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Decision Aiding

Inferring Electre's veto-related parameters from outranking examples

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Abstract

When considering Electre's valued outranking relations, aggregation/disaggregation methodologies have difficulties in taking discordance (veto) into account. We present a partial inference procedure to compute the value of the veto-related parameters that best restore a set of outranking statements (i.e., examples that an Electre model should restore) provided by a decision maker, given fixed values for the remaining parameters of the model.

This paper complements previous work on the inference of other preference-related parameters (weights, cutting level, category limits, ...), advancing toward an integrated framework of inference problems in Electre III and Tri methods. We propose mathematical programs to infer veto-related parameters, first considering only one criterion, then all criteria simultaneously, using the original version of Electre outranking relation and two variants. This paper shows that these inference procedures lead to linear programming, 0–1 linear programming, or separable programming problems, depending on the case.

Keywords: Multiple criteria analysis; Parameter inference procedures; Valued outranking relations; Electre; Discordance; Veto

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1. Introduction

The use of multiple criteria evaluation methods is often hindered by the need to provide precise values for many preference-related parameters whose role is not clear to the Decision Maker (DM). This paper addresses the problem of supporting the elicitation of parameter values in Electre models (for an overview of Electre methods see [13,14]) by a process of asking the DM to make holistic judgements concerning some pairs of alternatives, which he or she finds easy to compare. From these judgements, if consistent, a combination of parameter values may be inferred. The modest contribution of this paper concerns the inference of Electre's veto-related parameters, complementing (and contrasting with) previous work, which mostly focuses on how to infer criteria weights and often does not consider veto phenomena.

Aggregation/disaggregation methodologies, which have received much attention lately (see [5]), allow the inference of values for such parameters from holistic judgments (i.e., model results) that the DM is able to provide. Usually, when the underlying evaluation model is of the Electre type (cf. [6,9]) or of the value function type (cf. [4,18]), a mathematical programming problem is solved to find the combination of parameter values that best restores the examples of results proposed by the DM according to some error function to be minimized. This is particularly useful if the inference procedure is part of an interactive process, where the DM observes whether his/her result examples can be restored, and reacts accordingly. If the examples indicated by the DM can be restored, he/she may explore the complete set of results corresponding to the multiple combinations of parameter values that satisfy the imposed conditions (robustness analysis), which may help him/her provide further information. If not, the DM has to discover which of those examples are inconsistent, in order to withdraw some of them (inconsistency analysis). In a previous work (see [2]), we have proposed this type of methodology, integrating parameters inference, robustness analysis, and inconsistency analysis for decision aiding based on the Electre Tri method.

Inferring all the parameters in Electre simultaneously requires solving non-linear programs with non-convex constraints (see [9]), which is usually difficult. The current paper follows a different strategy, based on inferring a subset of the parameter values at a time, while maintaining the remaining ones fixed. These "partial" inference problems, besides simplifying the mathematical programs to be solved, present important advantages. First, they let the DM focus his/her attention on a subset of parameters at a time (e.g., concordance-related parameters, then discordance-related parameters, then returning to concordance, etc.). Second, they allow the DM to control the interactive process in an easier way. Namely, when there are many alternative combinations of parameter values that satisfy the requests of the DM, keeping a subset of the parameters temporarily constant prevents the solutions from being too disparate (even among alternative optima). Furthermore, the DM is less likely to encounter radically different solutions when progressing from one iteration to the next one, and is able to better understand the consequences of changing the examples that he/she provided. Indeed, we believe that inference programs should not be considered as a problem to be solved only once, but rather as problems to be solved several times in an interactive learning process, where the DM continuously revises the information he/she provides as he/she learns with the results of the inference programs.

Complementing a previous paper [7], which considered that the concordance-related parameters were the only variables, this paper now considers that all parameters are fixed except the discordance-related ones (Electre's veto thresholds). Both papers apply to the Electre methods that use valued outranking relations (Electre III and Tri), although this work has been motivated by the application to the Electre Tri method. The holistic information provided by the DM consists of pairs of alternatives (a,b) such that, to his/her opinion, "a outranks b" (i.e., a is at least as good as b, denoted aSb), or "a does not outrank b" (i.e., a is worse than b, denoted $\neg aSb$). The DM should indicate aSb as an example only if he/she feels confident about it. This confidence may stem from his/her knowledge of a and b, or from the fact that a and b were chosen with the purpose of making the comparison easy (e.g., their evaluations only differ in a couple of criteria). If the DM is not sure, then he/she will not indicate aSb as an outranking example, or will indicate it tentatively and later see what are the consequences.

This paper intends to present mathematical programs to infer Electre's veto thresholds from a set of "crisp" outranking statements, and their role in the elicitation process. The following section presents the original outranking relation of Electre III and Tri, as well as two variants [7]. Section 3 presents the inference problem in a general format, discussing the role of methods that infer only a subset of the model's parameters. Section 4 considers the problem of inferring the veto parameters for one criterion at a time, as a simplification of the more general problem. This simplification may nevertheless be useful in practice, since the veto parameters are not inter-related among different criteria. Section 5 deals with the more general problem of inferring the veto parameters for more than one criterion simultaneously. Finally, Section 6 presents illustrative examples, and a closing section offers a summary and some conclusions.

2. Valued outranking relations in Electre

In this section we recall how Electre III (see [12]) and Electre Tri (see [17,14]) build a valued outranking relation on the set of alternatives. Let $F = \{1,\ldots,n\}$ denote the set of criteria indices. Let A denote a finite set of alternatives characterized by their evaluations on criteria $g_1,\ldots,g_n$; $g_j(a)$ denotes the evaluation of an alternative $a \in A$ on criterion g_j. Without loss of generality, we will assume that the evaluations are coded in such a way that the higher the value, the better it is.

2.1. Outranking relations for a single criterion

Electre builds, for each criterion g_j, a valued outranking relation S_j restricted to a single criterion. For any ordered pair $(a,b) \in A^2$, $S_j(a,b)$ is defined by (2) on the basis of $g_j(a)$, $g_j(b)$ and two thresholds: indifference q_j and preference p_j ($0 \leqslant q_j < p_j$; note we consider $q_j < p_j$, although Electre also allows $q_j = p_j$). $S_j(a,b)$ represents the degree to which alternative a outranks (is at least as good as) b. In this paper, we consider the thresholds p_j and q_j as constant, although it is possible to consider them as affine functions (see [1]). For a more compact notation, we will write:

$$\Delta_j(b,a) = g_j(b) - g_j(a), \tag{1}$$

which is a constant value for each pair $(a,b) \in A^2$.

$$S_j(a,b) = \begin{cases} 0, & \text{if } \Delta_j(b,a) \geqslant p_j, \\ \frac{p_j - \Delta_j(b,a)}{p_j - q_j}, & \text{if } q_j < \Delta_j(b,a) < p_j, \\ 1, & \text{if } \Delta_j(b,a) \leqslant q_j. \end{cases} \tag{2}$$

2.2. Concordance relation

The valued concordance relation $C(a,b)$ is grounded on the relations S_j ($j \in F$) and represents the level of majority among the criteria in favor of the assertion "a is at least as good as b". When computing this majority level, each criterion g_j has a weight $w_j \geqslant 0$ representing its voting power. Without any loss of generality, we will consider $\sum_{j \in F} w_j = 1$. Therefore, $C(a,b)$ can be written as follows:

$$C(a,b) = \frac{1}{\sum_{j \in F} w_j} \sum_{j \in F} w_j S_j(a,b) = \sum_{j \in F} w_j S_j(a,b). \tag{3}$$

2.3. Non-discordance relations

Electre builds, for each criterion g_j, a valued discordance relation d_j restricted to that criterion. This relation $d_j(a,b)$ is defined by (4) on the basis of $g_j(a)$, $g_j(b)$, a veto threshold v_j and a preference threshold p_j ($p_j < v_j$; note we consider $p_j < v_j$, although Electre also allows $p_j = v_j$) (see Fig. 1). In this paper, we consider the thresholds v_j (as we already did for p_j and q_j) as constant, although it is possible to consider them as affine functions (see [1]).

$$d_j(a,b) = \begin{cases} 1, & \text{if } \Delta_j(b,a) \geqslant v_j, \\ \dfrac{\Delta_j(b,a) - p_j}{v_j - p_j}, & \text{if } p_j < \Delta_j(b,a) < v_j, \\ 0, & \text{if } \Delta_j(b,a) \leqslant p_j. \end{cases} \quad (4)$$

An overall valued non-discordance relation $ND(a,b)$ is grounded on $C(a,b)$ and on the relations d_j, $j \in F$; it represents the degree to which the minority criteria collectively oppose a veto to the assertion "a is at least as good as b". A classical way of defining $ND(a,b)$ is given in (5). $ND(a,b) = 0$ corresponds to a situation where the minority criteria are totally opposed to aSb whereas $ND(a,b) = 1$ means that none of the criteria oppose a veto to aSb.

$$ND(a,b) = \prod_{j \in \overline{F}} \frac{1 - d_j(a,b)}{1 - C(a,b)} \quad \text{where } \overline{F} = \{j \in F / d_j(a,b) > C(a,b)\}. \quad (5)$$

This expression is equivalent to (6):

$$ND(a,b) = \prod_{j \in F} ND_j(a,b), \quad (6)$$

where

$$ND_j(a,b) = \text{Min}\left\{1, \frac{1 - d_j(a,b)}{1 - C(a,b)}\right\}. \quad (7)$$

Let us remark that we can state $C(a,b) < 1$, as the case $C(a,b) = 1$ corresponds to a situation where no discordant criterion exists. As an alternative, Mousseau and Dias [7] have proposed the valued non-discordance relation defined by (8) and (9), where $u_j \in [p_j, v_j]$ is a new parameter for the jth criterion:

$$ND'(a,b) = \prod_{j \in F} ND'_j(a,b) = \prod_{j \in F} (1 - d'_j(a,b)), \quad (8)$$

$$d'_j(a,b) = \begin{cases} 1 & \text{if } \Delta_j(b,a) \geqslant v_j, \\ \dfrac{\Delta_j(b,a) - u_j}{v_j - u_j} & \text{if } u_j < \Delta_j(b,a) < v_j, \\ 0 & \text{if } \Delta_j(b,a) \leqslant u_j. \end{cases} \quad (9)$$

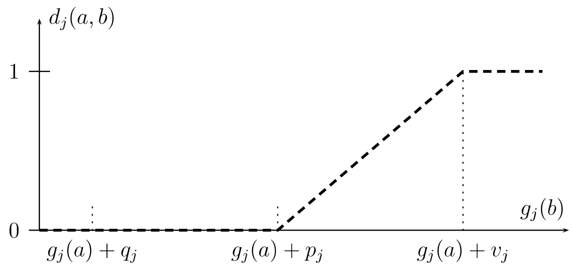

Fig. 1. Partial valued discordance relation.

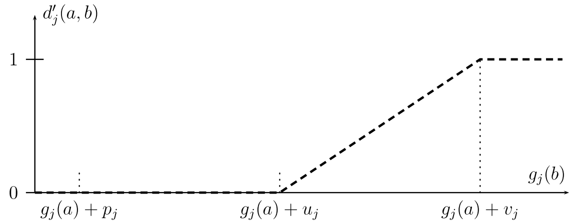

Fig. 2. Partial discordance relation $d'_j(a,b)$.

The new threshold u_j defines the difference of performances in favor of b where the discordance starts weakening the outranking relation (see Fig. 2). It can be considered either:

- as an additional preference parameter to be elicited directly through an interaction with the DM, or indirectly using a disaggregation procedure, or
- as a technical parameter (rather than a preference-related one) that defines the extent to which differences of evaluation $g_j(b) - g_j(a) < v_j$ should (or should not) weaken the concordance $C(a,b)$ in the definition of $S(a,b)$. This latter option should be used only when the DM does not wish to use the added flexibility offered by u_j, preferring to work with the thresholds v_j only (in such cases, a reasonable "rule-of-thumb" is to set automatically $u_j = p_j + 0.75(v_j - p_j)$ (see [7])).

A second alternative to define a valued non-discordance relation is the following (see [7]):

$$ND''(a,b) = \underset{j \in F}{\text{Min}}\, ND'_j(a,b). \tag{10}$$

2.4. Valued outranking relations

Electre combines the concordance and non-discordance relations in order to define the outranking relation S as shown in (11):

$$S(a,b) = C(a,b)ND(a,b), \tag{11}$$

or, according to the two alternative definitions,

$$S'(a,b) = C(a,b)ND'(a,b), \tag{12}$$

$$S''(a,b) = C(a,b)ND''(a,b). \tag{13}$$

From the valued outranking relation $S(a,b)$, it is possible to define a family of nested crisp outranking relations S_λ; these crisp relations correspond to λ-cuts of $S(a,b)$, where the cutting level $\lambda \in [0.5, 1]$ represents the minimum value for $S(a,b)$ so that $aS_\lambda b$ holds. The same applies when we consider $S'(a,b)$ or $S''(a,b)$ instead of $S(a,b)$.

3. Inference of parameter values from crisp outranking statements

The construction of the relation S (or S', or S'') involves determining the evaluation vector of the alternatives, and setting many parameters: the criteria weights, the various thresholds, and the cutting level.

DMs often find it difficult to provide precise values for all these preference parameters. Hence, "disaggregation approaches" have been proposed to infer the parameter values from holistic judgments.

Let us consider a decision process in which the DM is not able (or not willing) to assign directly values to the preference parameters involved in an outranking relation, but can state crisp statements about this relation for some specific pairs of alternatives (a,b), either positive (aSb) or negative ($\neg aSb$). Let us denote $S^+ = \{(a,b) \in A^2$ such that the DM stated $aSb\}$ and $S^- = \{(a,b) \in A^2$ such that the DM stated $\neg aSb\}$. Then, a combination of parameter values is able to restore the DM's request iff $S(a,b) \geqslant \lambda$, $\forall (a,b) \in S^+$ and $S(a,b) < \lambda$, $\forall (a,b) \in S^-$. The system of constraints below (14) has a solution if and only if there exists a combination of parameter values that yields all the crisp outranking statements in S^+ and S^-. Some additional constraints can be added to this system, in order to integrate explicit statements of the DM concerning the values of some parameters (similar systems can be defined replacing S by S' or S'').

$$\begin{cases} S(a,b) \geqslant \lambda, & \forall (a,b) \in S^+, \\ S(a,b) < \lambda, & \forall (a,b) \in S^-, \\ \lambda \in [0.5, 1], \\ v_j > p_j > q_j \geqslant 0, & \forall j \in F, \\ \sum_{j=1}^{n} w_j = 1; \quad w_j \geqslant 0, & \forall j \in F. \end{cases} \tag{14}$$

The idea of inferring all the parameters by maximizing the minimum slack for the above system of constraints was proposed by Mousseau and Slowinski [9] in the context of the Electre Tri method. However, the resulting mathematical program is very complex (non-linear and non-convex constraints). A solution to circumvent this difficulty is to formulate partial inference programs, where only a subset of the parameters are considered as variables, while the remaining ones are fixed. In partial inference problems, if no combination of values for the inferred parameters is able to restore the statements contained in S^+ and S^-, then the DM should wonder why. Then, he/she may either revise his/her statements or turn his/her attention to a different subset of parameters whose value may be inadequate. Instead of being forced to assign directly precise values for the models parameters, the DM may be supported by different partial inference tools for different subsets of parameters, although there is no guarantee that such a collection of tools constitutes an easy method to "solve" (14). Furthermore, there are situations where DMs need support concerning only a subset of the parameters: some parameters may have a natural definition that the DM does not want to change, or some parameter values may be imposed by some higher authority or may have been fixed in advance (e.g., criteria weights in some public procurement processes), and the DM may have more difficulties in setting some parameters than some others. Partial inference tools should be seen as elicitation aids, allowing the DM to fix some parameters (perhaps tentatively) with the purpose of focusing his/her attention on some others. Doumpos and Zopounidis [3] also proposed a sequential methodology to set the parameters of the Electre Tri method from a set of reference examples.

Among partial inference problems, previous research concerning Electre methods has focused mainly on inferring the weights and the cutting level. The problems involving the relation $S(a,b)$ can be solved using linear programs (LPs), but only if discordance is ignored, i.e., no veto phenomena occur and $ND(a,b) = 1$ (e.g., see [2,8,10] in the context of Electre Tri). However, when considering $S'(a,b)$ or $S''(a,b)$, the weights and the cutting level can be inferred using LP, even in the presence of discordance (see [7]). In the context of Electre Tri, a procedure exists to infer category limits, i.e., frontiers between categories and attached indifference and preference thresholds (everything else being fixed) (see [11]) using linear programs with 0–1 variables. In this paper, we are interested in the inference of veto thresholds, all other parameters being fixed (see Fig. 3).

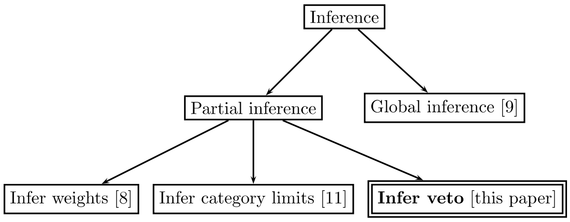

Fig. 3. Global vs. partial inference procedures.

4. Inference of veto-related parameters for a single criterion

In this subsection, we consider that all the parameters are fixed, except the veto threshold of one criterion (let i be its index). Indeed, contrary to the weights, which are interrelated in the coalitions entering the computation of concordance, veto thresholds can be set by considering one criterion at the time and usually refer to distinct units and distinct scales (e.g., a constraint like $v_1 \leqslant v_2$ will probably not make much sense). Hence, the DM may wish to focus on the veto power of criteria, one criterion at a time. Since v_i is the only variable, all the requests from the DM can be satisfied iff the system (15) has a solution:

$$\begin{cases} S(a,b) \geqslant \lambda, & \forall (a,b) \in S^+, \\ S(a,b) < \lambda, & \forall (a,b) \in S^-, \\ v_i > p_i. \end{cases} \tag{15}$$

4.1. Inference of v_i considering $S(a,b)$

In this subsection, we will show that inferring the value of a single veto threshold v_i considering all other parameters fixed and the use of relation $S(a,b)$ is a very simple problem. First, simple tests may allow one to find that some constraints are redundant or that the problem has no solution. Then, if the problem is not found to be impossible, each constraint derived from S^+ will be translated into a lower bound for v_i, whereas each constraint derived from S^- will be translated into an upper bound.

From (6) and (11), when only v_i is considered as variable, $S(a,b)$ is equal to $ND_i(a,b)$ (which is a function of v_i) multiplied by a constant value $K_i(a,b) = C(a,b)\prod_{j \in F \setminus \{i\}} ND_j(a,b)$. If this constant is lower than λ, then a will not outrank b, even if there is no discordance on the ith criterion, since $S(a,b) \leqslant K_i(a,b)$. Let us define a relation $aS_{-i}b$, meaning "aSb when there is no discordance on the ith criterion", or "aSb is possible for some values of v_i":

$$\begin{aligned} aS_{-i}b &\iff (ND_i(a,b) = 1 \Rightarrow aSb) \\ &\iff K_i(a,b) \geqslant \lambda. \end{aligned}$$

Since $K_i(a,b)$, $C(a,b)$, and λ are fixed constants, it is easy to perform the following preprocessing:

- if $\exists (a,b) \in S^+ : \neg aS_{-i}b$ or $\exists (a,b) \in S^- : C(a,b) = 1$ (which implies $S(a,b) = 1$), then the system (15) has no solution.
- all constraints associated with pairs $(a,b) \in S^+ : C(a,b) = 1$ or pairs $(a,b) \in S^- : \neg aS_{-i}b$ are redundant, because they will be respected for any value of v_i.

We now assume that this preprocessing has been performed. If the system was not found to be impossible and if the redundant constraints have been removed, then from (6), (7) and (11), the system (15) may be replaced by the following one (note that $K_i(a,b) \neq 0$ and $C(a,b) < 1$ for all the pairs considered):

$$\begin{cases} ND_i(a,b) = \text{Min}\left\{1, \frac{1-d_i(a,b)}{1-C(a,b)}\right\} \geqslant \frac{\lambda}{K_i(a,b)}, & \forall (a,b) \in S^+ : aS_{-i}b, \\ ND_i(a,b) = \text{Min}\left\{1, \frac{1-d_i(a,b)}{1-C(a,b)}\right\} < \frac{\lambda}{K_i(a,b)}, & \forall (a,b) \in S_i^- : aS_{-i}b, \\ v_i > p_i. \end{cases} \tag{16}$$

Noting that $aS_{-i}b \Rightarrow \lambda/K_i(a,b) \leqslant 1$, this system is equivalent to the following one:

$$\begin{cases} 1 - d_i(a,b) \geqslant (1 - C(a,b)) \frac{\lambda}{K_i(a,b)}, & \forall (a,b) \in S^+ : aS_{-i}b, \\ 1 - d_i(a,b) < (1 - C(a,b)) \frac{\lambda}{K_i(a,b)}, & \forall (a,b) \in S^- : aS_{-i}b, \\ v_i > p_i. \end{cases} \tag{17}$$

If we now define $B_i(a,b) = 1 - \frac{(1-C(a,b))\cdot\lambda}{K_i(a,b)}$, then the same system may be written as

$$\begin{cases} d_i(a,b) \leqslant B_i(a,b), & \forall (a,b) \in S^+ : aS_{-i}b, \\ d_i(a,b) > B_i(a,b), & \forall (a,b) \in S^- : aS_{-i}b, \\ v_i > p_i, \end{cases} \tag{18}$$

where each $d_i(a,b)$ is a function of v_i (recall that all other parameters are fixed) that yields a value in the interval $[0,1]$ (see (4) and Fig. 1).

Since $\lambda/K_i(a,b) \in]0,1]$ and $1 > C(a,b) \geqslant \lambda \geqslant 0.5$, we conclude that $1 > B_i(a,b) > 0$. From (4), each constraint derived from a pair $(a,b) \in S^+ : aS_{-i}b$ can be translated into a lower bound for v_i, and each constraint derived from a pair $(a,b) \in S^- : aS_{-i}b$ can be translated into an upper bound for v_i. Therefore, we may search for a solution to the system (19), knowing that $d_i(a,b) \in]0,1[$ iff $d_i(a,b) = \frac{\Delta_i(b,a)-p_i}{v_i-p_i}$:

$$\begin{cases} v_i \geqslant p_i + \frac{\Delta_i(b,a)-p_i}{B_i(a,b)}, & \forall (a,b) \in S^+ : aS_{-i}b, \\ v_i < p_i + \frac{\Delta_i(b,a)-p_i}{B_i(a,b)}, & \forall (a,b) \in S^- : aS_{-i}b, \\ v_i > p_i. \end{cases} \tag{19}$$

Let L_i denote the greatest of the lower bounds derived from S^+, and let U_i denote the lowest of the upper bounds derived from S^-. Then the system (19) has no solution if $U_i \leqslant \max\{L_i, p_i\}$. Otherwise any value for v_i in $[\max\{L_i, p_i\}, U_i]$ is acceptable, namely $v_i = \frac{U_i + \max\{L_i, p_i\}}{2}$.

4.2. Inference of u_i and v_i considering $S'(a,b)$

In the specific case of the outranking relation $S'(a,b)$ (see (12)), a new veto-related parameter u_i has been introduced (see (9) and Fig. 2). In this subsection we address the problem of inferring u_i and v_i simultaneously when both are considered as the only variables (if we considered that u_i is fixed and v_i is the only variable, then the process would be similar to the one followed in Section 4.1). First, simple tests may allow one to find that some constraints are redundant or that the problem has no solution. If the problem is not found to be impossible, we reach a system of inequalities with two variables (u_i and v_i) that produces a linear program whose solution (if it exists) solves this system. The notation hereafter is similar to Section 4.1:

- $K'_i(a,b) = C(a,b)\prod_{j\in F\setminus\{i\}} ND'_j(a,b)$ (the product of the factors that do not depend on u_i and v_i; this allows to write $S'(a,b) = ND'_i(a,b)K'_i(a,b)$);
- $aS'_{-i}b \Longleftrightarrow K'_i(a,b) \geqslant \lambda$ ($aS'_{-i}b$ iff $aS'b$ is possible for some values of u_i and v_i);
- $B'_i(a,b) = 1 - \lambda/K'_i(a,b)$.

Following the reasoning of Section 4.1, we may perform a similar preprocessing that may detect that the problem is infeasible (if $\exists(a,b) \in S^+ : \neg aS'_{-i}b$ or $\exists(a,b)\in S^- : C(a,b) = 1$) and allows one to remove redundant constraints (those associated with pairs $(a,b) \in S^+ : C(a,b) = 1$ or pairs $(a,b) \in S^- : \neg aS_{-i}b$). After that, inferring u_i and v_i amounts at solving the following system of inequalities, where the variables v_i and u_i affect $d'_i(a,b)$, whereas $B'_i(a,b)$ are constants:

$$\begin{cases} d'_i(a,b) \leqslant B'_i(a,b), & \forall (a,b) \in S^+ : aS'_{-i}b, \\ d'_i(a,b) > B'_i(a,b), & \forall (a,b) \in S^- : aS'_{-i}b, \\ v_i > u_i \geqslant p_i. \end{cases} \tag{20}$$

Since $\lambda/K'_i(a,b) \in]0,1]$, we conclude that $1 > B_i(a,b) \geqslant 0$. Let us define the auxiliary notation:

- $S^+_{(B>0)i} = \{(a,b) \in S^+ : aS'_{-i}b \wedge 1 > B'_i(a,b) > 0\}$;
- $S^+_{(B=0)i} = \{(a,b) \in S^+ : aS'_{-i}b \wedge B'_i(a,b) = 0\}$.

For each $(a,b) \in S^+_{(B=0)i}$ we have a constraint $d'_i(a,b) = 0$ (note that $d'_i(a,b)$ cannot be negative), which from (9) is equivalent to $\Delta_i(b,a) \leqslant u_i$. The remaining pairs $(a,b) \in S^+_{(B>0)i}$ and $(a,b) \in S^-_i : aS'_{-i}b$ constrain $d'_i(a,b)$ to be lower or higher (respectively) than a value in the interval]0,1[. Since $d'_i(a,b) \in]0,1[$ iff $d'_i(a,b) = \frac{\Delta_i(b,a)-u_i}{v_i-u_i}$, the following system is equivalent to the previous one:

$$\begin{cases} u_i \geqslant \Delta_i(b,a), & \forall (a,b) \in S^+_{(B=0)i}, \\ v_i \geqslant u_i + \frac{\Delta_i(b,a)-u_i}{B'_i(a,b)}, & \forall (a,b) \in S^+_{(B>0)i}, \\ v_i < u_i + \frac{\Delta_i(b,a)-u_i}{B'_i(a,b)}, & \forall (a,b) \in S^- : aS'_{-i}b, \\ v_i > u_i \geqslant p_i. \end{cases} \tag{21}$$

Any combination of values for u_i and v_i satisfying system (21) is acceptable. However, in order to find a "central" solution, we may solve the following LP, where ε is an arbitrary near-zero constant (to account for the strict inequalities) and the variables are v_i, u_i and σ:

$$\begin{aligned} \text{Max} \quad & \sigma \\ \text{s.t.} \quad & u_i \geqslant \Delta_i(b,a) + \sigma, \quad \forall (a,b) \in S^+_{(B=0)i}, \\ & v_i + u_i\left(\frac{1}{B'_i(a,b)} - 1\right) \geqslant \frac{\Delta_i(b,a)}{B'_i(a,b)} + \sigma, \quad \forall (a,b) \in S^+_{(B>0)i}, \\ & v_i + u_i\left(\frac{1}{B'_i(a,b)} - 1\right) \leqslant \frac{\Delta_i(b,a)}{B'_i(a,b)} - \sigma - \varepsilon, \quad \forall (a,b) \in S^- : aS'_{-i}b, \\ & v_i - \varepsilon \geqslant u_i \geqslant p_i. \end{aligned} \tag{22}$$

If the optimum value of the LP (22) is positive or null, then the system (20) has a solution, i.e., the optimum solution yields a value for v_i and u_i that respect all the statements provided by the DM. Otherwise, the system (20) has no solution.

4.3. Inference of u_i and v_i considering $S''(a,b)$

Finally, we consider the outranking relation $S''(a,b)$. Let us define $M_i(a,b) = \min_{j \in F \setminus \{i\}} ND'_j(a,b)$. We will follow a reasoning similar to that of Section 4.2 and will reach a similar system of inequalities to be solved in an identical manner.

As only v_i and u_i are considered as variables, $M_i(a,b)$ is a constant value such that

$$S''(a,b) = C(a,b) \cdot \min\{M_i(a,b), 1 - d'_i(a,b)\} \leqslant C(a,b)M_i(a,b). \tag{23}$$

Following the reasoning of Section 4.1, we may define a relation analogous to S_{-i} and S'_{-i}:

$$aS''_{-i}b \iff C(a,b)M_i(a,b) \geqslant \lambda \ (aS''_{-i}b \text{ iff } aS''b \text{ is possible for some values of} u_i \text{ and } v_i). \tag{24}$$

We perform a similar preprocessing that may detect that the problem is infeasible (if $\exists(a,b) \in S^+ : \neg aS''_{-i}b$ or $\exists(a,b) \in S^-:C(a,b) = 1$) and allows to remove redundant constraints (those associated with pairs $(a,b) \in S^+:C(a,b) = 1$ or pairs $(a,b) \in S^- : \neg aS''_{-i}b$). Hence, we consider the system

$$\begin{cases} S''(a,b) \geqslant \lambda, & \forall(a,b) \in S^+ : aS''_{-i}b, \\ S''(a,b) < \lambda, & \forall(a,b) \in S^- : aS''_{-i}b, \\ v_i > u_i \geqslant p_i. \end{cases} \tag{25}$$

From (23), each constraint $S''(a,b) \geqslant \lambda$ holds iff $M_i(a,b) \geqslant \lambda/C(a,b)$ and $1 - d'_i(a,b) \geqslant \lambda/C(a,b)$. However, $aS''_{-i}b \Rightarrow M_i(a,b) \geqslant \lambda/C(a,b)$; hence, $S''(a,b) \geqslant \lambda \iff 1 - d'_i(a,b) \geqslant \lambda/C(a,b), \forall(a,b) : aS''_{-i}b$. Therefore, if we now define $B''_i(a,b) = 1 - \lambda/C(a,b)$, then the system (25) can be written as

$$\begin{cases} d'_i(a,b) \leqslant B''_i(a,b), & \forall(a,b) \in S^+ : aS''_{-i}b, \\ d'_i(a,b) > B''_i(a,b), & \forall(a,b) \in S^- : aS''_{-i}b, \\ v_i > u_i \geqslant p_i. \end{cases} \tag{26}$$

This system is similar to the system (20) that we found in Section 4.2, hence we may proceed as proposed in that section: the process is the same, with a different definition for the relation S'_{-i}, and the constants $B'_i(a,b)$.

5. Inference of all veto-related parameters simultaneously

In this section we consider that all the parameters are fixed, except some of the veto thresholds, possibly all of them. This situation will occur when the DM does not wish to focus on the veto power of one criterion at a time. Let $V \subseteq F$ be the set of indices of the criteria whose veto thresholds are not fixed. The criteria whose indices are in $F \setminus V$ are either fixed or do not possess any veto power ($v_j = \infty$).

Being v_j, $j \in V$ the only variables, all the requests from the DM can be satisfied iff the system (27) has a solution:

$$\begin{cases} S(a,b) \geqslant \lambda, & \forall(a,b) \in S^+, \\ S(a,b) < \lambda, & \forall(a,b) \in S^-, \\ v_j > p_j, & \forall j \in V. \end{cases} \tag{27}$$

5.1. Inference of v_i considering $S(a,b)$ or $S'(a,b)$

In this subsection, we will propose a mathematical programming formulation to infer the value of several veto thresholds $v_j : j \in V$ considering all other parameters fixed and the use of relation $S(a,b)$. First, simple tests may allow one to find that some constraints are redundant or that the problem has no solution. If the problem is not found to be impossible, each constraint derived from S^+ will be translated into a lower bound for a product of several non-discordance indices, whereas each constraint derived from S^- will be translated into an upper bound for a similar product. By taking logarithms we transform these products into sums and reach a separable programming formulation to find values (if they exist) for the veto thresholds that respect those bounds. If we consider the use of relation $S'(a,b)$, the reasoning is similar but there are additional variables $u_j : j \in V$.

Let us consider the outranking $S(a,b)$ (see (11)) and define $K_V(a,b) = C(a,b)\prod_{j\in F\setminus V} ND_j(a,b)$, which is a constant value for (a,b). We can now write $S(a,b) = K_V(a,b)\prod_{j\in V} ND_j(a,b) \leqslant K_V(a,b)$. Let us define a relation $aS_{-V}b$, meaning "aSb is possible for some values of v_j, $j \in V$":

$$aS_{-V}b \Longleftrightarrow K_V(a,b) \geqslant \lambda. \tag{28}$$

Since $K_V(a,b)$, $C(a,b)$, and λ are fixed constants, it is easy to perform the following preprocessing:

- if $\exists(a,b) \in S^+ : \neg aS_{-V}b$ or $\exists(a,b) \in S^- : C(a,b) = 1$ (which implies $S(a,b) = 1$), then the system (27) has no solution.
- all constraints associated with pairs $(a,b) \in S^+ : C(a,b) = 1$ or pairs $(a,b) \in S^- : \neg aS_{-V}b$ are redundant, because they will be respected for any value of v_j, $j \in V$.

We now assume that this preprocessing has been performed. If the system was not found to be impossible and if the redundant constraints have been removed, then the system (27) may be replaced by the following one, with $B_V(a,b) = \lambda / K_V(a,b)$ (note that $K_V(a,b) \neq 0$ and $C(a,b) < 1$ for all the pairs considered):

$$\begin{cases} \prod_{j\in V} ND_j(a,b) \geqslant B_V(a,b), & \forall (a,b) \in S^+ : aS_{-V}b, \\ \prod_{j\in V} ND_j(a,b) < B_V(a,b), & \forall (a,b) \in S^- : aS_{-V}b, \\ v_j > p_j, & \forall j \in V. \end{cases} \tag{29}$$

This is a non-linear system of inequalities, where the variables are v_j, $j \in V$, as arguments of $ND_j(a,b)$ (see (7) and (4)). If $\Delta_j(b,a) \leqslant p_j$, then (by (4)) $d_j(a,b) = 0$ and (by (7)) $ND_j(a,b) = 1$, regardless of the value of v_j. Hence, if we denote $V_{ab} = \{j \in V : \Delta_j(b,a) > p_j\}$, we can write the system (29) above as

$$\begin{cases} \prod_{j\in V_{ab}} ND_j(a,b) \geqslant B_V(a,b), & \forall (a,b) \in S^+ : aS_{-V}b, \\ \prod_{j\in V_{ab}} ND_j(a,b) < B_V(a,b), & \forall (a,b) \in S^- : aS_{-V}b, \\ v_j > p_j, & \forall j \in V. \end{cases} \tag{30}$$

We will now transform this system using logarithms. Let us define:

$$f_j(a,b,v_j) = \log \max\{0.1, ND_j(a,b)\}, \quad \forall j \in V_{ab}. \tag{31}$$

This definition considers that $ND_j(a,b) = 0.1$, whenever its real value is below 0.1 (we need this to ensure we are taking the logarithm of a positive quantity). However, this change has no impact in the results. Indeed, if $ND_j(a,b) < 0.5$, for some $j \in V_{ab}$, then $S(a,b) < \lambda$, regardless of any other parameters, since $\lambda \geqslant 0.5$. The value 0.1 in the expression above is arbitrary: any value in $]0, 0.5[$ could replace it.

Now, the system (29) has a solution iff the following mathematical program has a non-negative optimal value (ε is an arbitrary near-zero constant to account for the strict inequalities):

$$\begin{aligned} \text{Max} \quad & \sigma \\ \text{s.t.} \quad & \sum_{j\in V_{ab}} f_j(a,b,v_j) \geqslant \log B_V(a,b) + \sigma, \quad \forall (a,b) \in S^+ : aS_{-V}b, \\ & \sum_{j\in V_{ab}} f_j\big(a,b,v_j\big) \leqslant \log B_V(a,b) - \sigma - \varepsilon, \quad \forall (a,b) \in S^- : aS_{-V}b, \\ & v_j \geqslant p_j + \varepsilon, \quad \forall j \in V. \end{aligned} \tag{32}$$

The advantage of using logarithms is that we obtain a separable non-linear program, which may be solved by 0–1 linear programming techniques. In the separable program (32), each function $f_j(a,b,v_j)$ may be approximated by a piecewise linear function of v_j. Since the feasible region is not convex, these problems may be solved by either introducing some integer (0–1) variables or using a special branch and bound technique for dealing with SOS2 (special ordered sets of variables where at most two consecutive ones are non-zero). For more details about separable programs, including how to formulate them and solve them using piecewise linear approximations and either integer 0–1 programming or SOS2 branch and bound, see [15] (Chapters 7 and 9) and [16] (Chapters 5 and 7). In [1], we provide specific details concerning the resolution of such programs.

If we considered the outranking relation $S'(a,b)$ (see (12)) instead of $S(a,b)$, the process would be analogous to the one described here, with the only difference being the use of $ND'_j(a,b)$ instead of $ND_j(a,b)$. The only significant consequence is that $ND'_j(a,b)$ depends also on the variables u_j ($j \in V$) (besides v_j), which increases the number of binary variables in the 0–1 linear programs to solve.

5.2. Inference of u_i and v_i considering $S''(a,b)$

In this subsection, we will propose a mathematical programming formulation to infer the value of several discordance-related thresholds $v_j, u_j : j \in V$ considering all other parameters fixed and the use of relation $S''(a,b)$. First, simple tests may allow to find that some constraints are redundant or that the problem has no solution. If the problem is not found to be impossible, each constraint derived from S^+ will be translated into a lower bound for the minimum non-discordance index among a set, whereas each constraint derived from S^- will be translated into an upper bound for a similar minimum. We present a 0–1 programming formulation to find values (if they exist) for $v_j, u_j : j \in V$ that respect those bounds, where the binary variables are used to cope with the upper bound constraints.

Considering the outranking relation $S''(a,b)$ (see (13)) and the variables u_j and v_j ($j \in V$), we will follow a similar reasoning, as we did for $S(a,b)$. The relation $aS''_{-V}b$ plays the same role as $aS_{-V}b$ in Section 5.1, but has a different definition:

$$aS''_{-V}b \Longleftrightarrow C(a,b) \min_{j \in F \setminus V} ND'_j(a,b) \geqslant \lambda. \tag{33}$$

We perform a similar preprocessing that may detect that the problem is infeasible (if $\exists (a,b) \in S^+ : \neg aS''_{-V}b$ or $\exists (a,b) \in S^- : C(a,b) = 1$) and allows to remove redundant constraints (those associated with pairs $(a,b) \in S^+ : C(a,b) = 1$ or pairs $(a,b) \in S^- : \neg aS''_{-V}b$). Hence, we consider the following system, with $B''(a,b) = \lambda / C(a,b)$ (recall the definition of $S''(a,b)$ (see (13)):

$$\begin{cases} \min_{j \in F} ND'_j(a,b) \geqslant B''(a,b), & \forall (a,b) \in S^+ : aS''_{-V}b, \\ \min_{j \in F} ND'_j(a,b) < B''(a,b), & \forall (a,b) \in S^- : aS''_{-V}b, \\ v_j > u_j \geqslant p_j, & \forall j \in V, \end{cases} \tag{34}$$

where the variables u_j and v_j ($j \in V$) are arguments of $ND'_j(a,b)$.

Noting that $ND'_j(a,b) = \min\{1, \max\{0, \frac{v_j - \Delta_j(b,a)}{v_j - u_j}\}\}$ (see (8) and (9)), and since $aS''_{-V}b \Rightarrow B''(a,b) \in]0,1]$, the system (34) is equivalent to the following one:

$$\begin{cases} \min_{j \in F} \frac{v_j - \Delta_j(b,a)}{v_j - u_j} \geqslant B''(a,b), & \forall (a,b) \in S^+ : aS''_{-V}b, & (35.a) \\ \min_{j \in F} \frac{v_j - \Delta_j(b,a)}{v_j - u_j} < B''(a,b), & \forall (a,b) \in S^- : aS''_{-V}b, & (35.b) \\ v_j > u_j \geqslant p_j, & \forall j \in V. \end{cases} \tag{35}$$

The values of $\frac{v_j - \Delta_j(b,a)}{v_j - u_j}$ are fixed for $j \in F \setminus V$ and variable for $j \in V$. Hence we may readily verify whether any of the fixed values makes a constraint (35.a) impossible to respect (hence there would be no solution) or makes a constraint (35.b) redundant (hence may be deleted).

We can now build a mathematical program to test if system (35) has a solution. Since $v_j > u_j$, each of the constraints (35.a) may be rewritten as:

$$\begin{aligned}&\min_{j\in F}\frac{v_j-\Delta_j(b,a)}{v_j-u_j}-B''(a,b)\geqslant 0\\&\iff\frac{v_j-\Delta_j(b,a)}{v_j-u_j}-B''(a,b)\geqslant 0,\quad\forall j\in F,\\&\iff(1-B''(a,b))\ v_j+B''(a,b)\ u_j\geqslant\Delta_j(b,a),\quad\forall j\in F.\end{aligned}\tag{36}$$

On the other hand, since $v_j > u_j$, each of the constraints (35.b) may be rewritten as:

$$\begin{aligned}&\min_{j\in F}\frac{v_j-\Delta_j(b,a)}{v_j-u_j}-B''(a,b)<0\\&\iff\exists j\in F:\frac{v_j-\Delta_j(b,a)}{v_j-u_j}-B''(a,b)<0\\&\iff\exists j\in F:(1-B''(a,b))\ v_j+B''(a,b)\ u_j<\Delta_j(b,a)\end{aligned}\tag{37}$$

$$\iff\begin{cases}(1-B''(a,b))\ v_j+B''(a,b)\ u_j+M\ \delta_{jab}<M+\Delta_j(b,a),\quad\forall j\in F,\\\sum_{j\in F}\delta_{jab}\geqslant 1,\\\delta_{jab}\in\{0,1\},\quad\forall j\in F,\end{cases}\tag{38}$$

where M is a large positive constant greater than $\frac{v_j-\Delta_j(b,a)}{v_j-u_j}$, $\forall j\in F,\forall(a,b)\in S^-:aS''_{-V}b$. The system (38) uses binary variables to account for the disjunctive nature of (37). Note that $\sum_{j\in F}\delta_{jab}\geqslant 1$ forces at least one of the binary variables δ_{jab} to be have the value 1, thus forcing (37). Considering these transformations, the system (35) has a solution iff the following 0–1 linear program has a non-negative optimal value (ε is an arbitrary near-zero constant):

$$\begin{aligned}\text{Max}\quad&\sigma\\\text{s.t.}\quad&(1-B''(a,b))\ v_j+B''(a,b)\ u_j\geqslant\Delta_j(b,a)+\sigma,\quad\forall j\in F,(a,b)\in S^+:aS''_{-V}b,\\&(1-B''(a,b))\ v_j+B''(a,b)\ u_j+M\cdot\delta_{jab}\leqslant M+\Delta_j(b,a)-\sigma-\varepsilon,\quad\forall j\in F,(a,b)\in S^-:aS''_{-V}b\\&\sum_{j\in F}\delta_{jab}\geqslant 1,\ \forall(a,b)\in S^-:aS''_{-V}b,\\&v_j-\varepsilon\geqslant u_j\geqslant p_j,\quad\forall j\in V,\\&\delta_{jab}\in\{0,1\},\quad\forall j\in F,(a,b)\in S^-:aS''_{-V}b,\ \sigma\text{ free.}\end{aligned}\tag{39}$$

6. Illustrative example

In this section, we present an example that illustrates the procedures presented in Sections 4 and 5. This example deals with a multiple criteria sorting problem using the pessimistic Electre Tri method (see [14] and [17]). Let us note, however, that this problem amounts to inferring a relation S from outranking statements (see Section 3), since a statement "*a is assigned to* C_k" is equivalent in the Electre Tri pessimistic procedure to $aSb_k\wedge\neg aSb_{k+1}$. Indeed, the methodology proposed in the previous sections applies when the DM expresses statements which lead to outranking statements; regardless of the context, e.g., sorting, choice or ranking of alternatives.

We consider a set of private companies (see Table 1) evaluated on the five following criteria. The evaluations on each criterion range on the interval $[0,100]$, with an increasing direction of preference, i.e., the more the better:

Table 1
Set of companies to be assigned a category

	g_1	g_2	g_3	g_4	g_5
a_1	80	75	80	50	35
a_2	85	31	45	95	6
a_3	75	95	90	70	45
a_4	90	55	80	80	34
a_5	95	90	85	49	75
⋮	⋮	⋮	⋮	⋮	⋮

g_1: macroeconomic indicator,
g_2: market position,
g_3: financial indicator,
g_4: management team,
g_5: past credit history.

The decision problem is to analyze the credit risk associated to a company. This problem may be formulated as a sorting problem which consists in assigning alternatives (corresponding to companies) to one of the four categories (C_1–C_4) that represents a credit risk level. In this decision problem the DM will be supported by the use of the Electre Tri method, considering the relation S'' (see (13)). The profiles separating the categories are given in Table 2:

C_1: very high risk (worst category),
C_2: high risk,
C_3: low risk,
C_4: very low (best category).

The DM needs support to determine the values for discordance-related parameters. In particular, she believes that criteria g_1, g_2 and g_3 should not have any veto power (hence the DM and analyst agree to set $v_1 = v_2 = v_3 = +\infty$), whereas criterion g_5 can have some veto power. She is unsure whether criterion g_4 should have any veto power or not. In terms of weights (w_j) involved in the computations of concordance, she wants to consider all criteria as equally important. She also wants to treat the criteria similarly in what concerns the imprecision of evaluations, fixing the indifference thresholds (q_j) and the preference thresholds (p_j) to the same values for the different criteria. Additionally, she considers that the thresholds (including the discordance thresholds), are constant from profile to profile (i.e., they do not vary with the performances of the alternatives being compared). This information is summarized in Table 3. Finally, she considered a cutting level of 0.54. Note that $\lambda = 0.54$ corresponds to a coalition of any of three criteria with a discordance index ND'' (see (10)) equal to 0.9, or better.

Initially, the DM considers three companies she knows well as examples (a_1, a_2 and a_3), stating that a_1 is a good example of C_3 (low risk), a_2 is a good example of C_2 (high risk) and a_3 is a good example of C_4 (very

Table 2
Category limits

	g_1	g_2	g_3	g_4	g_5
Limit(C_1–C_2): $g_j(b_1)$	25	25	30	30	15
Limit(C_2–C_3): $g_j(b_2)$	50	50	55	55	55
Limit(C_3–C_4): $g_j(b_3)$	75	75	80	85	80

Table 3
Weights and discrimination thresholds

	g_1	g_2	g_3	g_4	g_5
w_j	0.2	0.2	0.2	0.2	0.2
q_j	10	10	10	10	10
p_j	20	20	20	20	20

low risk). Stating these assignment examples is equivalent (in the Electre Tri assignment procedure) to the following outranking statements: a_1Sb_2 and $\neg(a_1Sb_3)$; a_2Sb_1 and $\neg(a_2Sb_2)$; a_3Sb_3. This leads to the following sets S^+ and S^- (see Section 3):

$$S^+ = \{(a_1,b_2),(a_2,b_1),(a_3,b_3)\}, \quad S^- = \{(a_1,b_3),(a_2,b_2)\}.$$

The following tables summarize the pairwise comparisons involved in S^+ and S^-:

$(a_1,b_2) \in S^+$

	g_1	g_2	g_3	g_4	g_5
$\Delta_j(b_2,a_1)$	-30	-25	-25	5	20
$C_j(a_1,b_2)$	1	1	1	1	0

$\Rightarrow C(a_1,b_2) = 0.8$,

$(a_1,b_3) \in S^-$

	g_1	g_2	g_3	g_4	g_5
$\Delta_j(b_3,a_1)$	-5	0	0	35	45
$C_j(a_1,b_3)$	1	1	1	0	0

$\Rightarrow C(a_1,b_3) = 0.6$,

$(a_2,b_1) \in S^+$

	g_1	g_2	g_3	g_4	g_5
$\Delta_j(b_1,a_2)$	-60	-6	-15	-65	9
$C_j(a_2,b_1)$	1	1	1	1	1

$\Rightarrow C(a_2,b_1) = 1$,

$(a_2,b_2) \in S^-$

	g_1	g_2	g_3	g_4	g_5
$\Delta_j(b_2,a_2)$	-35	19	10	-40	49
$C_j(a_2,b_2)$	1	0,1	1	1	0

$\Rightarrow C(a_2,b_2) = 0.62$,

$(a_3,b_3) \in S^+$

	g_1	g_2	g_3	g_4	g_5
$\Delta_j(b_3,a_3)$	0	-20	-10	15	35
$C_j(a_3,b_3)$	1	1	1	0,5	0

$\Rightarrow C(a_3,b_3) = 0.7$.

The DM wants to know whether it is possible for S'' to reproduce the statements in S^+ and S^- by considering that only criterion g_5 has some veto power (i.e., considering u_5 and v_5 as variables, and $v_j = +\infty$, $j = 1,\ldots,4$). Following section Section 4.3, we obtain the following system (after removing the constraint concerning the pair (a_2,b_1), which is redundant because $C(a_2,b_1) = 1$):

$$\begin{cases} d'_5(a_1,b_2) \leqslant B''_i(a_1,b_2) = 0.325, \\ d'_5(a_3,b_3) \leqslant B''_i(a_3,b_3) = 0.1, \\ d'_5(a_1,b_3) > B''_i(a_1,b_3) = 0.1, \\ d'_5(a_2,b_2) > B''_i(a_2,b_2) = 0.129, \\ u_5 > p_5, \\ v_5 \geqslant u_5. \end{cases}$$

This system may then be written as

$$\begin{cases} v_5 - 2.0769u_5 \geqslant 51.5385, \\ v_5 - 9u_5 \geqslant 350, \\ v_5 - 2.0769u_5 < 450, \\ v_5 - 6.75u_5 < 379.75, \\ u_5 > p_5, \\ v_5 \geqslant u_5. \end{cases} \tag{40}$$

A solution to this system can be obtained by linear programming (we consider $\varepsilon = 0.0001$):

$$\begin{aligned} \text{Max} \quad & \sigma \\ \text{s.t.} \quad & v_5 - 2.0769u_5 - \sigma \geqslant 51.5385, \\ & v_5 - 9u_5 - \sigma \geqslant 350, \\ & v_5 - 2.0769u_5 + \sigma \leqslant 450 - \varepsilon, \\ & v_5 - 6.75u_5 + \sigma \leqslant 379.75 - \varepsilon, \\ & u_5 \geqslant p_5 + \varepsilon, \\ & v_5 \geqslant u_5. \end{aligned} \tag{41}$$

This linear program yields as an optimal solution the following values: $\sigma = 50.00005$, $u_5 = 40$, and $v_5 = 40.0001$. Since σ is not negative, the inferred values for u_5 and v_5 reproduce all of the DM's requests. The DM might also reduce the number of variables in the problem, albeit at the cost of losing flexibility, by imposing $u_5 = (1-\alpha_5)p_5 + \alpha_5 v_5$ for a given value of α_5. In this case, solving the linear program would no longer be necessary, as each of the constraints would become an upper or lower bound on v_5. For instance, considering $\alpha_5 = 0.75$ yields the bounds $v_5 \in [39.3548, 52.2581]$.

After a discussion with the DM, the analyst adds two other examples, stating that actions a_4 and a_5 are also good examples of C_4 (very low risk).

$$S^+ = \{(a1, b2), (a2, b1), (a3, b3), (a4, b3), (a5, b3)\}, \quad S^- = \{(a_1, b_3), (a_2, b_2)\}.$$

The following tables summarize the new pairwise comparisons in S^+:

$(a_4, b_3) \in S^+$

	g_1	g_2	g_3	g_4	g_5
$\Delta_j(b_3, a_4)$	-15	20	0	5	46
$C_j(a_4, b_3)$	1	0	1	1	0

$\Rightarrow C(a_4, b_3) = 0.6,$

$(a_5, b_3) \in S^+$

	g_1	g_2	g_3	g_4	g_5
$\Delta_j(b_3, a_5)$	-20	-15	-5	36	5
$C_j(a_5, b_3)$	1	1	1	0	1

$\Rightarrow C(a_5, b_3) = 0.8.$

These new statements in S^+ lead to add two new constraints to the above system:

$$\begin{cases} d'_5(a_4, b_3) \leqslant B''_i(a_4, b_3) = 0.1 \iff v_5 - 9u_5 \geqslant 460, \\ d'_5(a_5, b_3) \leqslant B''_i(a_5, b_3) = 0.325 \iff v_5 - 2.0769u_5 \geqslant 15.3842. \end{cases}$$

The introduction of these two constraints to the linear program (41) leads to a new optimal solution: $\sigma = -5.00005$. This negative value means that there are no values for the parameters u_5 and v_5 such that all constraints are satisfied simultaneously.

Since considering u_5 and v_5 as the only variables does not allow the method to reproduce her requests, the DM accepts that criterion g_4 can also have veto power. Considering the variables are u_4, v_4, u_5, v_5, the system can be written as follows (recall Section 5.2), after removing the constraint concerning the pair (a_2, b_1), which is redundant because $C(a_2, b_1) = 1$:

$$\begin{cases} \min_{j\in F} ND'_j(a_1,b_2) \geqslant B''(a_1,b_2) = 0.675, \\ \min_{j\in F} ND'_j(a_3,b_3) \geqslant B''(a_3,b_3) = 0.9, \\ \min_{j\in F} ND'_j(a_4,b_3) \geqslant B''(a_4,b_3) = 0.9, \\ \min_{j\in F} ND'_j(a_5,b_3) \geqslant B''(a_5,b_3) = 0.675, \\ \min_{j\in F} ND'_j(a_1,b_3) < B''(a_1,b_3) = 0.9, \\ \min_{j\in F} ND'_j(a_2,b_2) < B''(a_2,b_2) = 0.871, \\ v_5 \geqslant u_5 > p_5, \\ v_4 \geqslant u_4 > p_4. \end{cases} \tag{42}$$

In this case, we may consider $F = \{4,5\}$ instead of $F = \{1,\ldots,5\}$, since the criteria g_1, g_2, and g_3 were not granted veto power by the DM. As described in Section 5.2, a solution to this system can be found by solving the 0–1 program:

$$\begin{aligned} &\text{Max} \quad \sigma \\ &\text{s.t.} \begin{cases} (a_1,b_2): \begin{cases} 0.325v_5 + 0.675u_5 - \sigma \geqslant 20, \\ 0.325v_4 + 0.675u_4 - \sigma \geqslant 5; \end{cases} \\ (a_3,b_3): \begin{cases} 0.1v_5 + 0.9u_5 - \sigma \geqslant 35, \\ 0.1v_4 + 0.9u_4 - \sigma \geqslant 15; \end{cases} \\ (a_4,b_3): \begin{cases} 0.1v_5 + 0.9u_5 - \sigma \geqslant 46, \\ 0.1v_4 + 0.9u_4 - \sigma \geqslant 5; \end{cases} \\ (a_5,b_3): \begin{cases} 0.325v_5 + 0.675u_5 - \sigma \geqslant 5, \\ 0.325v_4 + 0.675u_4 - \sigma \geqslant 36; \end{cases} \\ (a_1,b_3): \begin{cases} 0.1v_5 + 0.9u_5 - M\delta_{5a_1b3} - \sigma \geqslant -M - 45 + \varepsilon, \\ 0.1v_4 + 0.9u_4 - M\delta_{4a_1b3} - \sigma \geqslant -M - 35 + \varepsilon, \\ \delta_{5a_1b_3} + \delta_{4a_1b_3} \geqslant 1, \\ \delta_{5a_1b_3}, \delta_{4a_1b_3} \in \{0,1\}; \end{cases} \\ (a_2,b_2): \begin{cases} 0.129v_5 + 0.871u_5 - M\delta_{5a_2b_2} - \sigma \geqslant -M - 49 + \varepsilon, \\ 0.129v_4 + 0.871u_4 - M\delta_{4a_2b_2} - \sigma \geqslant -M + 40 + \varepsilon, \\ \delta_{5a_2b_2} + \delta_{4a_2b_2} \geqslant 1, \\ \delta_{5a_2b_2}, \delta_{4a_2b_2} \in \{0,1\}; \end{cases} \\ v_5 \geqslant u_5 \geqslant p_5 + \varepsilon; \\ v_4 \geqslant u_4 \geqslant p_4 + \varepsilon. \end{cases} \end{aligned} \tag{43}$$

This 0–1 program yields as an optimal solution the following values: $\sigma = 1.4999$, $u_4 = 31.7222$, $v_4 = 49.4999$, $u_5 = 47.4999$, and $v_5 = 47.5$. Since σ is not negative, the inferred values for u_4, v_4, u_5, v_5 reproduce all of the DM's requests. The DM might also reduce the number of variables in the problem, albeit at the cost of losing flexibility, by imposing $u_j = (1 - \alpha_{\ j})p_j + \alpha_{j\ vj}$ for a given values of α_j. For instance, considering $\alpha_j = 0.75$, the optimal solution would become $\sigma = 0.0427$ (the problem remains feasible, but there is a reduction in σ due to the loss of flexibility), $u_4 = 34.4747$, $v_4 = 39.2995$, $u_5 = 45.2027$, and $v_5 = 53.6035$.

During the course of the example, the DM has learned that she not only had to grant g_4 some veto power, but also that the veto threshold for that criterion needed to be relatively stringent (approximately twice the preference threshold). Criterion g_5 also needs to have some veto power, as the DM would discover if she tried to consider $v_5 = +\infty$ and u_4, v_4 as the only variables. Contrary to her initial thoughts, it is the performance of a_1 on g_4, and not on g_5, that does not allow the alternative to outrank b_3.

7. Conclusion

This paper presents a contribution to a methodology to infer the parameter values of an Electre model from crisp outranking statements provided by the DM (statements that the model should restore). This is a difficult problem when all parameters have to be inferred simultaneously, hence we limit ourselves to an interactive process of partial inference problems. Partial inference problems are frequently a wise choice when interacting with a DM, since they allow greater control and comprehension of the interactive process.

This paper focusses on the inference of the discordance-related parameters (veto thresholds), thus complementing previous work on the inference of the concordance-related parameters (weights, cutting level and limits of categories). In an earlier work [7] we proposed two variants of the original valued outranking relation S (denoted S' and S'') to simplify inference problems. In this paper, we show that regarding the inference of the veto thresholds, S (see (11)) and S' (see (12)) originate mathematical programs of similar complexity, while S'' (see (13)) yields simpler versions. Table 4 summarizes the type of mathematical programs corresponding to each situation. It shows that, contrary to our initial thoughts, there are relatively easy mathematical formulations for the non-convex problem of inferring multiple veto thresholds simultaneously. Such formulations may be solved by common solvers available nowadays, such as Microsoft Excel's solver. Although in this paper we have considered veto thresholds as independent from the performances, this table applies to the more general case where thresholds are affine functions of $g_j(\cdot)$ (see details in [1]).

It should be noted that the proposed inference procedures are not intended to be used as ‘‘machine learning’’ techniques providing ‘‘true parameter values’’, but rather as useful tools to propose possibly relevant sets of parameter values to the DM. The inference procedure should guide the DM in learning not only about the method itself (to avoid a ‘‘black box’’ effect), but particularly learning about his/her preferences. In a given situation, a result stating that the veto threshold of a given criterion is close its preference threshold may teach the DM that, contrary to her initial thoughts, the veto effect of that criterion should play an important role. If the method yields the result that the last example added by the DM makes the inference an infeasible problem, then she will learn that this example contradicts other constraints she had stated before. In another situation, the DM may learn that she must change more than one veto at the same time to satisfy her requests. Illustrative examples provided in Section 6 show how the inference procedures can be used as an elicitation aid in a sorting problem based on an Electre Tri model.

The examples did not illustrate how the inference tools proposed here can interact with other partial inference tools proposed before ([7,11]), in situations where the DM has difficulties in setting all the parameters. This is an important subject for future research, although we regard these partial inference methods

Table 4
Mathematical programs corresponding to the different veto inference problems

	S	S'	S''
Inference of weights and cutting level	Global (non-convex) programming	Linear programming	Linear programming
Inference of veto for a single criterion	Linear programming	Linear programming	Linear programming
Inference of veto for all the criteria	Separable non-linear programming	Separable non-linear programming	0–1 linear programming

more like a set of tools to be used at the DM's (or analyst's) discretion than as a single integrated method. The DM can use only some of tools or all of them, depending on the difficulties and resulting from the dialogue between DM and analyst. As general advice concerning the Electre Tri method, we deem that the DM should provisionally fix the profiles according to his/her experience and/or the range of performances displayed by the alternatives. Then, he/she could fix the veto thresholds conservatively (i.e., setting relatively high values) and try to infer the weights and cutting level. If the method is not able to match the requirements of the DM then some symptoms will become apparent. From these symptoms he/she may find that the profiles or the veto thresholds need to be changed. For instance, if many negative outranking statements are difficult to reconstitute, perhaps some veto threshold(s) should be lowered. The DM can study that using the veto inference tools proposed in this paper, and then go back to the weights and cutting level inference.

Empirical tests should be conducted in the context of real-world decision problems to find what are the best strategies, paying particular attention to behavioral aspects. Indeed, future research should try to assess how more comfortable DMs feel with these tools, as compared with direct elicitation of parameter values, rather than trying to prove that inference strategies converge to some elusive stable "true" parameter values (which we do not believe are hidden inside any DM's mind).

Another subject for future research is the development of a special-purpose solver to address the mathematical formulations in Table 4. Such a solver, in the particular case of relation S'', might then be incorporated in the commercial software IRIS (Reference: Documents of INESC Coimbra, No. 1/2002, also Documents du LAMSADE, No. 127, 2002), an existing decision support tool for multicriteria sorting problems based on Electre Tri, which is already capable of inferring weights and the cutting level.

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