchers have studied the problem of how to assign values to such parameters, but few of them have tried to analyse in detail what underlies the notion of importance of criteria and to give a clear formal definition of it. In this paper our purpose is to define a theoretical framework so as to analyse the notion of the importance of criteria under very general conditions. Within this framework it clearly appears that the importance of criteria is taken into account in very different ways in various aggregation procedures. This framework also allows us to shed new light on fundamental questions such as: Under what conditions is it possible to state that one criterion is more important than another? Are importance parameters of the various aggregation procedures dependent on or independent of the encoding of criteria? What are the links between the two concepts of the importance of criteria and the compensatoriness of preferences? This theoretical framework seems to us sufficiently general to ground further research in order to define theoretically valid elicitation methods for importance parameters.

KEY WORDS: decision aid; importance of criteria; weights; aggregation procedures

1. INTRODUCTION

This paper deals with Multiple-Criteria Decision Aid (MCDA). In the present work we will suppose defined

(1) a set A of alternatives,
(2) a family F of n semi-criteria $g_1, g_2, \ldots, g_n$ ($n \geqslant 3$) pertinent in a given decision context ($F = \{1, 2, \ldots, n\}$).

More precisely, we suppose that the preferences of the Decision Maker (DM) concerning the alternatives of A are formed and argued, in the context considered, by reference to n points of view adequately reflected by the criteria contained in F.

We denote by $g_j(a)$ the value attributed to the alternative a on the jth criterion; we will call this value the jth performance of a; $g_j(a)$ is a real number. The greater it is, the better the alternative a is for the DM. Specifically, we only suppose that $g_j(a) \geqslant g_j(b)$ implies that the alternative a is at least as good as the alternative b with respect to the jth point of view. When considering the only point of view formalized by g_j, the imprecision and/or the uncertainty and/or the inaccurate determination of performances may lead the DM to judge that a is indifferent to b when $g_i(a) = g_i(b), \forall i \neq j$, even if $g_j(a) \neq g_j(b)$. So as to account for such a statement, criteria contained in F are usually considered to be semi-criteria, i.e., such that

$$\begin{cases} aP_jb & \Leftrightarrow \quad g_j(a) > g_j(b) + q_j \\ aI_jb & \Leftrightarrow \quad |g_j(a) - g_j(b)| \leqslant q_j \end{cases}$$

where

(1) q_j, the indifference threshold, represents the maximum difference of performance compatible with an indifference situation,*
(2) aP_jb is to be interpreted as a preference for a over b on criterion g_j,
(3) aI_jb is to be interpreted as an indifference between a and b on criterion g_j.

P_j is called the preference relation restricted to the jth criterion (the same terminology holds for I_j). (I_j, P_j) defines a semi-order on A.

*In this paper we will suppose q_j constant; this is not restrictive, since any semi-order stemming from a semi-criterion using a variable threshold $q_j(g_j)$ can be represented by another semicriterion using a constant threshold (Roubens and Vincke, 1985).

CCC 1057–9214/96/020145–15

Accepted 12 June 1995

From a comprehensive point of view we refer to a position that accounts for the n points of view simultaneously. From such a comprehensive point of view, when considering two performance vectors $(g_1(a), g_2(a), \ldots, g_n(a))$ and $(g_1(b), g_2(b), \ldots, g_n(b))$, the DM can have difficulty for choosing, in a well-thought-out and stable way, one proposition from the three given below:

(1) b is strictly preferred to a (denoted by bPa);
(2) a is indifferent to b (denoted by aIb);
(3) a is strictly preferred to b (denoted by aPb).

These difficulties stem from zones of imprecision and conflict in the DM's mind. So as not to compel the DM to make arbitrary judgements, it is prudent and usual (Roy, 1985) to introduce a third relation in order to model the DM's preferences at a comprehensive level. This relation, incomparability, will be denoted by R and reflects the absence of positive reasons for choosing one of the three possibilities given above or the willingness to put off such a choice. Two alternatives a and b are said to be incomparable (aRb) if none of the three assertions aIb, aPb and bPa is valid. Thus for every pair of alternatives (a, b) one and only one of the following four assertions is valid:

$$\begin{cases} aPb \\ aIb \\ bPa \\ bRa \end{cases} \qquad (1)$$

Let us consider the outranking relation $S = P \cup I$, aSb being interpreted as 'a is at least as good as b'. Since $aPb \Rightarrow aSb \Rightarrow \text{not}[bPa]$, the four assertions of system (1) correspond to the following assertions:

$$\begin{cases} aPb \\ aSb \text{ and not}[aPb] \\ bPa \\ \text{not}[aSb] \text{ and not}[bPa] \end{cases} \qquad (2)$$

In this paper we consider a comprehensive preference system using three relations (I, P, R).* Let us recall that a standard approach to MCDA consists of grounding the construction of the comprehensive preferences on the restricted preferences (I_j and $P_j, \forall j \in F$) corresponding to the n criteria. This is done through a so-called Multiple-Criteria Aggregation Procedure (MCAP). In the MCAP all criteria are not supposed to play the same role; the criteria are commonly said not to have the same importance. This is why there are in the MCAP parameters that aim at specifying the role of each criterion in the aggregation of performances. We will call such parameters *importance parameters*. They aim at accounting for how the DM attaches importance to the points of view modelled by the criteria.

The nature of these parameters varies across MCAPs. The way of formalizing the relative importance of each criterion differs from one aggregation model to another. All this is done, for instance, by means of

(1) scaling constants in multiattribute utility theory (Keeney and Raiffa, 1976, 1993),
(2) a weak-order on F in lexicographic techniques or a complete pre-order on F in the ORESTE method (Roubens, 1982),
(3) intrinsic weights in PROMETHEE methods (Brans *et al.*, 1984),
(4) intrinsic weights combined with veto thresholds in ELECTRE methods (Roy and Bertier, 1973; Roy, 1991a,b),
(5) trade-offs or substitution rates in the Zionts and Wallenius (1976, 1983) procedures,
(6) aspiration levels in interactive methods (Vanderpooten and Vincke, 1989),
(7) eigen vectors of a pairwise comparison matrix in the AHP method (Saaty, 1980).

What exactly is covered by this notion of importance? What does a decision maker mean by assertions such as 'criterion g_j is more important than criterion g_i', 'criterion g_j has a much greater importance than criterion g_i'. etc? Let us recall by way of comparison that the assertion 'b is preferred to a' reflects the fact that if the decision maker must choose between b and a, he is supposed to decide in favour of b. Thus it is possible to test whether this assertion is valid or not. There is no similar possibility for comparing the importance of criteria. These remarks lead more generally to the question of what gives meaning in practice to this notion of importance. We will come back to this question at the end of the paper by showing that the framework presented here helps us to progress in the understanding of how to elicit this notion. Moreover, the way this notion is taken into account, within the framework of the various models mentioned above,

*The reader will find more details concerning the basic concepts of preference modelling in Vincke (1990).

by means of importance parameters reveals deep differences, as will be shown below.

This paper has a dual purpose:

(1) to propose a formal framework allowing us to give sense to the notion of relative importance of criteria under very general conditions;
(2) to use this formal framework to give some partial answers to questions such as
   (a) under what conditions is it possible to state that one criterion is more important than another?
   (b) are importance parameters of the various MCAPs dependent on or independent of the encoding of criteria?
   (c) what are the links between the two concepts of importance of criteria and the compensatoriness of preferences?

It seems to us that the proposed theoretical framework increases our capability to interpret the information obtained through interacting with the DM in order to express formally his positions concerning the respective role of the various criteria through numerical values assigned to importance parameters. This question is not addressed directly in this paper (certain aspects of this problem are studied in Mousseau (1993)). In any case, the theoretical framework proposed here highlights some basic traps to be avoided when assigning values to importance parameters.

We will introduce in Section 2 a formal definition of the notion of relative importance of criteria as well as the theoretical framework we propose to analyse it. In Section 3 we will present some results on the theoretical framework introduced. This will lead us to specify in Section 4 how the relative importance of criteria is taken into account in some basic aggregation procedures. In Section 5 we will show how the proposed theoretical framework sheds new light on certain fundamental problems. In the last section we will consider the two ways in which it is possible to give meaning to the notion of importance of criteria and show the interest of the proposed framework in both approaches.

2. THE THEORETICAL FRAMEWORK PROPOSED

Many authors have studied the problem of the elicitation of the relative importance of criteria (RIC),* but few have tried to give a precise definition of this notion.†

2.1. What information characterizes the notion of relative importance of criteria?

The first statement we should make when trying to give a formal definition of the notion of RIC is the following: the information underlying this notion is much richer than that contained in the importance parameters of the various multicriteria models. In fact, these parameters are mainly scalars and constitute a simplistic way of taking RIC into account, since this notion is by nature of a functional type.

In the comparison of two alternatives a and b when one or several criteria are in favour of a and one or several others in favour of b, the way each MCAP solves this conflict and determines the comprehensive preference refers to the importance attached to each criterion (and to the logic of the aggregation used). Thus the result of such conflicts (i.e., the comprehensive preference situation between a and b) constitutes the elementary data providing information on the relative importance of the criteria in conflict.

In order to delimit exhaustively the importance of a criterion, we should analyse the contribution of any preferences at the restricted level of a criterion to the comprehensive level (for each pair of alternatives). Nevertheless, when two alternatives are indifferent on criterion g_j, the comprehensive preference situation will generally give no significant information concerning the importance of this criterion.

When the preference model is of an (I, P, R) type, it does not seem restrictive to delimit the information concerning the relative importance of a criterion using only preference and outranking relations (see system (2) in Section 1). *On such a basis we will assume that the empirical content of the RIC notion refers to the nature and variety of cases in which a preference restricted to a given criterion g_j leads us to accept the same preference on the comprehensive level, or only an outranking in the same direction, or to refuse the inverse preference.* This conception may be synthesized as follows.

*A critial overview of the literature in the field can be found in Mousseau (1992).

†For an interesting exception see Podinovskii (1991, 1994).

Basic empirical hypothesis
The relative importance of criterion g_j is characterized by *

(1) the set of '*situations*' in which aP_jb and aPb hold simultaneously,
(2) the set of '*situations*' in which aP_jb and aSb hold simultaneously,
(3) the set of '*situations*' in which aP_jb and not $[bPa]$ hold simultaneously.

2.2. Notation

Let us denote by

(1) X_i the ordered set of possible performances on criterion g_i ($X_i \subset \mathbb{R}$),
(2) $X = \prod_{i=1}^{n} X_i$ the set of performance vectors,
(3) $\underline{x} = (x_1, x_2, \ldots, x_n) \in X$ a performance vector corresponding to an alternative a such that $g_i(a) = x_i, \forall i \in F$,
(4) $X_{-j} = \prod_{i \neq j} X_i$ the set of performance vectors from which the jth component is removed,
(5) $\underline{x}_{-j} = (x_1, x_2, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \in X_{-j}$.

In order to simplify the notation, we will denote by $\underline{x} = (\underline{x}_{-j}, x_j)$

$$(\underline{x}_{-j}, x_j)P(\underline{y}_{-j}, y_j) \quad \Leftrightarrow \quad \underline{x}P\underline{y}$$

The dominance relation Δ relative to F is defined by

$$\forall \underline{x}, \underline{y} \in X^2, \underline{x}\Delta\underline{y} \quad \Leftrightarrow \quad x_i \geqslant y_i, \forall i \in F$$

The data of the preference (I_j, P_j) restricted to the n viewpoints, together with the comprehensive preference system (I, P, R),† define what we call a preference structure Ψ on X.

Considering a preference structure Ψ on X, the following sets are defined for all $x_j, y_j \in X_j^2$ such that $x_jP_jy_j$:

$$\Delta_j^P(x_j, y_j) = \{(\underline{x}_{-j}, \underline{y}_{-j}) \in X_{-j}^2 \text{ such that } (\underline{x}_{-j}, x_j)P(\underline{y}_{-j}, y_j)\}$$

$$\Delta_j^S(x_j, y_j) = \{(\underline{x}_{-j}, \underline{y}_{-j}) \in X_{-j}^2 \text{ such that } (\underline{x}_{-j}, x_j)S(\underline{y}_{-j}, y_j)\}$$

$$\Delta_j^V(x_j, y_j) = \{(\underline{x}_{-j}, \underline{y}_{-j}) \in X_{-j}^2 \text{ such that not } [(\underline{y}_{-j}, y_j)P(\underline{x}_{-j}, x_j)]\}$$

The set $\Delta_j^P(x_j, y_j)$ (resp. $\Delta_j^S(x_j, y_j)$ and $\Delta_j^V(x_j, y_j)$) contains all combinations of performances ($\underline{x}_{-j}$ and $\underline{y}_{-j}$) that it is possible to combine with x_j and y_j when $x_jP_jy_j$ so as to obtain $\underline{x}P\underline{y}$ (resp., $\underline{x}S\underline{y}$ and *not* $[\underline{y}P\underline{x}]$) on the comprehensive preference level.

Since these sets are defined only $\forall x_j, y_j \in X_j^2$ such that $x_jP_jy_j$, we will not mention this condition of definition explicitly each time we refer to these sets.

2.3. Formal definition of the relative importance of criteria

The following definition is a simplification of the one proposed in Roy (1991a) analysing the more general case of pseudo-criteria. Using the notation given above, this definition formalizes the basic empirical hypothesis of Section 2.1.

Definition 1
The importance of a semi-criterion g_j is characterized by the sets $\Delta_j^P(x_j, y_j)$, $\Delta_j^S(x_j, y_j)$ and $\Delta_j^V(x_j, y_j)$ (defined $\forall x_j, y_j \in X_j^2$ such that $x_jP_jy_j$). The values attributed to the importance parameters must be grounded on the range and shape of the preceding sets.

This highlights the restricted nature of the importance parameters relative to the complexity of the notion of RIC.

Let us illustrate this definition with some important specific cases.

1. $\Delta_j^P(x_j, y_j) = X_{-j}^2$ means that $(\underline{x}_{-j}, x_j)P(y_{-j}, y_j)$ holds $\forall x_j, y_j \in X_j^2$ such that $x_jP_jy_j$ and $\forall(\underline{x}_{-j}, \underline{y}_{-j}) \in X_{-j}^2$. Criterion g_j imposes its viewpoint and is thus a dictator (see Section 4.2).
2. $\Delta_j^P(x_j, y_j) = X_{-j}^2(S) = \{(\underline{x}_{-j}, \underline{y}_{-j}) \in X_{-j}^2$ such that $\forall i \in F\backslash\{j\}, x_i \geqslant y_i - q_i$ (i.e., such that $x_iS_iy_i$)$\}$ means that the only cases in which

*In this characterization the contribution of criterion g_j to the preferences at the comprehensive level is considered only through the situations of strict preference at the restricted level of criterion g_j. This does not seem restrictive when all criteria are semi-criteria, but it can become restrictive when F contains pseudo-criteria (Roy and Vincke, 1984).

†The question of where the comprehensive preferences come from will be addressed in Section 6.

$x_jP_jy_j$ and $(\underline{x}_{-j}, x_j)P(\underline{y}_{-j}, y_j)$ hold simultaneously are the cases in which no other criterion is opposed to g_j $(x_iS_iy_i \Leftrightarrow \text{not}[y_iP_ix_i])$. In other words, criterion g_j imposes its decision on the comprehensive level only when no other criterion is opposed to it.

3. $\Delta_j^S(x_j, y_j) = X^2_{-j}(S)$ and $\Delta_j^P(x_j, y_j) \subset \Delta_j^S(x_j, y_j)$ mean that the importance of criterion g_j is reduced (relative to the preceding case), since even when no criterion is opposed to it, g_j does not always impose a strict preference on the comprehensive level.
4. $\Delta_j^S(x_j, y_j) = \Delta_j^V(x_j, y_j), \forall x_j, y_j \in X_j^2$, such that $x_jP_jy_j$ means that the only cases in which $x_jP_jy_j$ appears compatible with $\underline{y}P\underline{x}$ are those where $\underline{x}S\underline{y}$ holds. Let us remark that other cases can exist, particularly when a criterion g_j opposes a veto to $\underline{y}S\underline{x}$ (see Section 3.2.4).

3. INTERPRETATION AND RESULTS

3.1 Interpretation

Since $aPb \Rightarrow aSb \Rightarrow$ not[bPa], the following sequence of inclusions holds (see Section 3.2.1):

$$\forall x_j, y_j \in X_j^2, \Delta_j^P(x_j, y_j) \subseteq \Delta_j^S(x_j, y_j) \subseteq \Delta_j^V(x_j, y_j)$$

If F satisfies the cohesion axiom (see Appendix I), $\Delta_j^S(x_j, y_j) \supseteq X^2_{-j}(S)$. Moreover, $\Delta_j^P(x_j, y_j) \supseteq X^2_{-j}(S)$ holds if $\forall j \in F, [x_jP_jy_j$ and $x_i = y_i, \forall i \neq j] \Rightarrow \underline{x}P\underline{y}$.

From this fact it is possible to give a concrete interpretation of the differences (see Figure 1).

1. $C_j(x_j, y_j) = \{(\underline{x}_{-j}, \underline{y}_{-j}) \in \Delta_j^P(x_j, y_j)$ such that $(\underline{x}_{-j}, \underline{y}_{-j}) \notin X^2_{-j}(S)\}$ reflects the *ability of criterion g_j to* ***convince***, i.e., the cases in which the comprehensive preference between two alternatives is in favour of the partial preference on criterion g_j in spite of the opposition of other criteria.
2. $E_j(x_j, y_j) = \Delta_j^S(x_j, y_j) \backslash \Delta_j^P(x_j, y_j)$ reflects the *ability of criterion g_j to* ***equilibrate***, i.e., the cases in which the partial preference on criterion g_j allows us to obtain an indifference between the two alternatives in spite of the opposition of criteria.
3. $O_j(x_j, y_j) = \Delta_j^V(x_j, y_j) \backslash \Delta_j^S(x_j, y_j)$ reflects the *ability of criterion g_j is to* ***oppose***, i.e., the cases in which the partial preference on criterion g_j is opposed to other criteria in such a way that

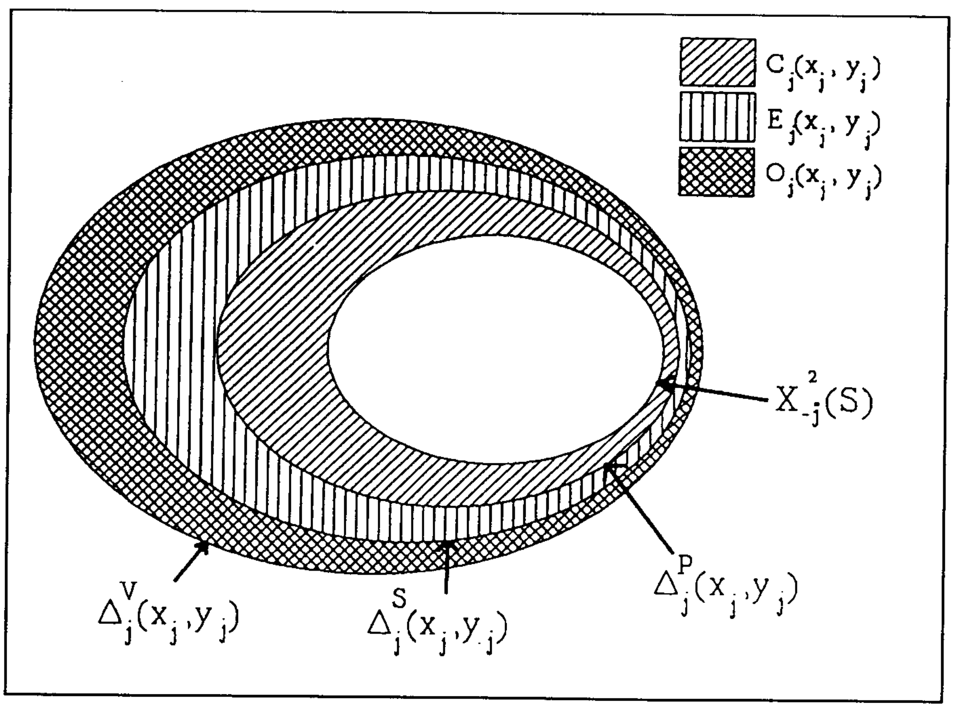

Figure 1. Interpretation of the sets $C_j(x_j, y_j)$, $E_j(x_jy_j)$ and $O_j(x_j, y_j)$ in the case where $\Delta_j^P(x_j, y_j) \supseteq X^2_{-j}(S)$

the two alternatives are incomparable on the comprehensive level (see Section 3.2.4).

The illustrations presented in Section 4 will allow the reader to verify that it is interesting to compare the relative sizes of the sets $C_j(x_j, y_j)$, $E_j(x_j, y_j)$ and $O_j(x_j, y_j)$ in order to analyse how these three different aspects of the notion of importance are taken into account in several MCAPs.

3.2. Results

In what follows, the family F of criteria will be supposed consistent, i.e., will verify the three axioms given in Appendix I (Roy and Bouyssou, 1993, Chap. 2). The proofs of the results presented in this subsection can be found in Appendix II.

3.2.1. Embedded nature of the sets $\Delta_j^H(x_j, y_j), H \in \{P, S, V\}$

Result 1

$\forall j \in F, \forall x_j, y_j \in X_j^2$ it holds that

$$\begin{cases} \Delta_j^P(x_j, y_j) \subseteq \Delta_j^S(x_j, y_j) \subseteq \Delta_j^V(x_j, y_j) \\ \Delta_j^S(x_j, y_j) \supseteq X_{-j}^2(S) \end{cases}$$

Moreover, if criterion g_j is inductive, i.e., such that $[g_i(a) = g_i(b), \forall i \neq j$, and $aP_jb] \Rightarrow aPb$, then

$$\Delta_j^P(x_j, y_j) \supseteq X_{-j}^2(S)$$

3.2.2. Expanding nature of the sets $\Delta_j^H(x_j, y_j), H \in \{P, S, V\}$

Result 2

$\forall x_j, y_j, u_j, v_j \in X_j^4$ such that $u_j \geqslant x_j \quad y_j \geqslant v_j$, x_j, P_j, y_j,

$$\Delta_j^H(x_j, y_j) \subseteq \Delta_j^H(u_j, v_j), \ \forall H \in \{P, S, V\}$$

3.2.3. Borderline of the sets $\Delta_j^H(x_j, y_j), H\{P, S, V\}$

Since Δ denotes the dominance relation relative to F (see Section 2.2), we denote by Δ_{-j} the dominance relation defined on $F\backslash\{j\}$.

Result 3

If $(\underline{x}_{-j}, \underline{y}_{-j}) \in \Delta_j^H(x_j, y_j)$, then $\forall(\underline{u}_{-j}, \underline{v}_{-j}) \in X_{-j}^2$ verifying

$$\begin{cases} \underline{u}_{-j}\, \Delta_{-j}\, \underline{x}_{-j} \\ \underline{y}_{-j}\, \Delta_{-j}\, \underline{v}_{-j} \end{cases}$$

it holds:

$$(\underline{u}_{-j}, \underline{v}_{-j}) \in \Delta_j^H(x_j, y_j) \ (H \in \{P, S, V\})$$

As a consequence, there exists a minimal subset from which it is possible to derive all the other elements of $\Delta_j^H(x_j, y_j)$. This subset will be called the borderline of $\Delta_j^H(x_j, y_j)$.

Definition 1*

$(\underline{x}_{-j}, \underline{y}_{-j})$ will be a borderline element of $\Delta_j^H(x_j, y_j)$ iff

$$\forall(\underline{u}_{-j}, \underline{v}_{-j}) \in \Delta_j^H(x_j, y_j)$$
$$\left.\begin{matrix} \underline{x}_{-j}\, \Delta_{-j}\, \underline{u}_{-j} \\ \underline{v}_{-j}\, \Delta_{-j}\, \underline{y}_{-j} \end{matrix}\right\} \Rightarrow (\underline{u}_{-j}, \underline{v}_{-j}) = (\underline{x}_{-j}, \underline{y}_{-j})$$

The knowledge of the set of all the borderline elements (i.e., the borderline of $\Delta_j^H(x_j, y_j)$) is sufficient to reconstruct the whole set $\Delta_j^H(x_j, y_j), H \in \{P, S, V\}$. An illustrative example of a borderline is given in Appendix III.

Let us denote by $C_j'(x_j, y_j)$, $E_j'(x_j, y_j)$ and $O_j'(x_j, y_j)$ the borderlines of $\Delta_j^P(x_j, y_j), \Delta_j^S(x_j, y_j)$ and $\Delta_j^V(x_j, y_j)$, respectively. If $\forall j \in F, X_j$ is discrete,† we have the following inclusions:

$$\forall j \in F, \forall x_j, y_j \in X_j \begin{cases} C_j'(x_j, y_j) \subset C_j(x_j, y_j) \\ E_j'(x_j, y_j) \subset E_j(x_j, y_j) \\ O_j'(x_j, y_j) \subset O_j(x_j, y_j) \end{cases}$$

Thus we will call these three borderlines the conviction, the equilibration and the opposition borderlines, respectively. Since the knowledge of these three borderlines allows us to characterize the form and extent of the sets $\Delta_j^H(x_j, y_j)$ $(H = P, S, V)$, we can claim, according to Definition 1, that all the information concern-

---

*In the case of a family of criteria containing some criteria for which X_j is continuous, if $\Delta_j^H(x_j, y_j)$ is an open set, it is necessary to replace this set by its topological closure.

†If not, the following inclusions still hold by substituting for C_j, E_j and O_j their topological closure.

ing the importance of the criterion g_j is contained in $C'_j(x_j, y_j)$, $E'_j(x_j, y_j)$ and $O'_j(x_j, y_j)$, $\forall x_j, y_j \in X_j^2$.

Consequently, when decision aid is grounded on a given MCAP, the numerical values attributed to the importance parameters of this MCAP are sufficient to specify the three borderlines. Hence the numerical values for importance parameters must be chosen in the light of the three resulting borderlines.

3.2.4. Opposition-incomparability result

Considering the comprehensive relational system (I, P, R) which underlies the sets $\Delta_j^H(x_j, y_j)$ $(H = P, S, V)$, we have the following result.

Result 4

$\forall j \in F, \forall x_j, y_j \in X_j^2$ such that $x_j\ P_j\ y_j$,

$$O'_j(x_j, y_j) = \varnothing \quad \Leftrightarrow \quad \nexists \underline{x}_{-j}, \underline{y}_{-j} \in X_{-j}^2 \text{ such that } (\underline{x}_{-j}, x_j) R (\underline{y}_{-j}, y_j)$$

As a consequence we have (see Sections 4.1–4.3)

$$\forall j \in F, \forall x_j, y_j \in X_j^2 \text{ such that } x_j P_j y_j,$$
$$\Delta_j^S(x_j, y_j) = \Delta_j^V(x_j, y_j) \quad \Leftrightarrow \quad R = \varnothing$$

4. CHARACTERIZATION OF THE SETS $\Delta_j^H(x_j, y_j)$ IN SOME BASIC MCAPs

For a given MCAP the sets $\Delta_j^H(x_j, y_j)$ $(\forall H \in \{P, S, V\})$ are completely defined when numerical values are attributed to importance parameters. Conversely, the knowledge of all the sets $\Delta_j^H(x_j, y_j)$ defines an MCAP in a unique way. In fact, the sets $\Delta_j^H(x_j, y_j)$ contain the comprehensive preference induced by any couple of performance vectors. There is a one-to-one correspondence between an MCAP (with given numerical values for its importance parameters) and the sets $\Delta_j^H(x_j, y_j)$.

So as to highlight the link between importance parameters and the shape of the sets $\Delta_j^H(x_j, y_j)$ (or their respective borderlines), we will characterize these sets below in the case of basic simple MCAPs.

4.1 Weighted sum

The weighted sum aggregation uses importance coefficients k_j $(k_j \geqslant 0)$ such that

$$\begin{cases} \underline{x}P\underline{y} & \Leftrightarrow \quad \sum_{i=1}^{n} k_i x_i > \sum_{i=1}^{n} k_i y_i \\ \underline{x}I\underline{y} & \Leftrightarrow \quad \sum_{i=1}^{n} k_i x_i = \sum_{i=1}^{n} k_i y_i \end{cases}$$

This MCAP implies that all criteria are true criteria. It follows that the sets $\Delta_j^H(x_j, y_j)$ are characterized by

$$\begin{cases} \Delta_j^P(x_j, y_j) = \{(\underline{x}_{-j}, \underline{y}_{-j}) \in X_{-j}^2 \\ \quad \text{such that } \sum_{i \neq j} k_i(x_i - y_i) > k_j(y_j - x_j)\} \\ \Delta_j^S(x_j, y_j) = \{(\underline{x}_{-j}, \underline{y}_{-j}) \in X_{-j}^2 \\ \quad \text{such that } \sum_{i \neq j} k_i(x_i - y_i) \geqslant k_j(y_j - x_j)\} \\ \Delta_j^V(x_j, y_j) = \Delta_j^S(x_j, y_j) \end{cases}$$

In this MCAP the importance is taken into account essentially through the ability of criteria to convince. The ability to equilibrate is very weak: the only couples $(\underline{x}_{-j}, \underline{y}_{-j})$ contained in $E_j(x_j, y_j)$ *and* $E'_j(x_j, y_j)$ are those verifying $\sum_{i \neq j} k_i(x_i - y_i) = k_j(y_j - x_j)$. No ability to oppose is attached to the criteria $(O_j(x_j, y_j) = O'_j(x_j, y_j) = \varnothing)$.

4.2. Lexicographic aggregation

In this MCAP the importance attached to each criterion is characterized by its position in a hierarchy such that g_1 is the most important criterion and g_n the least important one. More precisely, in this MCAP we have

$$\begin{cases} x_1 > y_1 \Rightarrow \underline{x}P\underline{y} \text{ whatever } \underline{x}_{-j}, \underline{y}_{-j} \\ x_1 = y_1, \ldots, x_{i-1} = y_{i-1}, x_i > y_i \Rightarrow \underline{x}P\underline{y} \\ \quad \text{whatever } x_{i+1}, \ldots, x_n, y_{i+1}, \ldots, y_n \\ x_1 = y_1, \ldots, x_n = y_n \Rightarrow \underline{x}I\underline{y} \end{cases}$$

This implies* that all criteria are true criteria $(q_j = 0)$. In this MCAP the importance parameter takes the form of a permutation on the set of criteria. The sets $\Delta_j^H(x_j, y_j)$ are characterized by

*A direct transposition of these formulae to the case of semi-criteria leads to a comprehensive preference relation which is not transitive.

$$\begin{cases} \Delta_1^P(x_1, y_1) = \Delta_1^S(x_1, y_1) = \Delta_1^V(x_1, y_1) = \\ \quad \{(\underline{x}_{-1}, \underline{y}_{-1}) \in X_{-1}^2\} \\ \Delta_2^P(x_2, y_2) = \Delta_2^S(x_2, y_2) = \Delta_2^V(x_2, y_2) = \\ \quad \{(\underline{x}_{-2}, \underline{y}_{-2}) \in X_{-2}^2 \text{ such that } x_1 \geqslant y_1\} \\ \Delta_3^P(x_3, y_3) = \Delta_3^S(x_3, y_3) = \Delta_3^V(x_3, y_3) = \\ \quad \{(\underline{x}_{-3}, \underline{y}_{-3}) \in X_{-3}^2 \text{ such that } (x_1 = y_1 \text{ and} \\ \quad x_2 \geqslant y_2) \text{ or } (x_1 > y_1)\} \end{cases}$$

etc.

In this MCAP the RIC is only taken into account through the ability of criteria to convince ($E_j(x_j, y_j) = O_j(x_j, y_j) = \varnothing$). We can observe that g_1 is a dictator.

4.3. Majority rule

This MCAP is defined by

$$\begin{cases} \underline{x}S\underline{y} & \Leftrightarrow \quad \sum_{i \in C[\underline{x}S\underline{y}]} k_i \geqslant s \\ \underline{x}P\underline{y} & \Leftrightarrow \quad \underline{x}S\underline{y} \text{ and not}[\underline{y}S\underline{x}] \end{cases}$$

where

(1) s is the majority level, $s \in]\frac{1}{2}, 1]$,
(2) $C[\underline{x}S\underline{y}]$ is the coalition of criteria such that $x_i S_i y_i$,
(3) k_j is the weight of criterion g_j, $\sum_{i=1}^{n} k_j = 1$.

In this MCAP it holds that

$$\begin{cases} \Delta_j^S(x_j, y_j) = \{(\underline{x}_{-j}, \underline{y}_{-j}) \in X_{-j}^2 \text{ such that} \\ \quad \sum_{i \neq j \text{ and } x_i \geqslant y_i - q_i} k_i \geqslant s - k_j\} \\ \Delta_j^V(x_j, y_j) = \Delta_j^S(x_j, y_j) \\ \Delta_j^P(x_j, y_j) = \{(\underline{x}_{-j}, \underline{y}_{-j}) \in \Delta_j^S(x_j, y_j) \text{ such that} \\ \quad \sum_{y_i \geqslant x_i - q_i} k_i < s - k_j\} \end{cases}$$

In this aggregation procedure the importance of criteria is taken into account through the ability of criteria both to convince and equilibrate (no ability to oppose is attached to criteria: $O_j(x_j, y_j) = \varnothing$).

4.4. ELECTRE I MCAP extended to semi-criteria*

In this MCAP the importance parameters are of two types: for each criterion g_j there is a weight k_j and a veto threshold v_j ($v_j \geq q_j$). $\underline{x}S\underline{y}$ holds iff

(1) the majority rule holds for a given majority level s,
(2) $\nexists i \in F$ such that $y_i > x_i + v_i$ (non-veto condition).

In this MCAP it holds that

$$\begin{cases} \Delta_j^S(x_j, y_j) = \{(\underline{x}_{-j}, \underline{y}_{-j}) \in X_{-j}^2 \text{ such that} \\ \quad \sum_{i \neq j \text{ and } x_i \geqslant y_i - q_i} k_i \geqslant s - k_j \text{ and} \\ \quad y_i \leqslant x_i + v_i, \forall i \neq j\} \\ \Delta_j^V(x_j, y_j) = \begin{cases} X_{-j}^2 & \text{if } x_j > y_j + v_j \\ \Delta_j^S(x_j, y_j) \cup V(X_{-j}^2) & \text{otherwise,} \\ \quad \text{with } V(X_{-j}^2) = \{(\underline{x}_{-j}, \underline{y}_{-j}) \\ \quad \in X_{-j}^2 \text{ such that} \\ \quad \exists i \in F \text{ verifying } x_i > y_i + v_i\} \end{cases} \\ \Delta_j^P(x_j, y_j) = \{(\underline{x}_{-j}, \underline{y}_{-j}) \in \Delta_j^S(x_j, y_j) \text{ such that} \\ \quad \sum_{y_i \geqslant x_i - q_i} k_i < s - k_j \text{ or } \exists i \in F \text{ such that} \\ \quad y_i > x_i + v_i\} \end{cases}$$

In this MCAP the importance of criteria is taken into account through all three aspects: ability to convince, to equilibrate and to oppose. This is due to the fact that in general

$$\begin{cases} C_j(x_j, y_j) \neq \varnothing \\ E_j(x_j, y_j) \neq \varnothing \\ O_j(x_j, y_j) \neq \varnothing \end{cases}$$

5. CONCERNING SOME FUNDAMENTAL QUESTIONS

5.1 How to give sense to the relation '*at least as important as*' between two criteria

Let us consider the sets $\Delta_j^H(x_j, y_j)$ ($H \in \{P, S, V\}$) corresponding to an MCAP and a particular importance parameter w_j. If when w_j increases (resp., decreases) the sets $\Delta_j^H(x_j, y_j)$ do not decrease, it seems reasonable to state that greater (resp., lesser) is w_j and greater is the importance given to criterion g_j in the considered MCAP. This consideration shows that variations of the value assigned to an importance parameter result in variations of the importance of the corresponding criterion in a constant direction. Unfortunately,

*See Roy and Bouyssou (1993) or Vincke (1992).

this link is not very useful for comparing the respective importance of two criteria.

In order to give sense to the relation '*at least as important as*' between two criteria, we propose to take the following empirical idea as a starting point.

Basic empirical hypothesis

The assertion 'g_j is at least as important as g_i' refers to the following statements.

1. The variety of cases in which xPy with $x_jP_jy_j$ and $y_iP_ix_i$ is greater than the variety of similar cases in which g_i is substituted for g_j and the reverse.
2. The variety of cases in which xSy with $x_jP_jy_j$ and $y_iP_ix_i$ is greater than the variety of similar cases in which g_i is substituted for g_j and the reverse.
3. The variety of cases in which not [yPx] with $x_jP_jy_j$ and $y_iP_iy_i$ is greater than the variety of similar cases in which g_i is substituted for g_j and the reverse.

This empirical hypotheses leads us to propose hereafter a formal condition for g_j to be at least as important as g_i grounded on the sets $\Delta_j^H(x_j, y_j)$ and $\Delta_i^H(x_i, y_i)$. However, we are confronted with difficulties linked to the fact that these two sets are neither defined on the same space (X^2_{-j} and X^2_{-i}, respectively) nor under the same conditions ($\forall x_j, y_j \in X_j^2$ such that $x_jP_jy_j$ and $\forall x_i, y_i \in X_i^2$ such that $x_iP_iy_i$, respectively). So as to circumvent these difficulties, we will introduce the following additional notations and definitions.

1. $$X_{-ij} = \prod_{k \neq i,j} X_k$$
2. $\Delta_j^H(x_j, y_j/y_iP_ix_i)$ is the restriction of $\Delta_j^H(x_j, y_j)$ to couples $(\underline{x}_j, \underline{y}_j)$ verifying $y_iP_ix_i$.
3. Let $D \subseteq X^2_{-j}$; we define $\mathrm{Proj}_i(D)$ as the subset of X^2_{-ij} obtained by deleting in each element of D the evaluations x_i and y_i.

On such a basis we propose the following condition.

Condition 1

A necessary condition for g_j to be at least as important as g_i is: $\forall H \in \{P, S, V\}, \forall z_i, t_i \in X_i^2$ such that $z_iP_it_i, \exists z_j, t_j$ verifying

$$\begin{cases} z_j P_j t_j \\ \mathrm{Proj}_j[\Delta_i^H(z_i, t_i/y_j P_j x_j)] \subset \\ \quad \mathrm{Proj}_i[\Delta_j^H(z_j, t_j/y_i P_i x_i)] \end{cases}$$

For $H = P$ this condition corresponds to the following statement: $\forall \underline{x}, \underline{y} \in X^2$ such that $\underline{x}P\underline{y}$ with $y_jP_jx_j$ and $x_iP_iy_i$ there exist $\underline{z}, \underline{t}$ such that $\underline{z}P\underline{t}$ with $z_h = x_h$, $t_h = y_h$, $\forall h \neq i, j$, $z_jP_jt_j$ and $t_iP_iz_i$.

This condition can lead to refusing conjointly both assertions 'g_j is as least as important as g_i' and 'g_i is as least as important as g_j'. This can be the case in an MCAP using two importance parameters for each criterion. Such a case (and more generally the following considerations) shows how difficult it is to give meaning to the empirical idea that one can have concerning the relative importance of two criteria.

The above condition cannot be considered as a sufficient one because it can be verified while the empirical idea of importance leads to rejecting this assertion. For example, it is the case in ELECTRE (see Section 4.4) with $k_i = k_j$ and $v_i = q_i$ and v_j much greater than q_j. Let us remark that if the ability to oppose a veto exists for g_j and not for g_i, then Condition 1 states that 'g_j is at least as important as g_i', the reverse being false. The difficulty arises when both criteria can effectively oppose their veto. A suggestion to formulate a necessary and sufficient condition may be found in Roy (1991a, Section 5.1).

When applying Condition 1 to the MCAP studied in Section 4, it appears that:

(1) in the weighted sum aggregation, criterion g_j is at least as important as criterion g_i only if $k_jL_j \geqslant k_iL_i$, with $L_j = g_j^* - g_j{}^*$;
(2) in the lexicographic aggregation, g_j is at least as important as g_i only if $j < i$;
(3) in the aggregation based on the majority rule, criterion g_j is at least as important as criterion g_i only if $k_j \geqslant k_1$.

5.2 Importance parameters dependent on or independent of the encoding of criteria

Defining a criterion consists of specifying how to take into account (for decision aid) the particular consequences attached to a given viewpoint. Thus encoding criterion j amounts to choosing a real value function g_j on the basis of which the preferences, relative to the attached viewpoint, may be argued.

However, there is a certain degree of freedom concerning how to define the function g_j in a convenient way. Hence there are monotonically increasing transformations ϕ such that g_j and $\phi(g_j)$ may be viewed as equally convenient in order to define the criterion, i.e., $\phi(g_j)$ could, in the decision context considered, replace g_j so as to describe, conceive and argue comprehensive preferences (the MCAP being unchanged). In this case ϕ is said to be an admissible transformation for g_j. Let Φ_j be the set of admissible transformations for g_j. Specifying the set Φ_j is necessary in order to specify the signification attached to each of the elements of X_j so as to ground comparisons of preference differences. Let us recall that (X_j, Φ_j) is called (Vansnick, 1990)

(1) an absolute scale when $\Phi_j = \{\text{identity}\}$,
(2) a ratio scale when $\Phi_j = \{\phi$ such that $\phi(g_j) = \alpha.g_j$ with $\alpha > 0\}$,
(3) an interval scale when $\Phi_j = \{\phi$ such that $\phi(g_j) = \alpha.g_j + \beta$ with $\alpha > 0\}$,
(4) an ordinal scale when $\Phi_j = \{\phi$ monotonically increasing$\}$.

The sets $\Delta_j^H(x_j, y_j)$ $(H \in \{P, S, V\})$ are subsets of X_{-j}^2. Consequently, X_{-j} does not depend on the ϕ chosen in Φ_j. For all $\phi \in \Phi_j$ the following invariance conditions may hold: $\Delta_j^H(x_j, y_j) = \Delta_j^H(\phi(x_j), \phi(y_j)), \forall H \in \{P, S, V\}, \forall x_j, y_j$. Considering a given MCAP, there is no guarantee that these invariance conditions hold, $\forall x_j, y_j$ and $\forall \phi \in \Phi_j$ without adapting the values of importance parameters to the encoding of X_j, i.e., to the chosen ϕ in Φ_j. This leads us to distinguish two cases for admissible transformations $(\in \Phi_j)$ adopted in a given context.

1. If these equalities hold without changing the values of importance parameters, the characterization of the sets $\Delta_j^H(x_j, y_j)$ is, in the MCAP under consideration, invariant when the encoding of g_j is changed (i.e., when we substitute $\phi(g_j)$ for g_j). Considering the definition proposed for importance parameters, it appears that these parameters are independent of the encoding of g_j; such parameters are said to be intrinisc to the significance axis of g_j.
2. In the opposite case it is necessary to adapt the value of some importance parameters in order to keep the sets $\Delta_j^H(x_j, y_j)$ unchanged when the encoding of g_j is transformed. In this case the importance parameters are said to be dependent on the encoding of the criterion.

Such terminology, when applied to the characterization of the sets $\Delta_j^H(x_j, y_j)$ for the MCAP studied in Section 4, leads to the following results.

1. The coefficients k_j of a weighted aggregation are dependent on the encoding of criteria.
2. The ranking of criteria that specifies the relative importance of criteria in the lexicographic aggregation is intrinsic to the significance axis of criteria.
3. The coefficients k_j of the MCAP based on the majority rule are independent of the encoding of criteria.
4. In the MCAP ELECTRE I the importance coefficients k_j are intrinsic while the veto thresholds are dependent on the encoding of criteria.

Hence the reader should be aware that it is inconsistent to assign values to importance parameters that are dependent on the encoding of the considered criterion without explicitly taking this encoding into account. Moreover, we should keep in mind that some MCAPs are appropriate only if Φ_j is restricted to a limited type of transformation (for example, the weighted sum is convenient only if $\Phi_j, j \in F$, are restricted to ϕ such that $\phi(g_j = \alpha.g_j + \beta$ with $\alpha > 0)$.

5.3 Importance invariant by translation

The formalism introduced in Section 3.3 leads us to highlight an interesting property: the invariance of the importance of a criterion by translation.

Definition 2

The importance of criterion g_j is invariant by translation if and only if

$$\forall \alpha \in \mathbb{R}, \forall x_j, y_j \in X_j, \Delta_j^H(x_j, y_j) = \Delta_j^H(x_j + \alpha, y_j + \alpha), \forall H \in \{P, S, V\}$$

When the importance of a criterion g_j is invariant by translation, a given difference between the performances of two alternatives has the same influence on the comprehensive preferences whatever the values of the performances. This definition may be restricted to a subset of X_j^2.

Two particular cases of non-invariance of the importance of a criterion by translation should be emphasized.

1. The condition $\forall\alpha \in \mathbb{R}$, $\forall x_j, y_j \in X_j^2$, $\Delta_j^H(x_j, y_j) \subseteq \Delta_j^H(x_j + \alpha, y_j + \alpha), \forall H = \{P, S, V\}$ characterizes the situation in which the importance of criterion g_j increases with its performance.
2. Reversing the inclusions in the preceding condition leads to characterizing the situation in which the importance of criterion g_j decreases with its performance.

When the importance parameters of a criterion are intrinsic to its significance axis, if invariance by translation holds for a specific encoding, this property still holds for any other admissible encoding. On the other hand, if certain importance parameters are dependent on the encoding, this property of invariance may be verified for certain encodings but becomes false for others.

5.4 Link between the notions of importance and compensation

The following considerations and definitions will demonstrate the strong relationship which exists between the notions of the relative importance of criteria and the compensatoriness of preferences. This last notion is, in its current use as well as in multicriteria methods, linked to the possibility of making substitutions between performances on different criteria. These substitutions allow us to compensate for a disadvantage on one or several criteria by a sufficent advantage on other criteria.

When an MCAP uses such ideas in order to solve conflicts between criteria, this MCAP is usually said to be of a compensatory nature; in the opposite case it is said to be non-compensatory. Consequently, a preference structure induced by an MCAP has a more or less compensatory nature.

More precisely, we will say that there is no possibility of compensation relative to criterion g_j if, in each indifference situation $\underline{x}I\underline{y}$ with $x_jP_jy_j$, when we destroy the indifference situation by substituting u_j for x_j and v_j for y_j (with $u_jP_jv_j$), it is not possible to re-establish it by modifying performances on other criteria ($g_i, i \neq j$).

The following definition formalizes this concept.

Definition 3
There is no possibility of compensation relative to criterion g_j iff $\forall x_j, y_j \in X_j^2$ verifying $x_jP_jy_j, \forall(\underline{x}_{-j}, \underline{y}_{-j}) \in E_j(x_j, y_j)$,

(1) if there exist $u_j, v_j \in X_j^2$ verifying $u_jP_jv_j$ and $(\underline{x}\_j, \underline{y}\_j) \notin E_j(u_j, , v_j)$,
(2) then $E_j(u_j, v_j) = \varnothing$.

On the contrary, there are some possibilities of compensation between the criteria g_j and g_k if there exist $\underline{x}$ and $\underline{y}$ such that $\underline{x}I\underline{y}$ for which it is possible to destroy this indifference situation by decreasing the difference in performances between their kth component and to re-establish it by increasing the difference of performances on their jth components.

The following definition formalizes this conception.

Definition 4
There are some possibilities of compensation between the criteria g_j and g_k iff

(1) $\exists x_j, y_j$ verifying $x_j\,P_j\,y_j, \exists(\underline{x}_{-j}, \underline{y}_{-j}) \in E_j(x_j, y_j)$,
(2) $\exists(\underline{x}^k_{-j}, \underline{y}^k_{-j}) \in X^2_{-j}$ deduced from $(\underline{x}_{-j}, \underline{y}_{-j})$ by decreasing the kth performance of $\underline{x}_{-j}$ and/or by increasing the kth performance of $\underline{y}_{-j}$ such that $(\underline{x}^k_{-j}, \underline{y}^k_{-j}) \notin E_j(x_j, y_j)$,
(3) $\exists u_j, v_j \in X^2_{-j}$ verifying $u_j\,P_j\,v_j$ such that $(\underline{x}^k_{-j}, \underline{y}^k_{-j}) \in E_j(u_j, v_j)$.

In the MCAP weighted sum (see Section 4.1) there are possibilities of compensation between any pair of criteria; in the other three MCAPs (see Sections 4.2–4.4) there is no possibility of compensation relative to criterion g_j ($\forall j \in F$).

Concerning Definition 4, two remarks are relevant.

(1) It follows from the definition that x_j, y_j, u_j and v_j are such that $x_j < u_j$ and $y_j > v_j$ (one of the inequalities being strict).
(2) This definition can easily be extended to possibilities of compensation between coalitions of criteria

Let us recall that the first formal definition of a totally non-compensatory (I, P) preference structure was given by Fishburn (1976) ($R = \varnothing$): an (I, P) preference structure Ψ is totally non-compensatory (TNC) iff

$$\forall x, y, u, v \in X^4, [P(x, y) = P(u, v) \text{ and } P(y, x) = P(v, u)] \quad \Leftrightarrow \quad [xPy \Leftrightarrow uPv]$$
$$\text{with } P(x, y) = \{i \in F / x_i P_i y_i\}$$

An (I, P) preference structure is TNC if two pairs of alternatives that have the same preferential profile are linked with the same comprehensive preference relation. In other words, a TNC MCAP considers preferences restricted to a single criterion g_j $(\forall j \in F)$ to be ordinal information when building comprehensive preferences.* The reader will easily verify that in a TNC (I, P) preference structure there is no possibility of compensation relative to any criterion (according to Definition 3).

Bouyssou (1986) proposes a definition of a minimally compensatory (I, P) preference structure: an (I, P) preference structure Ψ is minimally compensatory iff

$$\exists x, y, u, v \in X^4 \text{ such that}$$

$$\begin{cases} P(x, y) = P(u, v) \\ P(y, x) = P(v, u) \\ x_i = y_i \text{ and } u_i = v_i, \forall i \in I(x, y) \end{cases}$$

$$\text{with } [xSy \text{ and } vPu]$$

$$\text{with} \begin{cases} P(x, y) = \{i \in F / x_i P_i y_i\} \\ I(x, y) = \{i \in F / i \notin P(x, y) \text{ and } i \notin P(y, x)\} \end{cases}$$

This definition formally expresses phenomena similar to those taken into account in Definition 4. However, it does not specify between which criteria the compensation occurs. Moreover, the reader will easily verify that if there are some possibilities of compensation between two criteria g_j and g_k (Definition 4), then the corresponding preference structure is minimally non-compensatory.

The analysis of the concept of compensatoriness of preference structures and its relations with the notion of RIC is too vast a subject to be developed fully here. Definitions 3 and 4 lead us to think that the theoretical framework proposed in this paper constitutes a suitable basis on which to ground further research.

6. OPERATIONAL INTEREST OF THE PROPOSED FORMALISM

So as to give meaning to importance parameters of any type, it is necessary to refer to a preference structure Ψ (see Section 2.2). In other words, assigning numerical values to such parameters requires questioning such a structure. The way to do this differs according to whether we adopt a descriptivist or a constructivist approach.*

Adopting the *descriptivist approach* involves acknowledging that the way in which any two performance vectors $(\underline{x}_{-j}, x_j)$ and $(\underline{y}_{-j}, y_j)$ are compared on the comprehensive level is stable and well-defined in the mind of the DM prior to being questioned on his/her preference structure Ψ^*. The preference model built for decision aid generates a preference structure Ψ which is intended to give an exact an image as possible of Ψ^*. Towards this end the numerical values of importance parameters have to be chosen in such a way that the sets $\Delta_j^H(x_j, y_j)$, $H \in \{P, S, V\}$, induced from the adopted model match the pre-existing sets $\Delta_j^{*H}(x_j, y_j)$, $H \in \{P, S, V\}$. Hence adjustment techniques should be used to assign values to importance parameters. Several authors (see Mousseau (1992) for a review) talk in terms of estimating the numerical value of certain parameters, such as the weight w_j. This supposes that the sets $\Delta_j^H(x_j, y_j)$ and $\Delta_j^{*H}(x_j, y_j)$, $H \in \{P, S, V\}$ (or more simply their respective borderlines), admit of a common analytical representation.

The *constructivist approach*,† on the contrary, acknowledges that preferences are not entirely preformed in the decision maker's mind and that the very nature of the work involved in the modelling process (and *a fortiori* in decision aid) is to specify and even to modify pre-existing elements. The preference structure Ψ induced by the MCAP adopted for decision aid cannot be viewed here as the faithful image of a pre-existing preference structure Ψ^*, since it is considered ill-defined and unstable. Thus Ψ is here nothing other than the result of a set of rules deemed appropriate for comparing any performance vectors $(\underline{x}_{-j}, x_j)$ and $(\underline{y}_{-j}, y_j)$. The importance parameters appear as keys which allow us to differentiate the role played by each criterion in the preference model selected for use. The numerical values attributed to them correspond to a working hypothesis accepted for decision aid. They must be seen as suitable values with which it seems reasonable and instructive to work. So as to judge whether this

*This idea has been generalized by Bouyssou and Vansnick (1986). See also Vansnick (1986).

*For more details on this distinction see Roy (1993).

†An overview of behavioural decision research within this approach can be found in Paynes *et al.* (1992).

is the case with a given set of values, it is important to analyse the shape and variety of the sets $\Delta_j^H(x_j, y_j)$, $H \in \{P, S, V\}$, which result from it. One should check whether these values fit with the opinion of the DM (or of various stakeholders) regarding the fact that a particular $(\underline{x}_{-j}, \underline{y}_{-j})$ belongs or does not belong to one of the preceding sets. The analysis of these sets does not aim at discovering a unique set of values for importance parameters that fits the DM's opinion, but it can determine a domain containing an appropriate set of values for reasoning, investigating and communicating among the stakeholders in the decision-making process.

As shown in the examples of Section 4, the shape and variety of the sets $\Delta_j^H(x_j, y_j)$ $H \in \{P, S, V\}$, vary significantly according to the type of MCAP considered. Whatever the approach, descriptivist or constructivist, the preceding remark highlights the fact that the meaning of importance parameters (and consequently the appropriate manner for giving a numerical value to them) depends on the MCAP. The discussions presented in Section 5 show the danger of questioning the DM directly on values assigned to importance parameters. These discussions have proved that assigning values to certain importance parameters relying on the metaphor of weight results in serious misunderstandings (see Sections 5.1 and 5.2). Even when this metaphor is pertinent, its evocative power is different when used with intrinsic weights (referring to voting power) than with contingent weights (such as scaling constants for example). Specifically, this metaphor is of no use in evaluating parameters such as veto thresholds in the ELECTRE methods (Roy, 1991b) or coefficients k_{ij} that account for certain forms of dependences between criteria by adding terms of the form $k_{ij}.g_i(a).g_j(a)$ to an additive aggregation $\sum_{k \neq F} k_j.g_j(a)$ (Keeney and Raiffa, 1976). In order to determine suitable values for such parameters, it is possible to base an argument on the fact that a couple $(\underline{x}_{-j}, \underline{y}_{-j})$ belongs (or not) to a particular $\Delta_j^H(x_j, y_j)$.

7. CONCLUSIONS

The foregoing considerations show that the notion of relative importance of criteria is more complex than it is commonly assumed to be. The formal definition that we have proposed (see Section 2.3) seems sufficiently general when criteria are true criteria or semi-criteria. The examples from Section 4 and the considerations developed in Section 5 allow us to think that the sets $\Delta_j^H(x_j, y_j)$, $H \in \{P, S, V\}$ (certain results have been established in Section 3), constitute useful keys in order to study the significance of an MCAP and to clarify other notions closely linked to the notion of importance.

Importance parameters of any type give an account of a value system. Consequently, the values assigned to such parameters will always have a subjective character. As is true for any other subjective data, these values can be grasped only through communicating with certain stakeholders in the decision process. The definitions, results and comments presented in this paper should enable us to develop better means of conceptualizing this type of communication in order to understanding and decode it with greater insight within the framework of the MCAP selected for use.

APPENDIX I

By definition, a family F of n criteria $g_1, g_2, \ldots, g_n$ is consistent iff it verifies the following three axioms (these axioms express naturally requirements in a formal way):

Exhaustivity

If $\forall j \in F, g_j(a) = g_j(b)$, then

$$\begin{cases} cHa \Rightarrow cHb \\ aHc \Rightarrow bHc \end{cases}$$
$$\forall H \in \{P, I, R\}$$

Cohesion

If $\forall j \in F \backslash \{k\}$

$$\begin{cases} g_j(b^k) = g_j(b), g_k(b^k) \geqslant g_k(b) \\ g_j(a) = g_j(a_k), g_k(a) \geqslant g_k(a_k) \end{cases}$$

(one of the inequalities being strict), then $bPa \Rightarrow b^kPa_k$, $bIa \Rightarrow b^kSa_k$;
moreover, if

$$\begin{cases} g_k(a) = g_k(b) \\ a_k I_k b^k \end{cases}$$

then

$$\left\{\begin{array}{l} bHa \Rightarrow b^k Ha_k \\ aHb \Rightarrow a_k Hb^k \end{array}\right. \forall H \in \{P, I, R, S\}$$

Non-redundancy
$\forall j \in F, F \backslash \{j\}$ is either not exhaustive or not cohesive.

APPENDIX II: PROOFS

Proof of Result 1
The first two inclusions are justified by

$$xSy \Rightarrow \text{non}[yPx]$$
$$xPy \Rightarrow xSy$$

It follows from the first two of the above-mentioned axioms (exhaustivity and cohesion) that $\forall i \in F, g_i(a) = g_i(b), \forall i \neq j$, and $aP_jb \Rightarrow aSb$ (Roy and Bouyssou, 1993). Hence the last inclusion is true.

Proof of result 2
This property is an immediate consequence of the cohesion axiom which formally expresses a certain form of monotony of preferences: the higher the performance, the better the alternative.

Proof of result 3
The justification of this propery comes from the cohesion axiom. $H = P$: if$(\underline{x}_{-j}, \underline{y}_{-j}) \in \Delta_j^P(x_j, y_j)$, i.e., xPy, then $\forall \underline{u}_{-j}, \underline{v}_{-j} \in X_{-j}^2$ verifying

$$\left\{\begin{array}{l} \underline{u}_{-j}\, \Delta_{-j}\, \underline{x}_{-j} \\ \underline{y}_{-j}\, \Delta_{-j}\, \underline{v}_{-j} \end{array}\right.$$

we have

$$\left\{\begin{array}{l} \underline{u}\Delta\underline{x} \\ \underline{y}\Delta\underline{v} \end{array}\right. \text{with } x_j = u_j \text{ and } y_j = v_j$$

then, as a consequence of the cohesion axiom, it holds that uPv, i.e., $(\underline{u}_{-j}, \underline{v}_{-j}) \in \Delta_j^H(x_j, y_j)$ $(H \in \{P, S, V\})$. ■

Proof of Result 4
$\Rightarrow$ Let us suppose that $O_j'(x_j, y_j) = \varnothing$. Then $O_j(x_j, y_j) = \varnothing$, i.e., $\Delta_j^V(x_j, y_j) \backslash \Delta_j^S(x_j, y_j) = \varnothing$. $\nexists \underline{x}_{-j}, \underline{y}_{-j} \in X_{-j}^2$ verifying

$$\left\{\begin{array}{l} \text{not}[\underline{y}_{-j}, y_j)P(\underline{x}_{-j}, x_j)] \text{ and not}[(\underline{x}_{-j}, x_j)S(\underline{y}_{-j}, y_j)] \\ \text{i.e. not}[(\underline{y}_{-j}, y_j)P(\underline{x}_{-j}, x_j)] \text{ and} \\ \quad \text{not}[(\underline{x}_{-j}, x_j)P(\underline{y}_{-j}, y_j)] \text{ and} \\ \quad \text{not}[(\underline{x}_{-j}, x_j)I(\underline{y}_{-j}, y_j)] \end{array}\right.$$

Thus $\nexists \underline{x}_{-j}, \underline{y}_{-j} \in X_{-j}^2$ verifying $(\underline{x}_{-j}, x_j)R(\underline{y}_{-j}, y_j)$. ■

$\Leftarrow$ Let us suppose that $\nexists \underline{x}_{-j}, \underline{y}_{-j} \in X_{-j}^2$ such that $(\underline{x}_{-j}, x_j)R(\underline{y}_{-j}, y_j)$. Then there are only three preferential situations between $(\underline{x}_{-j}, x_j)$ and $(\underline{y}_{-j}, y_j)$: $(\underline{x}_{-j}, x_j)P(\underline{y}_{-j}, y_j)$, $(\underline{x}_{-j}, x_j)I(\underline{y}_{-j}, y_j)$ and $(\underline{y}_{-j}, y_j)P(\underline{x}_{-j}, x_j)$. However, $(\underline{x}_{-j}, x_j)S(\underline{y}_{-j}, y_j) \Leftrightarrow (\underline{x}_{-j}, x_j)P(\underline{y}_{-j}, y_j)$ or $(\underline{x}_{-j}, x_j)I(\underline{y}_{-j}, y_j)$. Then it holds that $(\underline{x}_{-j}, x_j)S(\underline{y}_{-j}, y_j) \Rightarrow$ not$[(\underline{y}_{-j}, y_j)\ P(\underline{x}_{-j}, x_j)]$. Then $\Delta_j^S(x_j, y_j) = \Delta_j^V(x_j, y_j)$, i.e., $O_j(x_j, y_j) = \varnothing$. Thus $O_j'(x_j, y_j) = \varnothing$. ■

APPENDIX III: ILLUSTRATION OF THE CONCEPT OF BORDERLINE

Let us illustrate the interpretation of the borderline through a bi-criteria example in which each criterion is evaluated on a discrete scale containing five levels $(AP_iBP_iCP_iDP_iE, i = 1, 2)$.

Table 1 shows an example of the bordline of $\Delta_1^P(x_1 = A, y_1 = B)$.

Table I. Borderline of $\Delta_1^P(x_1 = A, y_1 = B)$

	$x_2 = A$	$x_2 = B$	$x_2 = C$	$x_2 = D$	$x_2 = E$
$y_2 = A$	$\in$	$\in$	$\boldsymbol{\in}$	$\notin$	$\notin$
$y_2 = B$	$\in$	$\in$	$\boldsymbol{\in}$	$\notin$	$\notin$
$y_2 = C$	$\in$	$\in$	$\in$	$\boldsymbol{\in}$	$\notin$
$y_2 = D$	$\in$	$\in$	$\in$	$\in$	$\boldsymbol{\in}$
$y_2 = E$	$\in$	$\in$	$\in$	$\in$	$\in$

Key: $\in$, element of $\Delta_1^P(x_1 = A, y_1 = B)$; $\notin$, non-element of $\Delta_1^P(x_1 = A, y_1 = B)$; $\boldsymbol{\in}$, element of the borderline of $\Delta_1^P(x_1 = A, y_1 = B)$.

The reader will easily verify that if he/she knew all the shaded cells, he/she could deduce all the other elements of $\Delta_1^P(x_1 = A, y_1 = B)$.

REFERENCES

Bouyssou, D., 'Some remarks on the notion of compensation in MCDM', *Eur. J. Oper. Res.*, **26**, 150–160 (1986).

Bouyssou, D. and Vansnick, J.-Cl., 'Noncompensatory and generalized noncompensatory preference structures', *Theory Decis.*, **21**, 251–266 (1986).

Brans, J. P., Mareschal, B. and Vincke, Ph., 'PROMETHEE: a new family of outranking methods in multicriteria analysis', in Brans, J. P. (ed.), *Operational Research '84*, Amsterdam: Elsevier, 408–421 (1984).

Fishburn, P. C., 'Noncompensatory preferences', *Syntheses*, **33**, 393–403 (1976).

Keeney, R. L. and Raiffa, H., *Decision with Multiple Objectives: Preferences and Value Tradeoffs*, New York: John Wiley & Sons, 1976, 2nd edn, 1993.

Mousseau, V., 'Analyse et classification de la littérature traitant de l'importance relative des critères en aide multicritère à la décision, *RAIRO/Operations Research*, **26**(4), 367–389 (1992).

Mousseau, V., 'Problémes liés à l'èvaluation de l'importance relative des critères en aide multicritère à la décision: réflexions théoriques, expérimentations et implémentations informatiques', *Thèse de Doctorat*, Université Paris-Dauphine, 1993.

Paynes, J. W., Bettman, J. R. and Johnson, J. E., 'Behavioral decision research: a constructive processing perspective', *Ann. Rev. Psychol.*, **43**, 87–131 (1992).

Podinovskii, V. V., 'Criteria importance theory', in Lewandowski, A. and Volkovich, V. (eds), *Multiobjective Problems of Mathematical Programming*, LNEMS 351, Springer Verlag, 1991.

Podinovskii, V. V., 'Criteria importance theory', *Math. Social Sci.*, **27**, 237–253 (1994).

Roubens, M., 'Preference relations on actions and criteria in multicriteria decision making', *Eur. J. Oper. Res.*, **10**, 51–55 (1982).

Roubens, M. and Vincke, Ph., *Preference Modelling*, Berlin: Springer, 1985.

Roy, B., *Méthodologie Multicritère d'Aide à la Décision*, Paris: Economica, 1985.

Roy, B., 'Concerning the notion of importance of criteria: its formal integration into multiple criteria decision aid', *Res. Paper 1*, LAMSADE, Université Paris-Dauphine, 1991a.

Roy, B., 'The outranking approach and the foundations of ELECTRE methods', *Theory Decis.*, **31**, 49–73 (1991b).

Roy, B., 'Decision science or decision-aid science', *Eur. J. Oper. Res.*, **66**, 184–203 (1993).

Roy, B. and Bertier, P., 'La méthode ELECTRE II — une application au média-planning', in Ross, M. (ed.), *OR'72*, Amsterdam: North-Holland, 1973. pp. 291–302.

Roy, B. and Bouyssou, D., *Aide Multicritère à la Décision: Méthodes et Cas*, Paris: Economica, 1993.

Roy, B. and Vincke, Ph., 'Relational systems of preferences with one or more pseudo-criteria: some new concepts and results', *Manag. Sci.*, **30**, 1323–1334 (1984).

Saaty, T. L., *The Analytic Hierarchy Process*, New York: McGraw-Hill, 1980.

Vanderpooten, D. and Vincke, Ph., 'Description and analysis of some representative interactive multicriteria procedures', *Math. Comput. Model.*, **12**, 1221–1238 (1989).

Vansnick, J.-Cl., 'On the problem of weights in multiple criteria decision making (the noncompensatory approach)', *EJOR*, **24**, 288–294 (1986).

Vansnick, J.-Cl., 'Measurement theory and decision aid', in Bana e Costa, C. A. (ed.), *Readings in Multiple Criteria Decision Aid*, Berlin: Springer, 1990, pp. 81–100.

Vincke, Ph., 'Basic concepts of preference modelling', in Bana e Costa, C. A. (ed.), *Readings in Multiple Criteria Decision Aid*, Berlin: Springer, 1990, pp. 101–118.

Vincke, Ph., *Multicriteria Decision Aid*, Chichester: Wiley, 1992.

Zionts, S. and Wallenius, J., 'An interactive programming method for solving the multiple criteria problem', *Manag. Sci.*, **22**, 652–663 (1976).

Zionts, S and Wallenius, J., 'An interactive multiple objective linear programming method for a class of underlying nonlinear utility functions', *Manag. Sci.*, **29**, 519–529 (1983).