Inferring Electre's veto-related parameters from outranking examples

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Résumé

Lorsque l'on considère les relations de surclassement valuées d'Electre, il est difficile d'intégrer la discordance (effets de veto) aux méthodologies d'agrégation-désagrégation. Nous présentons une procédure d'inférence partielle pour calculer la valeur des seuils de veto qui restitue *au mieux* des affirmations de surclassement fournies par un décideur (*i.e.*, des exemples que le modèle Electre doit restituer).

Cet article poursuit des travaux antérieurs sur l'inference des autres paramètres préférentiels (coefficients d'importance, niveau de coupe, limites des catégories, ...) en direction d'une approche intégrée de l'inférence dans les méthodes Electre. Nous proposons des programmes mathématiques pour inférer les paramètres liés au veto, tout d'abord dans le cas d'un seul veto, puis dans le cas de veto sur plusieurs critères, et ceci en utilisant la relation de surclassemnt floue d'Electre III et deux variantes.

Ces programmes d'inférence partielle sont considérés comme des routines à utiliser de façon répétitive dans un processus de désagrégation interactif dans lequel le décideur revise l'information qu'il fournit à mesure qu'il obtient des résultats et apprend à propos de ses préférences.

Mots-Clés : *Relations de Surclassement floues, Effet de veto, Inférences des paramètres, Electre*

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Abstract

When considering Electre's valued outranking relations, aggregation/disaggregation methodologies have difficulties in taking discordance (veto) into account. We present a partial inference procedure to compute the value of the veto-related parameters that best restore a set of outranking statements, provided by a decision maker (*i.e.*, examples that an Electre model should restore).

This paper complements previous work on the inference of other preference-related parameters (weights, cutting level, category limits, ...), advancing toward an integrated approach to inference problems in Electre III and Tri methods. We propose mathematical programs to infer veto-related parameters, first considering only one criterion, then all criteria simultaneously, using the original version of Electre outranking relation and two variants. Depending on the case, these inference procedures lead to linear programming, 0-1 linear programming, or separable programming problems.

The partial inference programs are considered as problems to be solved several times in an interactive process, where decision makers continuously revise the information they provide as they learn from the results.

Keywords: *Parameter Inference Procedures, Valued Outranking Relations, Electre, discordance, veto*

Introduction

The use of multiple criteria evaluation methods is often hindered by the need to provide precise values for many preference-related parameters whose role is not clear to the Decision Maker (DM). Aggregation/disaggregation methodologies, which have received much attention lately (see *EJOR* 30(2), [3]), allow to infer values for such parameters from holistic judgments (i.e., model results) that the DM is able to provide. Usually, when the underlying evaluation model is of the Electre type (cf. [4], [6]) or of the value function type (cf. [2], [15]), a mathematical programming problem is solved to find the combination of parameter values that best restores the examples of results proposed by the DM, by minimizing some error function. This is particularly useful if the inference procedure is part of an interactive process, where the DM observes whether his/her result examples can be restored, and reacts accordingly. If the examples indicated by the DM can be restored, he/she may explore the complete set of results corresponding to the multiple combinations of parameter values that satisfy the imposed conditions (robustness analysis), which may help him/her provide further information. If not, the DM has to discover which of those examples are inconsistent, in order to withdraw some of them (inconsistency analysis).

[1] have proposed this type of methodology, integrating parameters inference, robustness analysis, and inconsistency analysis for decision aiding based on the Electre Tri method (for an overview of Electre methods see [10] and [11]). However, the implementation they proposed was limited to the situations where Electre's veto thresholds were not taken into account. This is due to the difficulty of inferring all the parameters in Electre simultaneously (see [6]), which requires solving non-linear programs with non-convex constraints. The current paper complements [5] following a different strategy, based on inferring a subset of the parameter values at a time, while maintaining the remaining ones fixed.

These "partial" inference problems, besides simplifying the mathematical programs to be solved, present important advantages. First, they let the DM focus his/her attention on a subset of parameters at a time (e.g., concordance-related parameters, then discordance-related parameters, then returning to concordance, etc.). Second, they allow the DM to control and drive the interactive process in an easier way. Namely, when there are many alternative combinations of parameter values that satisfy all the requests of the DM, keeping a subset of the parameters temporarily constant prevents the solutions from being too disparate (even among alternative optima). Furthermore, the DM is less likely to encounter radically different solutions when progressing from one iteration to the next one, and is able to better understand the consequences of changing the examples. Indeed, we believe that inference programs should not be considered as a problem to be solved only once, but rather as problems to be solved several times in an interactive learning process, where the DM(s) continuously revise the information they provide as they learn from the results of the inference programs.

While [5] considers that the concordance-related parameters were the only variables, this paper now considers that all parameters are fixed except the discordance-related ones (Electre's veto thresholds). Both papers apply to the Electre methods that use valued outranking relations (Electre III and Tri), although this work has been motivated by its application to the Electre Tri method. The holistic information provided by the DM consists of pairs of alternatives (a, b) such that, to his/her opinion, "a outranks b" (or "a does not outrank b"). Examples of results from the Electre Tri method may easily be translated into outranking statements of this kind. This paper intends to present how to infer Electre's veto thresholds from a set of such "crisp" outranking statements.

The following section presents the original outranking relation, as well as two variants we proposed in [5]. Section 2 presents the inference problem in a general format. Section 3 considers the problem of inferring the veto parameters for one criterion at a time, as a simplification of the more general problem that may nevertheless be useful in practice, since the veto parameters are not inter-related among different criteria. Section 4 deals with the more general problem of inferring the veto parameters for more than one criterion simultaneously. Finally, Section 5 presents illustrative examples, and a closing section offers a summary and some conclusions.

1 The Valued Outranking Relation in Electre

In this section we recall how Electre III (see [9]) and Electre Tri (see [14], [11]) build a valued outranking relation on the set of alternatives. Let A denote a finite set of alternatives characterized by their evaluations on n criteria $g_1, ..., g_n$. $g_j(a)$ denotes the evaluation of an alternative $a \in A$ on criterion g_j. Without any loss of generality, we will assume that the evaluations are coded in such a way that the higher the value, the better it is.

1.1 Outranking relations for a single criterion

Electre builds, for each criterion g_j, a valued outranking relation S_j restricted to a single criterion. For any ordered pair $(a, b) \in A^2$, $S_j(a, b)$ is defined by (1) on the basis of $g_j(a)$, $g_j(b)$ and two thresholds functions: indifference $q_j(g_j)$ and preference $p_j(g_j)$ ($0 \leq q_j(g_j) < p_j(g_j)$[1]). $S_j(a, b)$ represents the degree to which alternative a outranks (is at least as good as) b (see Figure 1).

$$S_j(a,b) = \frac{p_j(g_j(a)) \ - \ min\{g_j(b) - g_j(a), p_j(g_j(a))\}}{p_j(g_j(a)) \ - \ min\{g_j(b) - g_j(a), q_j(g_j(a))\}} \tag{1}$$

1.2 Concordance relation

The valued concordance relation $C(a, b)$ is grounded on the relations S_j ($j = 1, 2, ..., n$) and represents the level of majority among the criteria in favor of the assertion "a is at least as good as b". When computing this majority level, each criterion g_j has a weight $w_j \geq 0$ representing its voting power. Without any loss of generality, we will consider $\sum_{j=1}^{n} w_j = 1$. Therefore, $C(a, b)$ can be written as follows:

$$C(a,b) = \frac{1}{\sum_{j=1}^{n} w_j} \sum_{j=1}^{n} w_j.S_j(a,b) = \sum_{j=1}^{n} w_j.S_j(a,b) \tag{2}$$

[1]We will consider $q_j(g_j) < p_j(g_j)$, although Electre also allows $q_j(g_j) = p_j(g_j)$.

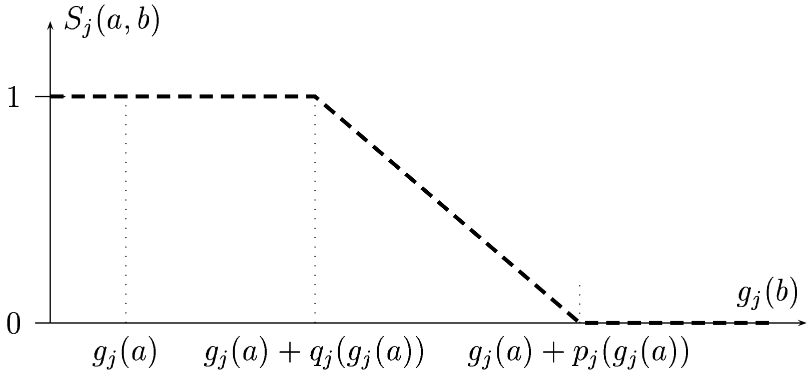

Figure 1: Partial valued outranking relation

1.3 Non-discordance relations

Electre builds, for each criterion g_j, a valued discordance relation d_j restricted to that criterion. This relation $d_j(a,b)$ is defined by (3) on the basis of $g_j(a)$, $g_j(b)$, a veto threshold function $v_j(g_j)$ and a preference threshold function $p_j(g_j)$ ($p_j(g_j) < v_j(g_j)$[2]) (see Figure 2).

$$d_j(a,b) = 1 - \frac{v_j(g_j(a)) - min\{g_j(b) - g_j(a), v_j(g_j(a))\}}{v_j(g_j(a)) - min\{g_j(b) - g_j(a), p_j(g_j(a))\}} \tag{3}$$

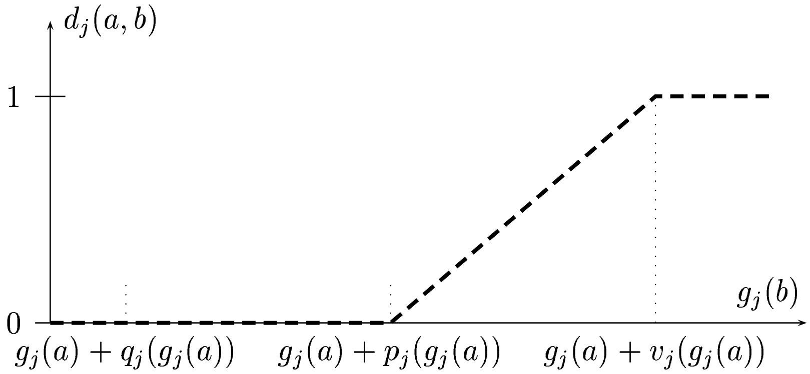

Figure 2: Partial valued outranking relation

An overall valued non-discordance relation $ND(a,b)$ is grounded on $C(a,b)$ and on the relations d_j, $j = 1,2,...,n$; it represents the degree to which the minority criteria collectively oppose a veto to the assertion "a is at least as good as b". A classical way of defining $ND(a,b)$ is given in (4). $ND(a,b) = 0$ corresponds to a situation where the minority criteria are totally opposed to aSb whereas $ND(a,b) = 1$ means that none of the criteria oppose a veto to aSb.

$$ND(a,b) = \prod_{j \in \overline{F}} \frac{1 - d_j(a,b)}{1 - C(a,b)} \text{ where } \overline{F} = \{j \in F / d_j(a,b) > C(a,b)\} \tag{4}$$

This expression is equivalent to (5):

[2]We will consider $p_j(g_j) < v_j(g_j)$, although Electre also allows $p_j(g_j) = v_j(g_j)$.

$$ND(a,b) = \prod_{j \in F} ND_j(a,b), \tag{5}$$

where[3]:

$$ND_j(a,b) = Min\left\{1, \frac{1 - d_j(a,b)}{1 - C(a,b)}\right\}. \tag{6}$$

As an alternative, [5] propose the valued non-discordance relation defined by (7)-(8), where $\alpha_j \in]0,1[$ is a constant value for the j-th criterion:

$$ND'(a,b) = \prod_{j \in F} ND'_j(a,b) \tag{7}$$

$$ND'_j(a,b) = min\left\{1, \frac{1 - d_j(a,b)}{1 - \alpha_j}\right\} \tag{8}$$

This new definition is equivalent to (9)-(10), when $u_j(g_j) = p_j(g_j) + \alpha_j.(v_j(g_j) - p_j(g_j))$:

$$ND'(a,b) = \prod_{j \in F} 1 - d'_j(a,b) \tag{9}$$

$$d'_j(a,b) = 1 - \frac{v_j(g_j(a)) - min\{g_j(b) - g_j(a), v_j(g_j(a))\}}{v_j(g_j(a)) - min\{g_j(b) - g_j(a), u_j(g_j(a))\}} \tag{10}$$

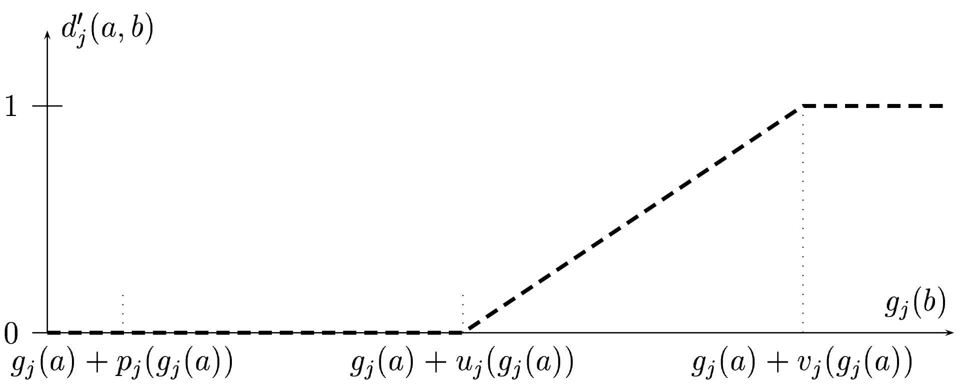

Figure 3: Partial discordance relation $d'_j(a,b)$

The new threshold $u_j(g_j)$ defines the difference of performances in favor of b where the discordance starts weakening the outranking relation (see Figure 3). It can be considered either:

- as an additional preference parameter to be elicited directly through an interaction with the DM, or indirectly using a disaggregation procedure, or

[3] Let us remark that we can state $C(a,b) < 1$, as the case $C(a,b) = 1$ corresponds to a situation where no discordant criterion exists.

- as a fixed technical parameter (rather than a preference-related one) that defines the extent to which differences of evaluation $g_j(b) - g_j(a) < v_j(g_j(a))$ should (or should not) weaken the concordance $C(a, b)$ in the definition of $S(a, b)$ (a reasonable value for α_j is 0.75, as discussed in [5]).

A second alternative proposed by [5] to define a valued non-discordance relation is the following:

$$ND''(a, b) = Min_{j \in F} ND'_j(a, b) \tag{11}$$

1.4 Valued outranking relations

Electre combines the concordance and non-discordance relations in order to define the outranking relation S as shown in (12)

$$S(a, b) = C(a, b) \ . \ ND(a, b), \tag{12}$$

or, according to the two alternative definitions,

$$S'(a, b) = C(a, b) \ . \ ND'(a, b) \tag{13}$$

$$S''(a, b) = C(a, b) \ . \ ND''(a, b) \tag{14}$$

From the valued outranking relation $S(a, b)$, it is possible to define a family of nested crisp outranking relations S_λ; these crisp relations correspond to λ-cuts of $S(a, b)$, where the cutting level $\lambda \in [0.5, 1]$ represents the minimum value for $S(a, b)$ so that $aS_\lambda b$ holds. The same applies when we consider $S'(a, b)$ or $S''(a, b)$ instead of $S(a, b)$.

2 Inference of parameter values from crisp outranking statements

The construction of the relation S (S', or S'') involves determining the evaluation vector of the alternatives, and setting many parameters: the criteria weights, the various thresholds, and the cutting level. DMs often find it difficult to provide precise values for all these preference parameters. Hence, “disaggregation approaches” have been proposed to infer the parameter values from holistic judgments.

Let us consider a decision process in which the DM(s) is/are not able (or not willing) to assign directly values to the preference parameters involved in an outranking relation, but can state crisp statements about this relation for some specific pairs of alternatives (a, b), *i.e.*, either aSb or $\neg aSb$. Let us denote $S^+ = \{(a, b) \in A^2$ such that the DM(s) stated $aSb\}$ and $S^- = \{(a, b) \in A^2$ such that the DM(s) stated $\neg aSb\}$. Then, a combination of parameter values is able to restore the DM(s)' request iff $S(a, b) \geq \lambda,\ \forall (a, b) \in S^+$ and $S(a, b) < \lambda,\ \forall (a, b) \in S^-$, which may be written as $S(a, b) - \lambda \geq 0,\ \forall (a, b) \in S^+$ and $\lambda - S(a, b) - \varepsilon \geq 0,\ \forall (a, b) \in S^-$

(ε being a small positive value). The system of constraints below (15)-(19) has a solution if and only if there exists a combination of parameter values that yields all the crisp outranking statements in S^+ and S^-. Some additional constraints can be added to this system, in order to integrate explicit statements of the DM(s) concerning the values of some parameters.

$$0 \leq S(a,b) - \lambda, \quad \forall (a,b) \in S^+ \tag{15}$$
$$0 \leq \lambda - S(a,b) - \varepsilon, \quad \forall (a,b) \in S^- \tag{16}$$
$$\lambda \in [0.5, 1] \tag{17}$$
$$v_j(g_j) > p_j(g_j) > q_j(g_j) \geq 0, \quad \forall j \in F \tag{18}$$
$$\textstyle\sum_{j=1}^{n} w_j = 1; \quad w_j \geq 0, \ \forall j \in F \tag{19}$$

The idea of inferring all the parameters by maximizing the minimum slack for the above system of constraints was proposed by [6] in the context of the Electre Tri method. However, the resulting mathematical program is very complex (nonlinear and nonconvex constraints). A solution to circumvent this difficulty is to formulate partial inference programs, where only a subset of the parameters are considered as variables, while the remaining ones are fixed. In partial inference problems, if no combination of values for the inferred parameters is able to restore the statements contained in S^+ and S^-, then the DMs should wonder why. Then, they may either revise their statements or turn their attention to a different subset of parameters that they suspect has been fixed to an inadequate level.

Among partial inference problems, previous research concerning Electre methods has focused mainly on inferring the weights and the cutting level. The problems involving the relation $S(a,b)$ can be solved using linear programs (LPs), only if discordance is ignored, i.e. no veto phenomena occur and $ND(a,b) = 1$ (e.g., see [7], [1] in the context of Electre Tri). However, when considering $S'(a,b)$ or $S''(a,b)$, the weights and the cutting level can be inferred using LP, even in the presence of discordance (see [5]).

In the context of Electre Tri, a procedure exists to infer category limits, *i.e.*, frontiers between categories and attached indifference and preference thresholds (everything else being fixed), assuming that no veto phenomena occur (see [8]). This procedure involves linear programs with 0-1 variables.

We next present an approach to infer veto-related thresholds, when all other parameters have been fixed. For a more compact notation, we will write:

$$\Delta_j(b,a) = g_j(b) - g_j(a), \tag{20}$$

which is a constant value for each pair $(a,b) \in A^2$.

3 Inference of veto-related parameters for a single criterion

In this section, we consider that all the parameters are fixed, except the veto threshold of one criterion (let i be its index). Indeed, contrarily to the weights, the veto thresholds are not interdependent, *i.e.*, fixing the veto threshold for one criterion is not influenced by the value of the veto thresholds for the remaining criteria. Hence, the DMs may wish to focus their attention on the veto power of criteria, one criterion at a time.

3.1 Inference of $v_i(g_i)$ considering $S(a,b)$

Considering the outranking relation $S(a,b)$ (see (12)) and the sets S^+ and S^- as defined in the previous section, there exists a function $v_i(g_i)$ satisfying all the requests from the DMs iff the system (21)-(23) has a solution when $v_i(g_i)$ is the only variable (recall that ε is a positive near-zero value):

$$0 \leq S(a,b) - \lambda, \quad \forall (a,b) \in S^+ \tag{21}$$
$$0 \leq \lambda - S(a,b) - \varepsilon, \quad \forall (a,b) \in S^- \tag{22}$$
$$v_i(g_i) \geq p_i(g_i) + \varepsilon \tag{23}$$

From (5) and (12), when only $v_i(g_i)$ is considered as variable, $S(a,b)$ is equal to $ND_i(a,b)$ multiplied by a constant value $K_i(a,b) = C(a,b).\prod_{j \in F \setminus \{i\}} ND_j(a,b)$. Hence, the system (21)-(23) may be written equivalently as:

$$K_i(a,b).ND_i(a,b) \geq \lambda, \quad \forall (a,b) \in S^+ \tag{24}$$
$$K_i(a,b).ND_i(a,b) \leq \lambda - \varepsilon, \quad \forall (a,b) \in S^- \tag{25}$$
$$v_i(g_i) \geq p_i(g_i) + \varepsilon \tag{26}$$

As $ND_i(a,b) \in [0,1]$, it holds $S(a,b) = ND_i(a,b).K_i(a,b) \leq K_i(a,b), \forall (a,b)$. This originates the following two remarks:

- Every pair $(a,b) \in S^+$ such that $K_i(a,b) < \lambda$ implies $S(a,b) < \lambda$. Consequently, these statements can not be reproduced by $S(a,b)$ whatever the veto threshold on g_i.
- Every pair $(a,b) \in S^-$ such that $K_i(a,b) \leq \lambda - \varepsilon$ implies $S(a,b) \leq \lambda - \varepsilon$. Consequently, these statements are correctly reproduced by $S(a,b)$ whatever the veto threshold on g_i.

Now, let us define the sets S_i^+ and S_i^- as in (27). S_i^+ contains the positive outranking statements provided by the DM for which a "correct" replication by $S(a,b)$ depends on the discordance on criterion g_i. S_i^- contains the negative outranking statements (not aSb) for which a "correct" replication by $S(a,b)$ depends on the discordance on criterion g_i.

$$S_i^+ = \left\{(a,b) \in S^+ : K_i(a,b) \geq \lambda\right\}, \qquad S_i^- = \left\{(a,b) \in S^- : K_i(a,b) > \lambda - \varepsilon\right\} \tag{27}$$

It follows from the preceding that:

- if S_i^+ does not coincide with S^+, then there exists a pair $(a,b) \in S_i^+ \setminus S^+$ such that the constraint $K_i(a,b).ND_i(a,b) \geq \lambda$ is violated whatever the value of $ND_i(a,b)$, *i.e.*, the system (24)-(26) has no solution;

- if S_i^- does not coincide with S^-, then every pair $(a,b) \in S_i^- \backslash S^-$ is such that the constraint $K_i(a,b).ND_i(a,b) \leq \lambda - \varepsilon$ is respected, regardless of $ND_i(a,b)$, *i.e.*, the constraint is superfluous in the system (24)-(26).

Hence, after checking that S_i^+ coincides with S^+ (otherwise there is no solution), it is safe to consider S_i^+ and S_i^- instead of S^+ and S^-, as follows (note that $K_i(a,b) \neq 0, \forall (a,b) \in S_i^+ \cup S_i^-$, which is obvious from (27), since $\lambda \geq 0.5$):

$$
\begin{aligned}
ND_i(a,b) &= Min\left\{1, \frac{1-d_i(a,b)}{1-C(a,b)}\right\} \geq \frac{\lambda}{K_i(a,b)}, \quad \forall (a,b) \in S_i^+ && (28)\\
ND_i(a,b) &= Min\left\{1, \frac{1-d_i(a,b)}{1-C(a,b)}\right\} \leq \frac{\lambda-\varepsilon}{K_i(a,b)}, \quad \forall (a,b) \in S_i^- && (29)\\
v_i(g_i) &\geq p_i(g_i) + \varepsilon. && (30)
\end{aligned}
$$

Now, (27) implies that $\lambda / K_i(a,b) \in]0,1], \forall (a,b) \in S_i^+$ and $(\lambda - \varepsilon)/K_i(a,b) \in]0,1[, \forall (a,b) \in S_i^-$. Therefore, the system (28)-(30) is equivalent to the following one:

$$
\begin{aligned}
1 - d_i(a,b) &\geq (1 - C(a,b)) \, . \frac{\lambda}{K_i(a,b)}, \quad \forall (a,b) \in S_i^+ && (31)\\
1 - d_i(a,b) &\leq (1 - C(a,b)) \, . \frac{\lambda-\varepsilon}{K_i(a,b)}, \quad \forall (a,b) \in S_i^- && (32)\\
v_i(g_i) &\geq p_i(g_i) + \varepsilon && (33)
\end{aligned}
$$

If we now define:

$$
U_i(a,b) = 1 - \frac{(1 - C(a,b)) \, .\lambda}{K_i(a,b)}, (a,b) \in S_i^+, \qquad (34)
$$

$$
L_i(a,b) = 1 - \frac{(1 - C(a,b)) \, .(\lambda - \varepsilon)}{K_i(a,b)}, (a,b) \in S_i^-, \qquad (35)
$$

then the same system may be written as:

$$
\begin{aligned}
d_i(a,b) &\leq U_i(a,b), \quad \forall (a,b) \in S_i^+ && (36)\\
d_i(a,b) &\geq L_i(a,b), \quad \forall (a,b) \in S_i^- && (37)\\
v_i(g_i) &\geq p_i(g_i) + \varepsilon && (38)
\end{aligned}
$$

where each $d_i(a,b)$ is a function of $v_i(g_i(a))$ (recall that all other parameters are fixed) that yields a value in the interval $[0,1]$ (see (3) and Figure 2).

From (27) and (34)-(35), it is obvious that both $U_i(a,b) \in [0,1[$ and $L_i(a,b) \in]0,1[$. Let us define the following partitions of the set S_i^+:

$$
S^+_{(U>0)i} = \left\{(a,b) \in S_i^+ : 1 > U_i(a,b) > 0\right\}, \quad S^+_{(U=0)i} = \left\{(a,b) \in S_i^+ : U_i(a,b) = 0\right\} \qquad (39)
$$

In (36), each constraint derived from $S^+_{(U=0)i}$ requires $d_i(a,b)=0$ (note that $d_i(a,b)$ can not be negative), hence it is either impossible to respect (if $\Delta_i(b,a) > p_i(g_i(a))$) or irrelevant (if $\Delta_i(b,a) \leq p_i(g_i(a))$), regardless of the value of $v_i(g_i(a))$. In the first case, the system (36)-(38) has no solution; in the second case, the constraint can be ignored. Each constraint derived from $S^+_{(U>0)i}$ requires $0 < d_i(a,b) < 1$, hence from (3) it can be translated into a lower bound for $v_i(g_i(a))$. Similarly, in (37), each constraint derived from S^-_i requires $0 < d_i(a,b) < 1$, hence from (3) it can be translated into an upper bound for $v_i(g_i(a))$. Therefore, either we find that the system has no solution, or we may search for a solution to the system (40)-(42), knowing that $d_i(a,b) \in]0,1[$ iff $d_i(a,b) = \frac{\Delta_i(b,a)-p_i(g_i(a))}{v_i(g_i(a))-p_i(g_i(a))}$:

$$v_i(g_i(a)) \geq p_i(g_i(a)) + \frac{\Delta_i(b,a) - p_i(g_i(a))}{U_i(a,b)}, \quad \forall (a,b) \in S^+_{(U>0)i} \tag{40}$$

$$v_i(g_i(a)) \leq p_i(g_i(a)) + \frac{\Delta_i(b,a) - p_i(g_i(a))}{L_i(a,b)}, \quad \forall (a,b) \in S^-_i \tag{41}$$

$$v_i(g_i) \geq p_i(g_i) + \varepsilon \tag{42}$$

Usually, $v_i(g_i(a))$ is: (i) a constant value v_i^c; (ii) a proportion of the evaluation $v_i(g_i(a)) = v_i^p.g_i(a)$ (which requires that $g_i(a) > 0$); or (iii) an affine function $v_i(g_i(a)) = v_i^c + v_i^p.g_i(a)$.

(i) let L_i denote the greatest of the lower bounds derived from S_i^+, and let U_i denote the lowest of the upper bounds derived from S_i^-. Then the system (40)-(42) has no solution if $U_i < \max\{L_i, p_i(g_i)+\varepsilon\}$. Otherwise any value for v_i^c in $[\max\{L_i, p_i(g_i)+\varepsilon\}, U_i]$ is acceptable, namely $v_i^c = \frac{U_i + \max\{L_i, p_i(g_i)+\varepsilon\}}{2}$.

(ii) the process is similar, after we divide all the constraints in (40)-(42) by $g_i(a)$.

(iii) we may solve the following LP, where the variables are v_i^c, v_i^p, and σ.

$$\max \quad \sigma \quad \text{s.t.} \tag{43}$$

$$v_i^c + v_i^p.g_i(a) \geq p_i(g_i(a)) + \frac{\Delta_i(b,a) - p_i(g_i(a))}{U_i(a,b)} + \sigma, \quad \forall (a,b) \in S^+_{(U>0)i} \tag{44}$$

$$v_i^c + v_i^p.g_i(a) \leq p_i(g_i(a)) + \frac{\Delta_i(b,a) - p_i(g_i(a))}{L_i(a,b)} - \sigma, \quad \forall (a,b) \in S^-_i \tag{45}$$

$$v_i^c + v_i^p.g_i(a) \geq p_i(g_i(a)) + \varepsilon, \quad \forall a \in A \tag{46}$$

If the optimum value of the LP (43)-(46) is positive or null, then the system (40)-(42) has a solution, i.e., the optimum solution yields a function $v_i(g_i(a))$ that respects all the statements provided by the DM(s). Otherwise, the system (40)-(42) has no solution.

3.2 Inference of $u_i(g_i)$ and $v_i(g_i)$ considering $S'(a,b)$

In the specific case of the outranking relation $S'(a,b)$ (see (13)), a new veto-related parameter $u_i(g_i)$ has been introduced (see (10) and Figure 3). In this subsection we address the problem of inferring $u_i(g_i)$ and $v_i(g_i)$ simultaneously ($v_i(g_i) > u_i(g_i) \geq p_i(g_i)$). Note that if we considered that $u_i(g_i)$ is fixed and $v_i(g_i)$ is the only variable, then the process would be quite similar to the one followed in §3.1.

We will start by introducing some notation similar to the one introduced in §3.1:

- $K'_i(a,b) = C(a,b).\prod_{j\in F\setminus\{i\}} ND'_j(a,b)$ (the product of the factors that do not depend on $u_i(g_i)$ and $v_i(g_i)$);
- $S'^+_i = \{(a,b) \in S^+ : K'_i(a,b) \geq \lambda\}$ (if $S'^+_i \neq S^+$ then the problem has no solution);
- $S'^-_i = \{(a,b) \in S^- : K'_i(a,b) > \lambda - \varepsilon\}$(all constraints from $S^-\setminus S'^-_i$ are redundant);
- $L'_i(a,b) = 1 - (\lambda - \varepsilon)/K'_i(a,b), (a,b) \in S'^-_i$ (lower bound for $d'_i(a,b)$);
- $U'_i(a,b) = 1 - \lambda/K'_i(a,b), (a,b) \in S'^+_i$ (upper bound for $d'_i(a,b)$);
- $S'^+_{(U>0)i} = \left\{(a,b) \in S'^+_i : 1 > U'_i(a,b) > 0\right\}$;
- $S'^+_{(U=0)i} = \left\{(a,b) \in S'^+_i : U'_i(a,b) = 0\right\}$.

Following the reasoning of §3.1, inferring $u_i(g_i)$ and $v_i(g_i)$ amounts at solving the following system of inequalities, where the variables $v_i(g_i)$ and $u_i(g_i)$ affect $d'_i(a,b)$, whereas $U'_i(a,b)$ and $L'_i(a,b)$ are constants:

$$\begin{aligned} d'_i(a,b) &\leq U'_i(a,b), \quad \forall (a,b) \in S'^+_i && (47)\\ d'_i(a,b) &\geq L'_i(a,b), \quad \forall (a,b) \in S'^-_i && (48)\\ v_i(g_i) - \varepsilon &\geq u_i(g_i) \geq p_i(g_i). && (49)\end{aligned}$$

For each $(a,b) \in S'^+_{(U=0)i}$ we have a constraint $d'_i(a,b) = 0$ (note that $d'_i(a,b)$ can not be negative), which from (10) is equivalent to $\Delta_i(b,a) \leq u_i(g_i(a))$. The remaining pairs $(a,b) \in S'^+_{(U>0)i}$ and $(a,b) \in S'^-_i$ constrain $d'_i(a,b)$ to be lower or higher (respectively) than a value in the interval $]0,1[$. Since $d'_i(a,b) \in]0,1[$ iff $d'_i(a,b) = \frac{\Delta_i(b,a)-u_i(g_i(a))}{v_i(g_i(a))-u_i(g_i(a))}$, the following system is equivalent to the previous one:

$$\begin{aligned} u_i(g_j(a)) &\geq \Delta_i(b,a), \quad \forall (a,b) \in S'^+_{(U=0)i} && (50)\\ v_i(g_i(a)) &\geq u_i(g_i(a)) + \frac{\Delta_i(b,a) - u_i(g_i(a))}{U'_i(a,b)}, \quad \forall (a,b) \in S'^+_{(U>0)i} && (51)\\ v_i(g_i(a)) &\leq u_i(g_i(a)) + \frac{\Delta_i(b,a) - u_i(g_i(a))}{L'_i(a,b)}, \quad \forall (a,b) \in S'^-_i && (52)\\ v_i(g_i) - \varepsilon &\geq u_i(g_i) \geq p_i(g_i). && (53)\end{aligned}$$

Assuming that the thresholds are affine functions, let $v_i(g_i(a)) = v_i^c + v_i^p.g_i(a)$, $u_i(g_i(a)) = u_i^c + u_i^p.g_i(a)$, and $p_i(g_i(a)) = p_i^c + p_i^p.g_i(a)$. In this case we may solve the following LP, where the variables are v_i^c, v_i^p, u_i^c, u_i^p, and σ:

$$\text{Max} \quad \sigma \quad \text{s.t.} \tag{54}$$
$$u_i^c + u_i^p.g_i(a) \geq \Delta_i(b,a) + \sigma, \quad \forall (a,b) \in S'^{+}_{(U=0)i} \tag{55}$$
$$v_i^c + v_i^p.g_i(a) + (u_i^c + u_i^p.g_i(a))\left(\frac{1}{U'_i(a,b)} - 1\right) \geq \frac{\Delta_i(b,a)}{U'_i(a,b)} + \sigma, \ \forall (a,b) \in S'^{+}_{(U>0)i} \tag{56}$$
$$v_i^c + v_i^p.g_i(a) + (u_i^c + u_i^p.g_i(a))\left(\frac{1}{L'_i(a,b)} - 1\right) \leq \frac{\Delta_i(b,a)}{L'_i(a,b)} - \sigma, \ \forall (a,b) \in S'^{-}_i \tag{57}$$
$$v_i^c + v_i^p.g_i(a) - \varepsilon \geq u_i^c + u_i^p.g_i(a) \geq p_i(g_i(a)), \quad \forall a \in A \tag{58}$$

If the optimum value of the LP (54)-(58) is positive or null, then the system (47)-(49) has a solution, *i.e.*, the optimum solution yields a function $v_i(g_i(a))$ and a function $u_i(g_i(a))$ that respect all the statements provided by the DM(s). Otherwise, the system (47)-(49) has no solution. The affine case includes the constant and proportional ones: it is a matter of fixing $v_i^p = u_i^p = p_i^p = 0$ in the constant case, or fixing $v_i^c = u_i^c = p_i^c = 0$ in the proportional case.

NOTE: As discussed in §1.3, the new parameter $u_i(g_i)$ can also be interpreted as function of a technical parameter α, when the DMs do not wish to regulate α_i separately for different criteria. In that case, $u_i(g_i) = p_i(g_i) + \alpha.(v_i(g_i) - p_i(g_i))$, where $\alpha \in [0,1[$ is a constant for all the criteria. Hence, we need only to replace $u_i(g_i)$ by this expression in the system (50)-(53). Both the constant and proportional cases can be solved easily without LP techniques, since each constraint will translate into a bound on v_i^c or v_i^p, respectively. The affine case may be addressed replacing $u_i^c + u_i^p.g_i(a)$ by $p_i(g_i(a)) + \alpha.(v_i^c + v_i^p.g_i(a) - p_i(g_i(a))$ in the LP (54)-(58). The variables are now v_i^c, v_i^p, and σ.

3.3 Inference of $u_i(g_i)$ and $v_i(g_i)$ considering $S''(a,b)$

Let us now consider the outranking relation $S''(a,b)$, in a situation where the DMs are interested in inferring $u_i(g_i)$ and $v_i(g_i)$ simultaneously ($v_i(g_i) > u_i(g_i) \geq p_i(g_i)$) for a given criterion g_i. Let us define $M_i(a,b) = \min_{j \in F \setminus \{i\}} ND'_j(a,b)$. As only $v_i(g_i)$ is considered as variable, $M_i(a,b)$ is a constant value such that

$$S''(a,b) = C(a,b).\min\{M_i(a,b), 1 - d'_i(a,b)\} \leq C(a,b).M_i(a,b). \tag{59}$$

If we define the sets

$$S''^{+}_i \ = \ \left\{(a,b) \in S^+ : C(a,b) > 0 \wedge M_i(a,b) \geq \lambda / C(a,b)\right\}, \tag{60}$$
$$S''^{-}_i \ = \ \left\{(a,b) \in S^- : C(a,b) > 0 \wedge M_i(a,b) > (\lambda - \varepsilon)/C(a,b)\right\}, \tag{61}$$

then, as in §3.1, if $S''^{+}_i \neq S^+$ then there is no solution; furthermore, the constraints from the pairs in $S^- \setminus S''^{-}_i$ are redundant. Therefore, when $S''(a,b)$ replaces $S(a,b)$ in the system (21)-(23), we may consider the following system instead:

$$S''(a,b) \ \geq \ \lambda, \quad \forall (a,b) \in S''^{+}_i \tag{62}$$
$$S''(a,b) \ \leq \ \lambda - \varepsilon, \quad \forall (a,b) \in S''^{-}_i \tag{63}$$
$$v_i(g_i) - \varepsilon \ \geq \ u_i(g_i) \geq p_i(g_i) \tag{64}$$

From (59), each constraint $S''(a,b) \geq \lambda$ holds iff $M_i(a,b) \geq \lambda/C(a,b)$ and $1 - d_i'(a,b) \geq \lambda/C(a,b)$. However, $M_i(a,b) \geq \lambda/C(a,b)$ is true for every $(a,b) \in S_i''^{+}$; hence, $S''(a,b) \geq \lambda \Leftrightarrow 1 - d_i'(a,b) \geq \lambda/C(a,b), \forall (a,b) \in S_i''^{+}$. On the other hand, each constraint $S''(a,b) \leq \lambda - \varepsilon$ holds iff $M_i(a,b) \leq (\lambda-\varepsilon)/C(a,b)$ or $1 - d_i'(a,b) \leq (\lambda-\varepsilon)/C(a,b)$. However, $M_i(a,b) \leq (\lambda-\varepsilon)/C(a,b)$ is false for every $(a,b) \in S_i''^{-}$; hence, $S''(a,b) \leq \lambda - \varepsilon \Leftrightarrow 1 - d_i'(a,b) \leq (\lambda-\varepsilon)/C(a,b), \forall (a,b) \in S_i''^{-}$. Therefore, if we now define the bounds for $d_i'(a,b)$ as

$$U_i''(a,b) = 1 - \lambda/C(a,b), (a,b) \in S_i''^{+}, \quad L_i''(a,b) = 1 - (\lambda-\varepsilon)/C(a,b), (a,b) \in S_i''^{-}, \qquad (65)$$

then the system (62)-(64) can be written as:

$$\begin{aligned} d_i'(a,b) &\leq U_i''(a,b), \quad \forall (a,b) \in S_i''^{+} && (66)\\ d_i'(a,b) &\geq L_i''(a,b), \quad \forall (a,b) \in S_i''^{-} && (67)\\ v_i(g_i) - \varepsilon &\geq u_i(g_i) \geq p_i(g_i). && (68) \end{aligned}$$

This system is similar to the system (47)-(49) that we found in §3.2, hence we may proceed as proposed in that section. In summary, the process is the same as when we use $S'(a,b)$, with a different definition for the sets $S_i'^{+}$and $S_i'^{-}$, and the bounds $U_i'(a,b)$ and $L_i'(a,b)$.

NOTE: In the particular case where $u_j(g_j) = p_j(g_j) + \alpha.(v_j(g_j) - p_j(g_j))$, for a fixed α, the process is also the same as when we use $S'(a,b)$ (see §3.2),with a different definition for the sets $S_i'^{+}$and $S_i'^{-}$, and the bounds $U_i'(a,b)$ and $L_i'(a,b)$.

4 Inference of all veto-related parameters simultaneously

In this section we consider that all the parameters are fixed, except some of the veto thresholds, possibly all of them. This situation will occur when the DMs do not wish to focus on the veto power of one criterion at a time. Let $V \subseteq F$ contain the indices of the criteria whose veto threshold is not fixed. The criteria whose indices are in $F \backslash V$ are either fixed or do not possess any veto power ($v_j(g_j) = \infty$).

4.1 Inference of $v_i(g_i)$ considering $S(a,b)$ or $S'(a,b)$

Considering the outranking relation $S(a,b)$ (see (12)), there exists a set of functions $v_j(g_j)$ satisfying all the requests from the DMs iff the system (21)-(23) has a solution when the variables are $v_j(g_j)$, $j \in V$. Let us define $B_V(a,b) = C(a,b).\prod_{j\in F\backslash V} ND_j(a,b)$, which is a constant value for (a,b). We can now write $S(a,b) = B_V(a,b).\prod_{j\in V} ND_j(a,b)$. Let us also define:

$$S_V^{+} = \Big\{(a,b) \in S^{+} : B_V(a,b) \geq \lambda\Big\}, \qquad S_V^{-} = \Big\{(a,b) \in S^{-} : B_V(a,b) > \lambda - \varepsilon\Big\} \qquad (69)$$

From this definition and the fact that $S(a,b) \leq B_V(a,b)$, it follows that:

- if S_V^+ does not coincide with S^+, then there exists a pair $(a,b) \in S_V^+ \backslash S^+$ such that $S(a,b) \leq B_V(a,b) < \lambda$ whatever the value of $\prod_{j \in V} ND_j(a,b)$, *i.e.*, the constraint $S(a,b) \geq \lambda$ is always violated and the system (21)-(23) has no solution;

- if S_V^- does not coincide with S^-, then every pair $(a,b) \in S_V^- \backslash S^-$ is such that $B_V(a,b) \leq \lambda - \varepsilon$, which implies that the constraint $S(a,b) \leq \lambda - \varepsilon$ is respected, regardless of $\prod_{j \in V} ND_j(a,b)$, *i.e.*, the constraint is superfluous in the system (21)-(23).

Since $B_V(a,b) \neq 0, \forall (a,b) \in S_V^+ \cup S_V^-$, we can also define $LB_V(a,b) = \lambda / B_V(a,b)$ and $UB_V(a,b) = (\lambda - \varepsilon)/B_V(a,b)$. Hence, after checking that S_V^+ coincides with S^+ (otherwise there is no solution), it is safe to consider S_V^+ and S_V^- instead of S^+ and S^-, as follows:

$$\prod_{j \in V} ND_j(a,b) \geq LB_V(a,b), \quad \forall (a,b) \in S_V^+ \tag{70}$$
$$\prod_{j \in V} ND_j(a,b) \leq UB_V(a,b), \quad \forall (a,b) \in S_V^- \tag{71}$$
$$v_j(g_j) \geq p_j(g_j) + \varepsilon, \quad \forall j \in V \tag{72}$$

This is a nonlinear system of inequalities, where the variables are $v_j(g_j)$, $j \in V$, as arguments of $ND_j(a,b)$ (see §1.3):

$$ND_j(a,b) = \min\left\{1, \frac{1 - d_j(a,b)}{1 - C(a,b)}\right\} = \min\left\{1, \frac{\min\left\{1, \max\left\{0, \frac{v_j(g_j(a)) - \Delta_j(b,a)}{v_j(g_j(a)) - p_j(g_j(a))}\right\}\right\}}{1 - C(a,b)}\right\} \tag{73}$$

If $\Delta_j(b,a) \leq p_j(g_j(a))$, then $ND_j(a,b) = 1$, regardless of $v_j(g_j)$. Hence, if we denote $V_{ab} = \{j \in V : \Delta_j(b,a) > p_j(g_j(a))\}$, we can write the system above as:

$$\prod_{j \in V_{ab}} ND_j(a,b) \geq LB_V(a,b), \quad \forall (a,b) \in S_V^+ \tag{74}$$
$$\prod_{j \in V_{ab}} ND_j(a,b) \leq UB_V(a,b), \quad \forall (a,b) \in S_V^- \tag{75}$$
$$v_j(g_j) \geq p_j(g_j) + \varepsilon, \quad \forall j \in V \tag{76}$$

Note also that:

$$ND_j(a,b) = \min\left\{1, \frac{\max\left\{0, \frac{v_j(g_j(a)) - \Delta_j(b,a)}{v_j(g_j(a)) - p_j(g_j(a))}\right\}}{1 - C(a,b)}\right\}, \ \forall j \in V_{ab} \tag{77}$$

We will now transform this system using logarithms. Let us define:

$$f_j(a,b,v_j) = \log \max\{0.1, ND_j(a,b)\}, \ \forall j \in V_{ab} \tag{78}$$

This definition considers that $ND_j(a,b) = 0.1$, whenever its real value is below 0.1 (we need this to ensure we are taking the logarithm of a positive quantity). However, note that if $ND_j(a,b) < 0.5$, for some $j \in V_{ab}$, then $S(a,b) < \lambda$, regardless of any other parameters, since $\lambda \geq 0.5$. Hence, this modification will not influence the results (any value in]0,0.5[could replace 0.1).

Now, the system (74)-(76) has a solution iff the following mathematical program has a non-negative optimal value:

$$\max \quad \sigma \quad \text{s.t.} \tag{79}$$

$$\sum_{j\in V_{ab}} f_j\left(a,b,v_j\right) \geq \log LB_V(a,b)+\sigma, \quad \forall (a,b)\in S_V^+ \tag{80}$$

$$\sum_{j\in V_{ab}} f_j\left(a,b,v_j\right) \leq \log UB_V(a,b)-\sigma, \quad \forall (a,b)\in S_V^- \tag{81}$$

$$v_j(g_j) \geq p_j(g_j)+\varepsilon, \quad \forall j\in V \tag{82}$$

The advantage of using logarithms is that we obtain a separable nonlinear program, which may be solved by 0-1 linear programming techniques (see Appendix A).

If we considered the outranking relation $S'(a,b)$ (see (13)) instead of $S(a,b)$, the process would be analogous to the one described here, with the difference of placing $ND'_j(a,b)$ instead of $ND_j(a,b)$. The only significant consequence is that $ND'_j(a,b)$ depends also on the variables $u_j(g_j)$ $(j\in V)$ (besides $v_j(g_j)$), which increases the number of binary variables in the 0-1 linear programs to solve.

4.2 Inference of $u_i(g_i)$ and $v_i(g_i)$ considering $S''(a,b)$

Considering the outranking relation $S''(a,b)$ (see (14)) and the variables $u_j(g_j)$ and $v_j(g_j)$ $(j\in V)$, we will follow the same reasoning, as compared with $S(a,b)$. The sets $S_V^{+\prime\prime}$ and $S_V^{-\prime\prime}$ play the same role as S_V^+ and S_V^- in §4.1, but have different definitions:

$$S_V^{+\prime\prime} = \left\{(a,b)\in S^+ : C(a,b).\min_{j\in F\backslash V} ND'_j(a,b)\geq\lambda\right\}, \tag{83}$$

$$S_V^{-\prime\prime} = \left\{(a,b)\in S^- : C(a,b).\min_{j\in F\backslash V} ND'_j(a,b)>\lambda-\varepsilon\right\} \tag{84}$$

Note that $S''(a,b)<\lambda$, $\forall(a,b)\in S^+\backslash S_V^{+\prime\prime}$ and $S''(a,b)\leq\lambda-\varepsilon$, $\forall(a,b)\in S^-\backslash S_V^{-\prime\prime}$. Let us define $LB''(a,b)=\lambda/C(a,b)$ and $UB''(a,b)=(\lambda-\varepsilon)/C(a,b)$, which play the role of $LB_V(a,b)$ and $UB_V(a,b)$, respectively, in §4.1. After checking that $S_V^{+\prime\prime}=S^+$ (otherwise there is no solution), the inference of values for $u_j(g_j)$ and $v_j(g_j)$ $(j\in V)$ may be performed by solving the following system:

$$\min_{j\in F} ND'_j(a,b) \;\geq\; LB''(a,b), \quad \forall(a,b)\in S_V^{+\prime\prime} \tag{85}$$

$$\min_{j\in F} ND'_j(a,b) \;\leq\; UB''(a,b), \quad \forall(a,b)\in S_V^{-\prime\prime} \tag{86}$$

$$v_j(g_j)-\varepsilon \;\geq\; u_j(g_j)\geq p_j(g_j), \quad \forall j\in V \tag{87}$$

where the variables $u_j(g_j)$ and $v_j(g_j)$ $(j\in V)$ are arguments of $ND'_j(a,b)$. According to the definitions above, it holds that $LB''(a,b)\in]0,1]$ and $UB''(a,b)\in]0,1[$.

Let us now define the notation $f(a,b,u_j,v_j)=\frac{v_j(g_j(a))-\Delta_j(b,a)}{v_j(g_j(a))-u_j(g_j(a))}$, which allows us to write $ND'_j(a,b)=\min\{1,\max\{0,f(a,b,u_j,v_j)\}\}$. The system (85)-(87) is equivalent to the following one:

$$\min_{j\in F} f(a,b,u_j,v_j) \;\geq\; LB''(a,b), \quad \forall(a,b)\in S_V^{+\prime\prime} \tag{88}$$

$$\min_{j\in F} f(a,b,u_j,v_j) \;\leq\; UB''(a,b), \quad \forall(a,b)\in S_V^{-\prime\prime} \tag{89}$$

$$v_j(g_j)-\varepsilon \;\geq\; u_j(g_j)\geq p_j(g_j), \quad \forall j\in V \tag{90}$$

where:

- constraints (88) are respected if $\min_j f(.) \geq 1$ and are violated if $\min_j f(.) \leq 0$; in the first case $\min_j ND'_j(a,b) = 1$, which respects (85); in the second case, $\min_j ND'_j(a,b) = 0$, which violates (85) (note that $LB''(a,b) \in]0,1[$ for $(a,b) \in S_V^{+\prime\prime}$);
- constraints (89) are violated if $\min_j f(.) \geq 1$ and are respected if $\min_j f(.) \leq 0$; in the first case $\min_j ND'_j(a,b) = 1$, which violates (86); in the second case, $\min_j ND'_j(a,b) = 0$, which respects (86);
- the values of $f(a,b,u_j,v_j)$ are fixed for $j \in F \backslash V$ and variable for $j \in V$, hence we may readily verify that one of the fixed values makes a constraint from (88) impossible to respect (hence there would be no solution) or makes a constraint from (89) redundant (hence may be deleted).

We can now build a mathematical program to test if system (88)-(90) has a solution. We propose different types of formulations to deal with the different types of threshold functions (affine, proportional and constant).

Affine case: $v_j(g_j(a)) = v_j^c + v_j^p g_j(a)$ **and** $u_j(g_j(a)) = u_j^c + u_j^p g_j(a)$
Each of the constraints (88) has the form $\min_{j \in F} f(a,b,u_j,v_j) \geq LB''(a,b)$, for some $(a,b) \in S_V^{+\prime\prime}$, which may be written equivalently (since $v_j^c + v_j^p g_j(a) > u_j^c - u_j^p g_j(a)$) as:

$$\min_{j \in F} f(a,b,u_j,v_j) - LB''(a,b) \geq 0 \tag{91}$$

$$\Leftrightarrow \min_{j \in F} \frac{v_j^c + v_j^p g_j(a) - \Delta_j(b,a)}{v_j^c + v_j^p g_j(a) - u_j^c - u_j^p g_j(a)} - LB''(a,b) \geq 0 \tag{92}$$

$$\Leftrightarrow \frac{v_j^c + v_j^p g_j(a) - \Delta_j(b,a)}{v_j^c + v_j^p g_j(a) - u_j^c - u_j^p g_j(a)} - LB''(a,b) \geq 0,\ \forall j \in F \tag{93}$$

$$\Leftrightarrow (1 - LB''(a,b)) \,.\, \left(v_j^c + v_j^p g_j(a)\right) + LB''(a,b).\, \left(u_j^c + u_j^p g_j(a)\right) \geq \Delta_j(b,a),\ \forall j \in F \tag{94}$$

On the other hand, each of the constraints (89) has the form $\min_{j \in F} f(a,b,u_j,v_j) \leq UB''(a,b)$, for some $(a,b) \in S_V^{-\prime\prime}$, which may be written equivalently (since $v_j^c + v_j^p g_j(a) > u_j^c - u_j^p g_j(a)$) as:

$$\min_{j \in F} f(a,b,u_j,v_j) - UB''(a,b) \leq 0 \tag{95}$$

$$\Leftrightarrow \min_{j \in F} \frac{v_j^c + v_j^p g_j(a) - \Delta_j(b,a)}{v_j^c + v_j^p g_j(a) - u_j^c - u_j^p g_j(a)} - UB''(a,b) \leq 0 \tag{96}$$

$$\Leftrightarrow \exists j \in F : \frac{v_j^c + v_j^p g_j(a) - \Delta_j(b,a)}{v_j^c + v_j^p g_j(a) - u_j^c - u_j^p g_j(a)} - UB''(a,b) \leq 0 \tag{97}$$

$$\Leftrightarrow \exists j \in F : (1 - UB''(a,b)) \,.\, \left(v_j^c + v_j^p g_j(a)\right) + UB''(a,b).\, \left(u_j^c + u_j^p g_j(a)\right) \leq \Delta_j(b,a) \tag{98}$$

$$\Leftrightarrow \begin{cases} (1 - UB''(a,b)) \,.\, \left(v_j^c + v_j^p g_j(a)\right) + UB''(a,b).\, \left(u_j^c + u_j^p g_j(a)\right) + M.\delta_{jab} \\ \qquad \leq M + \Delta_j(b,a),\ \forall j \in F \\ \sum_{j \in F} \delta_{jab} \geq 1 \\ \delta_{jab} \in \{0,1\},\ \forall j \in F \end{cases} \tag{99}$$

where M is a large positive constant greater than $f(a,b,u_j,v_j)$, $\forall j \in F$, $\forall (a,b) \in S_V^{-\prime\prime}$. The system (99) uses 0-1 (binary) variables to account for the disjunctive nature of (98). Note that $\sum_{j\in F} \delta_{jab} \geq 1$ forces at least one of the binary variables δ_{jab} to be have the value 1, thus forcing (98).

Considering these transformations, the system (88)-(90) has a solution iff the following 0-1 linear program has a non-negative optimal value:

$$\max \quad \sigma \tag{100}$$

$$\text{s.t.} \quad \begin{aligned}(1-LB''(a,b))\,.\left(v_j^c + v_j^p g_j(a)\right) + LB''(a,b).\left(u_j^c + u_j^p g_j(a)\right) \\ \geq \Delta_j(b,a) + \sigma, \quad \forall j \in F, (a,b) \in S_V^{+\prime\prime}\end{aligned} \tag{101}$$

$$\begin{aligned}(1-UB''(a,b))\,.\left(v_j^c + v_j^p g_j(a)\right) + UB''(a,b).\left(u_j^c + u_j^p g_j(a)\right) + M.\delta_{jab} \\ \leq M + \Delta_j(b,a) - \sigma, \ \forall j \in F, (a,b) \in S_V^{-\prime\prime}\end{aligned} \tag{102}$$

$$\sum_{j\in F} \delta_{jab} \geq 1, \ \forall (a,b) \in S_V^{-\prime\prime} \tag{103}$$

$$v_j^c + v_j^p g_j(a) - \varepsilon \geq u_j^c + u_j^p g_j(a) \geq p_j(g_j(a)), \quad \forall j \in V, a \in A. \tag{104}$$

$$\delta_{jab} \in \{0,1\}, \ \forall j \in F, (a,b) \in S_V^{-\prime\prime}, \ \ \sigma \text{ free.} \tag{105}$$

Proportional case: $v_j(g_j(a)) = v_j^p g_j(a)$ **and** $u_j(g_j(a)) = u_j^p g_j(a)$
The program to solve is the same as in the affine case, but setting $v_j^c = 0$ and $v_j^c = 0$, for all $j \in F$.

Constant case: $v_j(g_j(a)) = v_j^c$ **and** $u_j(g_j(a)) = u_j^c$
The program to solve is the same as in the affine case, but setting $v_j^p = 0$ and $v_j^p = 0$, for all $j \in F$.

5 Illustrative example

In this section, we present an example that illustrates the procedures presented in §3 and §4. This example deals with a multiple criteria sorting problem using the Electre Tri method (see [11] and [14]). Within this framework, we infer the veto thresholds (the value of all other parameters being fixed) from assignment examples using Electre Tri pessimistic assignment rule. Let us note that this problem amounts at inferring a relation S from outranking statements (see §2) as in it holds a is assigned to C_k iff $aSb_k \wedge \neg aSb_{k+1}$.

5.1 Scheme of the experiment

The experiment will proceed as follows. First, we define an Electre Tri model by specifying all preference parameters (including veto thresholds). Applying this model allows us to assign each alternative to a specific category. Next, we consider these results as assignment examples. These examples are used as an input of the inference procedures defined in §3 and §4 where we consider one/several veto-related parameters unknown. This experimental scheme (see Figure 4) enables us to compare the inferred model to the initial one, to analyze the behavior of the inference procedures.

Let us consider a sorting problem in which alternatives have to be assigned to three categories, *good* $\succ$ *medium* $\succ$ *bad*, defined by two profiles, b_1 and b_2 ($B = \{1, 2\}$), taking into account their evaluations on 7 criteria, $g_1, \ldots, g_7$ ($F = \{1, \ldots, 7\}$). The evaluations on each criterion take their values in the interval [0,100]. The Electre Tri model is defined by the values for preference parameters specified in Table 1. Electre Tri pessimistic assignment rule (using these parameters and posing $\lambda = 0.61$) defines the 6 assignment examples given in Table 2.

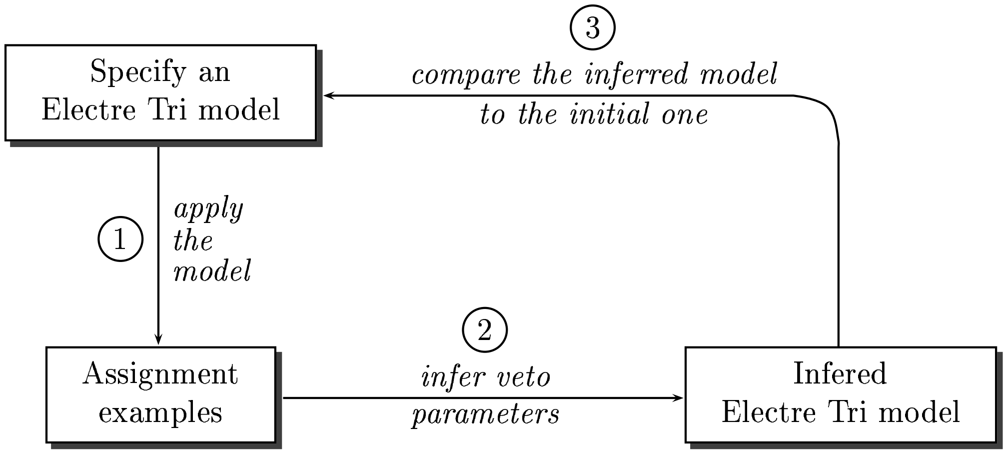

Figure 4: Experimental scheme

	g_1	g_2	g_3	g_4	g_5	g_6	g_7
w_j	0.143	0.143	0.143	0.143	0.143	0.143	0.143
$g_j(b_1)$	33	33	33	33	33	33	33
$q_j(b_1)$	3	3	3	3	3	3	3
$p_j(b_1)$	5	5	5	5	5	5	5
$v_j(b_1)$	33	33	33	∞	∞	∞	∞
$g_j(b_2)$	66	66	66	66	66	66	66
$q_j(b_2)$	3	3	3	3	3	3	3
$p_j(b_2)$	5	5	5	5	5	5	5
$v_j(b_2)$	33	33	33	∞	∞	∞	∞

Table 1: preference parameters

These assignment examples result in the following positive and negative outranking statements: $S^+ = \{(a_1, b_1),\ (a_2, b_2),\ (a_3, b_2),\ (a_6, b_1)\}$ and $S^- = \{(a_1, b_2),\ (a_4, b_1),\ (a_5, b_1),\ (a_6, b_2)\}$.

	g_1	g_2	g_3	g_4	g_5	g_6	g_7	Category
a_1	39	55	59	65	65	65	65	C_2
a_2	45	55	59	65	65	65	65	C_3
a_3	49	55	59	65	65	65	65	C_3
a_4	7	27	29	25	65	65	65	C_1
a_5	6	27	29	65	65	65	65	C_1
a_6	8	6.5	29	65	65	65	65	C_2

Table 2: Set of assignment examples

5.2 inferring one veto using the relation S

We first infer the veto parameters for criterion g_1, *i.e.*, $v_1(b_1)$ and $v_1(b_2)$. Following the method defined in §3.1, the sets S^+ and S^- lead to $v_1(b_1) \in [32.7, 35.4]$ and $v_1(b_2) \in [30.2, 39.7]$. These intervals result from the computations given in Table 3 and Table 4. As expected, the values for $v_1(b_1) = 33$ and $v_1(b_2) = 33$ in the initial model (the one from which the assignment examples were derived) are contained in the computed intervals.

S^+							
a_i	b_h	$g_1(a_i)$	$g_1(b_h)$	$K_1(a_i, b_h)$	Inconsistent ?	$U_1(a_i, b_h)$	$v_1^{min}(b_h)$
a_1	b_1	39	33	1.000	no	1.000	-6.00
a_2	b_2	45	66	0.625	no	0.634	30.24
a_3	b_2	45	66	0.625	no	0.634	23.93
a_6	b_1	8	33	0.6875	no	0.723	32.67

Table 3: Computation of lower bounds for $v_1(b_h)$

S^-							
a_i	b_h	$g_1(a_i)$	$g_1(b_h)$	$K_1(a_i, b_h)$	Redundant ?	$L_1(a_i, b_h)$	$v_1^{max}(b_h)$
a_1	b_2	39	66	0.625	no	0.634	39.70
a_4	b_1	7	33	0.5625	yes	-	-
a_5	b_1	6	33	0.6875	no	0.723	35.44
a_6	b_2	8	66	0	yes	-	-

Table 4: Computation of upper bounds for $v_1(b_h)$

5.3 inferring one veto using the relation S'

In the case of the outranking relation S' (see (13)), a new veto-related parameter u_i is introduced (see (10) and Figure 3). If we pose $u_i = p_j + 0.75(v_j - p_j)$ (v_i being the only variable), then the calculations are very similar to the preceding case and lead to the following intervals: $v_1(b_1) \in [31.3, 33.3]$ and $v_1(b_2) \in [26.2, 34.1]$.

If we consider the threshold u_1 as a parameter to be inferred, and we impose both thresholds to be constant ($i.e.$, $u_1(b_1) = u_1(b_2) = u_1^c$ and $v_1(b_1) = v_1(b_2) = v_1^c$), the mathematical program to be solved is the following:

$$\left\{ \begin{array}{ll} \text{Max } \sigma & \text{s.t.} \\ & 1.56\, u_1^c \; + \; v_1 \; - \; \sigma \; \geq \; -15.38 \\ & 40.66\, u_1^c \; + \; v_1 \; - \; \sigma \; \geq \; 875 \\ & 40.66\, u_1^c \; + \; v_1 \; - \; \sigma \; \geq \; 708.33 \\ & 7.87\, u_1^c \; + \; v_1 \; - \; \sigma \; \geq \; 221.77 \\ & -40.39\, u_1^c \; - \; v_1 \; - \; \sigma \; \geq \; -1117.55 \\ & -7.86\, u_1^c \; - \; v_1 \; - \; \sigma \; \geq \; -239.21 \\ & u_j^c \geq p_j(g_j) + \varepsilon \\ & v_j^c \geq u_j^c \end{array} \right. \qquad (106)$$

In this mathematical program, the four first constraints result from the positive outranking statements in S^+ (a_1Sb_1, a_2Sb_2, a_3Sb_2, a_6Sb_1) and the two following ones result from negative outranking in S^- ($\neg a_4Sb_1$ and $\neg a_6Sb_2$, the two other negative outranking statements being redundant). At the optimum $v_1^c = 25.6$ and $u_1^c = 25.5$.

5.4 inferring one veto using the relation S''

To illustrate the case of the outranking relation S'' we pose $u_i = p_j + 0.75(v_j - p_j)$ (v_i being the only variable). In this case, the calculations are similar to §5.2 and lead to the following intervals: $v_1(b_1) \in [30.7, 33.3]$ and $v_1(b_2) \in [26.2, 34.1]$.

5.5 inferring several veto using the relation S''

Let us suppose that the DM wants to infer all veto simultaneously. In this case, we will suppose that $u_j = p_j + 0.75(v_j - p_j)$ (v_j being the only variables). As the DM considers that only the first three criteria can have a veto effect, we infer v_1, v_2 and v_3 from the assignment examples. In this case, the mathematical program to be solved is given below. At the optimum, it holds: $v_1(b_1) = 32.0$, $v_1(b_2) = 27.5$, $v_2(b_1) = 33.9$, $v_2(b_2) = 14.3$, $v_3(b_1) = 5.0$ and $v_3(b_2) = 22.2$.

$$
\begin{cases}
\text{Max } \sigma \quad \text{s.t.} \\
0.61(1.25+0.75v_1(b_1))+0.39v_1(b_1)-\sigma \ \geq\ -6 \\
0.61(1.25+0.75v_2(b_1))+0.39v_2(b_1)-\sigma \ \geq\ -22 \\
0.61(1.25+0.75v_3(b_1))+0.39v_3(b_1)-\sigma \ \geq\ -26 \\
0.976(1.25+0.75v_1(b_2))+0.024v_1(b_2)-\sigma \ \geq\ 21 \\
0.976(1.25+0.75v_2(b_2))+0.024v_2(b_2)-\sigma \ \geq\ 11 \\
0.976(1.25+0.75v_3(b_2))+0.024v_3(b_2)-\sigma \ \geq\ 17 \\
0.976(1.25+0.75v_1(b_2))+0.024v_1(b_2)-\sigma \ \geq\ 17 \\
0.976(1.25+0.75v_2(b_2))+0.024v_2(b_2)-\sigma \ \geq\ 11 \\
0.976(1.25+0.75v_3(b_2))+0.024v_3(b_2)-\sigma \ \geq\ 7 \\
0.8873(1.25+0.75v_1(b_1))+0.1127v_1(b_1)-\sigma \ \geq\ 25 \\
0.8873(1.25+0.75v_2(b_1))+0.1127v_2(b_1)-\sigma \ \geq\ 26.5 \\
0.8873(1.25+0.75v_3(b_1))+0.1127v_3(b_1)-\sigma \ \geq\ 4 \\
-0.97584(1.25+0.75v_1(b_2))-0.02416v_1(b_2)-M\delta_{1a_1b_2}-\sigma \ \geq\ -M-27 \\
-0.97584(1.25+0.75v_2(b_2))-0.02416v_2(b_2)-M\delta_{2a_1b_2}-\sigma \ \geq\ -M-11 \\
-0.97584(1.25+0.75v_3(b_2))-0.02416v_3(b_2)-M\delta_{3a_1b_2}-\sigma \ \geq\ -M-7 \\
\delta_{1a_1b_2}+\delta_{2a_1b_2}+\delta_{3a_1b_2} \ \geq\ 1 \\
-0.8871(1.25+0.75v_1(b_1))-0.1129v_1(b_1)-M\delta 1a_5b_1-\sigma \ \geq\ -M-27 \\
-0.8871(1.25+0.75v_2(b_1))-0.1129v_2(b_1)-M\delta 2a_5b_1-\sigma \ \geq\ -M-6 \\
-0.8871(1.25+0.75v_3(b_1))-0.1129v_3(b_1)-M\delta 3a_5b_1-\sigma \ \geq\ -M-4 \\
\delta_{1a_5b_1}+\delta_{2a_5b_1}+\delta_{3a_5b_1} \ \geq\ 1 \\
v_j(b_h) \ \geq\ p_j(b_h)+\varepsilon \ j=1,2,3,\ h=1,2 \\
v_j(b_2)-v_j(b_1) \ \geq\ g_j(b_1)-g_j(b_2) \ j=1,2,3 \\
\delta j a_i b_h \in \{0,1\}, j=1,2,3,\ i=1,...,6,\ h=1,2
\end{cases}
$$

Conclusion

This paper presents a contribution to a methodology to infer the parameter values of an Electre model from crisp outranking statements provided by the DM (statements that the model should restore). This is a difficult problem when all parameters have to be inferred simultaneously, hence we limit ourselves to an interactive process of partial inference problems. Partial inference problems are frequently a wise choice as regards the interaction with a DM, since they allow greater control and understanding of the interactive process.

This paper focusses on the inference of the discordance-related parameters (veto thresholds), thus complementing previous work on the inference of the concordance-related parameters (weights, cutting level and limits of categories). In [5] we used two variants of the original valued outranking relation S (denoted S' and S'') to simplify inference problems. In this paper, we show that regarding the inference of the veto thresholds, S and S' originate mathematical programs of similar complexity, while S'' yields simpler versions. Table 5 summarizes the type of mathematical programs corresponding to each situation.

	S	S'	S''
Inference of weights and cutting level	Global (non-convex) programming	Linear programming	Linear programming
Inference of veto for a single criterion	Linear programming	Linear programming	Linear programming
Inference of veto for all the criteria	Separable nonlinear programming	Separable nonlinear programming	0-1 linear programming

Table 5: Mathematical programs corresponding to the different veto inference problems

Illustrative examples provided in §5 show how the inference procedures can be used in a sorting problem based on an Electre Tri model. Further research should be conducted in order to study empirically the behavior of the inference procedures.

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Appendix A: Using 0-1 linar programming to solve the separable programming problems

In the separable program (79)-(82), each function $f_j(a, b, v_j)$ may be approximated by a piecewise linear function. Each function $f_j(a, b, v_j)$ has only one non-fixed parameter if $v_j(g_j(a)) = v_j^c$ (constant case) or $v_j(g_j(a)) = v_j^p g_j(a)$ (proportional case), and has two non-fixed parameters if $v_j(g_j(a)) = v_j^c + v_j^p g_j(a)$ (affine case). Since the feasible region is not convex, these problems may be solved either introducing some integer (0-1) variables or using a special branch and bound technique for dealing with SOS2 (special ordered sets of variables where at most two consecutive ones are non-zero) are used. [12] (chapter 7 and 9) and [13] (chapter 5 and 7) overview separable programming, including how to formulate them and solve them using either integer 0-1 programming or SOS2 branch and bound.

For instance, suppose we are inferring veto thresholds as constants and using the v_j^c-axis points $v_{j1}, ..., v_{jz}$ (such that $p_j(g_j) = v_{j1} \leq ... \leq v_{jz}$) for the piecewise linear approximations of $f_j(a, b, v_j)$, such that:

$$\begin{aligned} v_j^c &= v_{j1}\lambda_{j1} + ... + v_{jz}\lambda_{jz}, && (107)\\ f_j(a, b, v_j) &\simeq f_j(a, b, v_{j1})\lambda_{j1} + ... + f_j(a, b, v_{jz})\lambda_{jz}, && (108)\\ \lambda_{j1} + ... + \lambda_{jz} &= 1, \quad \lambda_{j1}, ..., \lambda_{jz} \geq 0 && (109) \end{aligned}$$

Then, we would solve the following mathematical program, where c_{abji} denotes the constant value $f_j(a, b, v_{ji})$ and the variables are σ and λ_{ji} ($j \in V$; $i \in \{1, ..., z\}$):

$$\begin{aligned} &\max \quad \sigma \quad \text{s.t.} && (110)\\ &\textstyle\sum_{j \in V_{ab}} \sum_{i=1}^{z} c_{abji}\lambda_{ji} \geq \log LB_V(a, b) + \sigma, \quad \forall (a, b) \in S_V^+ && (111)\\ &\textstyle\sum_{j \in V_{ab}} \sum_{i=1}^{z} c_{abji}\lambda_{ji} \leq \log UB_V(a, b) - \sigma, \quad \forall (a, b) \in S_V^- && (112)\\ &\textstyle\sum_{i=1}^{z} v_{ji}\lambda_{ji} \geq p_j^c + \varepsilon, \quad \forall j \in V, && (113)\\ &\lambda_{j1} + ... + \lambda_{jz} = 1, \quad \lambda_{j1}, ..., \lambda_{jz} \geq 0, \quad \forall j \in V, && (114)\\ &\lambda_{j1}, ..., \lambda_{jz} \text{ is a SOS2, } i.e., \text{ at most two consecutive values can be positive.} && (115) \end{aligned}$$

The last constraint may be dealt with using either branch and bound (see [13], §7.3) or integer 0-1 programming (a binary variable must be associated with each λ_{ji}). The solution of this mathematical program is an approximation to the global optimum of (79)-(82), which can be made as good as needed by using a finer grid, i.e. points $v_{j1}, ..., v_{jz}$ closer to each other. A strategy that avoids using a large z number of points is to solve the problem with a coarse grid, and then solve it again with a finer grid of points around the optimum values that were found.

Here, we have developed the constant case where $v_j(g_j(a)) = v_j^c$. The proportional case is similar, but the affine case is more complex since each $f_j(a, b, v_j)$ depends on two variables: v_j^c and v_j^p. Hence, the piecewise linear approximation of these functions would require a bidimensional grid using z^2 points per criterion, which would increase the number of variables.