USING ASSIGNMENT EXAMPLES TO INFER CATEGORY LIMITS FOR THE ELECTRE TRI METHOD

An Ngo The and Vincent Mousseau
Lamsade, University of Paris Dauphine
Place du Mal De Lattre de Tassigny
75 775 Paris cedex 16 - FRANCE

Abstract :

Given a finite set of alternatives, the sorting (or assignment) problem consists in the assignment of each alternative to one of the pre-defined categories. In this paper, we are interested in multiple criteria sorting problems and, more precisely, in the existing method ELECTRE TRI. This method requires the elicitation of preferential parameters (importance coefficients, thresholds, profiles,...) in order to construct the decision-maker's (DM) preference model.

A direct elicitation of these parameters being sometimes difficult, Mousseau & Slowinksi proposed an interactive aggregation-disaggregation approach that infer ELECTRE TRI parameters indirectly from holistic information, i.e., assignment examples. In this approach, the determination of ELECTRE TRI parameters that best restore the assignment examples is formulated through a non-linear optimization program. Also in this direction, Mousseau *et al.* considered the subproblem of the determination of the importance coefficients only (the thresholds and category limits being fixed). This subproblem leads to solve a linear program (rather that non-linear in the global inference model).

We pursue the idea of partial inference model by considering the complementary subproblem which determines the category limits (the importance coefficients being fixed). With some simplification, it also leads to solve a linear program. Together with the result of Mousseau *et al.*, we have a couple of complementary models which can be combined in an interactive approach inferring the parameters of an ELECTRE TRI model from assignment examples. In each interaction, the DM can revise his/her assignment examples, to give additional information and to choose which parameters to fix before the optimization phase restarts.

Keywords : Assignment problem, ELECTRE TRI, Category limit elicitation, Inference procedure.

1 Introduction

According to (Roy, 1985), real world decision problems using multiple criteria decision aid can be classified in three basic problematics : choice, sorting and ranking (see also (Bana e Costa, 1996)). The sorting problematic consists in formulating the decision problem in terms of the assignment of a set of alternatives $A = \{a_1, a_2, ..., a_N\}$ to one of the pre-defined categories $C_1, C_2, ..., C_p, C_{p+1}$. The assignment of an alternative a to the appropriate category relies on the intrinsic value of a, and not on the comparison of a with other alternatives.

In this paper, we are interested in the multiple criteria sorting problematic and, more precisely, in the ELECTRE TRI method (see (Yu, 1992), (Mousseau *et al.*, 2000) and (Roy & Bouyssou, 1993)). The implementation of this method requires the determination of several parameters such as : limit profiles between consecutive categories, importance coefficients of criteria, discrimination thresholds, ... The set of these parameters (that we will call an ELECTRE TRI model in this paper) is used to construct a preference model that the Decision Maker (DM) accept as a working hypothesis. In many situations, it is difficult for the DM to determine these values ; a direct evaluation of these parameters requires an important cognitive effort.

To overcome this difficulty, (Mousseau & Slowinski, 1998) proposed an indirect approach in order to infer these parameters from assignment examples through a certain form of regression on assignment examples. This approach corresponds to an aggregation-disaggregation methodology (see (Jacquet-Lagrèze & Siskos, 1982), (Jacquet-Lagrèze & Siskos, 2001)) to elicit preferences by interaction on holistic preferences.

(Mousseau & Slowinski, 1998) propose a global inference model which infers all ELECTRE TRI parameters simultaneously starting from assignment examples. In this approach, the determination of the parameters' values that best fit the assignment examples results from the resolution of a non-linear mathematical program. This optimization procedure is integrated in an interactive tool that enable the DM to react on the set of obtained parameters and to get insights on his/her preferences. In the continuation of this idea, (Mousseau *et al.*, 2001) proposed a partial inference approach consisting in the introduction of a subproblem that infers the importance coefficients and the cutting level only. In this case, the mathematical program to be solved becomes linear. (Dias & Mousseau, 2002) considers the inference of veto related parameters only.

Our work also account for the idea of inferring a subset of ELECTRE TRI parameters from assignment examples. We consider the problem of determining the definition of categories (limit profiles and discrimination thresholds), the importance coefficients being fixed. Subject to some restrictions, the corresponding mathematical program to be solved is linear. Following (Mousseau *et al.*, 2001) and (Mousseau & Slowinski, 1998), we aim at enriching the approaches to determine parameters of an ELECTRE TRI model (see Figure 1).

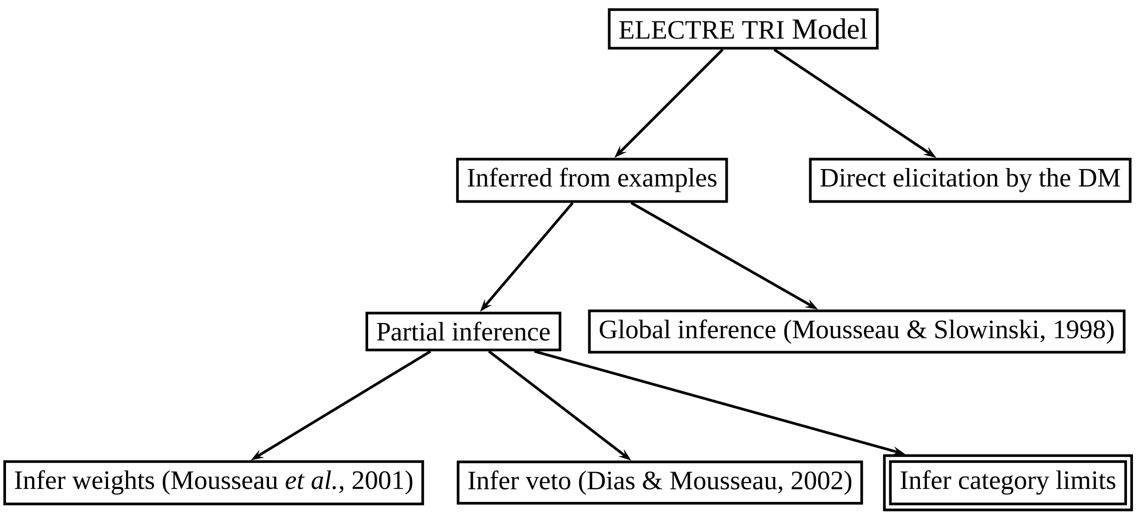

FIG. 1 – Different approaches to determine ELECTRE TRI parameters

Figure 1 reveals the fact that the use of our inference model can be considered in a broader scheme where all ELECTRE TRI parameters are to be inferred. In such a situation, the partial inference models (the inference of importance coefficients (Mousseau *et al.*, 2001), inference of veto thresholds (Dias & Mousseau, 2002) and inference of category limits (this paper)) can be used iteratively. At each iteration, the DM can revise his/her assignment examples, give additional information and choose which parameters to fix before the optimization phase restarts.

This paper presents a new inference procedure that determine the category limits from assignment examples. This procedure is validated by numerical results obtained in a laboratory experiment aiming at testing the operational usefulness of the category limits' inference procedure in an interactive process. The paper is organized as follows. In section 2, we recall briefly the ELECTRE TRI method and the general approach used by the inference tool. In the next two sections, we present the two phases of the category limit inference model. In section 5, we consider some variations of the model when more information is available or when a strong consistency is required. Section 6 is dedicated to the experimental design and the empirical results. A final section groups conclusions.

2 The inference procedure

2.1 Brief presentation of the ELECTRE TRI method

We give here a very brief overview of the ELECTRE TRI method and define some notations that will be used. A complete description can be found in (Roy & Bouyssou, 1993). The corresponding software is described in (Mousseau *et al.*, 2000)

ELECTRE TRI is a multiple criteria sorting method used to assign alternatives to predefined ordered categories. The assignment of an alternative a results from the comparison of a with the profiles defining the limits of the categories. Let A denote the set of alternatives to be assigned and $A^* \subset A, A^* = \{a_1, a_2, ...a_n\}$ denote a subset of alternatives that the DM intuitively assigns to a category or a range of categories (A^* contains the assignment examples given by the DM) and let $K = \{1, 2, ...n\}$ be the set of indices of the alternatives from A^*. Let F denote the set of the indices of the criteria $g_1, g_2, ..., g_m$ (F=$\{1, 2, ..., m\}$), k_j the importance coefficient of the criterion g_j, B the set of indices of the profiles defining $p+1$ categories (B=$\{1, 2, ..., p\}$), b_h being the upper limit of category C_h and the lower limit of category C_{h+1}, $h = 1, 2, ..., p$. Each profile b_h is characterized by its performances $g_j(b_h)$ and its thresholds $p_j(b_h)$ (preference thresholds), $q_j(b_h)$ (indifference thresholds) and $v_j(b_h)$ (veto thresholds). In what follows, we will assume, without any loss of generality, that preferences increase with the value on each criterion and that $\sum_{j \in F} k_j = 1$.

Further on, we use $a \to C_h$ to denote that the alternative a is assigned to the category C_h, when necessary, $a \to_{DM} C_h$ is used to highlight the fact that the assignment is stated by the DM.

ELECTRE TRI builds a fuzzy outranking relation S whose meaning is "at least as good as". Preferences restricted to the significance axis of each criterion are defined through pseudo-criteria (see (Roy & Vincke, 1984) for details on this double thresholds preference representation). Beside the intra-criterion preferential information, represented by the indifference and preference thresholds, $q_j(b_h)$ and $p_j(b_h)$, the construction of S also makes use of two types of inter-criterion preferential information :

– the set of weight-importance coefficients ($\{k_j, j \in F\}$) is used in the concordance test when computing the relative importance of the coalitions of criteria being in favor of the assertion

aSb_h (and b_hSa) ;

– the set of veto thresholds ($\{v_j(b_h), j \in F, h \in B\}$) is used in the discordance test ; $v_j(b_h)$ represents the smallest difference $g_j(b_h) - g_j(a)$ incompatible with the assertion aSb_h (and b_hSa).

As the assignment of alternatives to categories does not result directly from the relation S, an exploitation phase is necessary ; it requires the relation S to be "defuzzyfied" using a so-called λ-cut : the assertion aSb_h (b_hSa respectively) is considered to be valid if the credibility index of the fuzzy outranking relation is greater than a "cutting level" λ such that $\lambda \in (0.5, 1]$. This λ-cut determines the preference situation between a and b_h.

Two assignment procedures (optimistic and pessimistic) are available, their role being to analyze the way in which an alternative a compares to the profiles so as to determine the category to which a should be assigned. The result of these two assignment procedures differs when the alternative a is incomparable with at least one profile b_h.

2.2 Scheme of the general inference procedure

The general scheme of the inference procedure (see Figure 2) is to find an ELECTRE TRI model as compatible as possible with the assignment examples (A^*) given by the DM. The compatibility between the ELECTRE TRI model and the assignment examples is understood as an ability of the ELECTRE TRI method using this model to reassign the alternatives from A^* in the same way as the DM did.

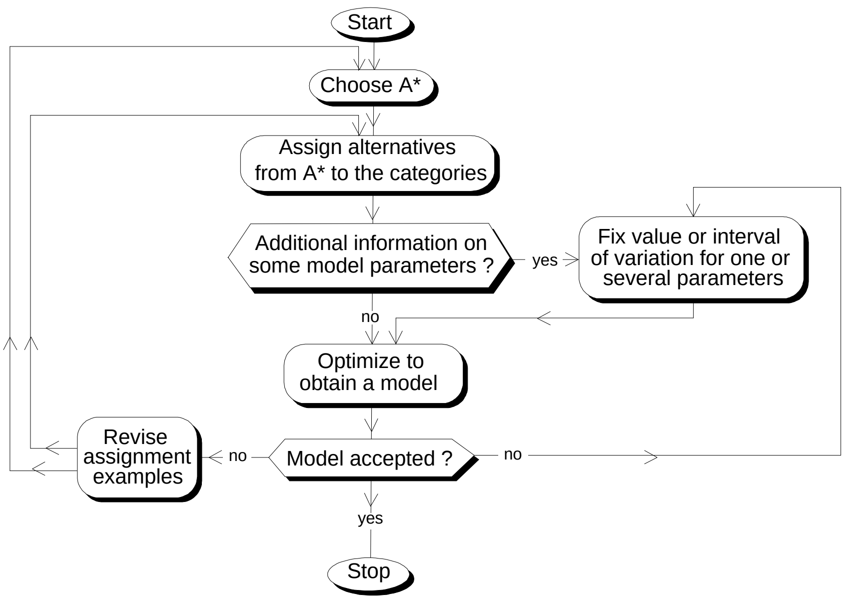

FIG. 2 – General scheme of the inference procedure

In order to minimize the differences between the assignments made by ELECTRE TRI and the assignments made by the DM, an optimization procedure is used. The DM can tune up the model in the course of an interactive procedure. He/she may either revise the assignment examples or fix values (or intervals of variation) for some model parameters. The DM may modify (A^*) as well as introduce some more constraints concerning the profiles.

When the model is not perfectly compatible with the assignment examples, the procedure should be able to detect all "hard cases", i.e., the alternatives for which the assignment computed by the model strongly differs from the DM's assignment. The DM could then be asked to reconsider his/her judgment. This general scheme is applicable for the global inference ((Mousseau & Slowinski, 1998)) as well as partial inference procedures ((Mousseau *et al.*, 2001) and this paper). For more discussion on the procedure and its interest, see (Mousseau & Slowinski, 1998) and (Mousseau *et al.*, 2001).

Almost all disaggregation procedures (including ours) are grounded on mathematical programs in which various objective functions can be considered as equally acceptable to infer the values of the preference parameters. Moreover, there might be multiple optimal solutions to the program. It is possible to deal with such difficulty by performing (near)post-optimal analysis. This enables to identify several solutions to be presented to the decision maker, rather than presenting a single solution.

In our work, we propose to deal with this issue in a slightly different and complementary way. We consider the inference procedure as a tool to be integrated in an interactive process in which the decision maker should react to the output of the inference procedure by stating some additional constraints on the parameter values, modifying the assignment examples, ... In such an interactive process, the decision maker will get insights on the possible values for the preference parameters in relation with the assignment examples the model should restitute.

2.3 Formulation of the problem

In what follows, we will confine our analysis to the case were the pessimistic assignment procedure is used and no veto phenomenon occurs ($v_j(b_h) = \infty, \forall j \in F, \forall h \in B$). As the importance coefficients are fixed, the inferred parameters are the category limits (i.e. the limit profiles : $g_j(b_h)$ as well as the thresholds $q_j(b_h)$ and $p_j(b_h), j \in F, h \in B$) and the cutting level λ.

It is difficult to infer the ELECTRE TRI category limits directly, therefore, the computation is decomposed in two phases :

– Phase 1 : partial concordance indices $c_j(a, b_h)$, $c_j(b_h, a)$, $j \in F$, $h \in B$ are determined by means of a linear program.
– Phase 2 : $g_j(b_h), p_j(b_h), q_j(b_h), \forall j \in F, \forall h \in B$, will then be reconstructed from the indices computed in phase 1.

3 Phase 1 : determination of partial concordance indices

3.1 Notations and hypothesis

In the ELECTRE TRI method, the construction of the outranking relation S is based on the aggregation-disaggregation paradigm which is materialized by the partial concordance indices, $c_j(a_k, b_h)$, $c_j(b_h, a_k)$, $j \in F$, $k \in K, h \in B$, and then by the global concordance indices $\sigma(a_k, b_h)$, $\sigma(b_h, a_k)$, $k \in K, h \in B$. We recall that the hypothesis of no veto is assumed in our approach. The following observations are straightforward from ELECTRE TRI :

$$\left.\begin{array}{l}\sigma(a_k, b_h) = c(a_k, b_h) = \sum_{j \in F} k_j c_j(a_k, b_h) \\ \sigma(b_h, a_k) = c(b_h, a_k) = \sum_{j \in F} k_j c_j(b_h, a_k)\end{array}\right\} \forall k \in K, \forall h \in B \qquad (1)$$

$$c_j(a_k,b_h), c_j(b_h,a_k) \in [0,1], \forall j \in F, \forall k \in K, \forall h \in B \tag{2}$$

When g_j is a quasi-criterion (i.e. $p_j = q_j$), (2) becomes

$$c_j(a_k,b_h), c_j(b_h,a_k) \in \{0,1\}, \forall j \in F, \forall k \in K, \forall h \in B \tag{3}$$

$$\left.\begin{array}{l} g_j(a_k) < g_j(a_l) \Rightarrow \left\{ \begin{array}{l} c_j(a_k,b_h) \leq c_j(a_l,b_h) \\ c_j(b_h,a_k) \geq c_j(b_h,a_l) \end{array}\right. \\ g_j(a_k) = g_j(a_l) \Rightarrow \left\{ \begin{array}{l} c_j(a_k,b_h) = c_j(a_l,b_h) \\ c_j(b_h,a_k) = c_j(b_h,a_l) \end{array}\right. \end{array}\right\} \forall j \in F, \forall k,l \in K, \forall h \in B \tag{4}$$

$$\left.\begin{array}{l} c_j(a_k,b_{h+1}) \leq c_j(a_k,b_h) \\ c_j(b_h,a_k) \leq c_j(b_{h+1},a_k) \end{array}\right\} \forall j \in F, \forall k \in K, h = 1,2,...,p-1 \tag{5}$$

$$\left.\begin{array}{l} c(a_k,b_{h+1}) = \sum_{j\in F} k_j c_j(a_k,b_{h+1}) \leq \sum_{j\in F} k_j c_j(a_k,b_h) = c(a_k,b_h) \\ c(b_h,a_k) = \sum_{j\in F} k_j c_j(b_h,a_k) \leq \sum_{j\in F} k_j c_j(b_{h+1},a_k) = c(b_{h+1},a_k) \end{array}\right\} \forall k \in K, h = 1,2,...,p-1 \tag{6}$$

The ELECTRE TRI pessimistic assignment rule proceeds as follows :

a) compare a successively to b_i, for i=p,p-1, ..., 0,
b) b_h being the first profile such that aSb_h,
assign a to category C_{h+1} ($a \rightarrow C_{h+1}$).

Hence, the pessimistic assignment rule assigns the alternative a_k to the category C_{h_k} ($a_k \rightarrow C_{h_k}$) iff

$$\left\{ \begin{array}{l} c(a_k,b_{h_k-1}) \geq \lambda \\ c(a_k,b_{h_k}) < \lambda \end{array}\right. \tag{7}$$

3.2 Results justifying the inference model

Definition 3.1. *For each criterion $g_j, j \in F$ and each profile $b_h, h \in B$, the function $\phi_{jh}(x)$ is called the category limit characterization function.*

$$\phi_{jh}(x) = \left\{ \begin{array}{ll} 0 & if\ \ x \leq g_j(b_h) - p_j(b_h) \\ \frac{x-g_j(b_h)+p_j(b_h)}{p_j(b_h)-q_j(b_h)} & if\ \ g_j(b_h)-p_j(b_h) < x < g_j(b_h)-q_j(b_h) \\ 1 & if\ \ g_j(b_h)-q_j(b_h) \leq x \leq g_j(b_h)+q_j(b_h) \\ \frac{g_j(b_h)+p_j(b_h)-x}{p_j(b_h)-q_j(b_h)} & if\ \ g_j(b_h)+q_j(b_h) < x < g_j(b_h)+p_j(b_h) \\ 0 & otherwise \end{array}\right. \tag{8}$$

It is obvious from the definition 3.1 that :

$$\phi_{jh}(a_k) = \min\{c_j(a_k,b_h), c_j(b_h,a_k)\}^{\,1} \tag{9}$$

This function plays a central role in our approach. As illustrated in Figure 3, it represents a fuzzy membership of the relation $(a_k I_j b_h)$. It also represents all the partial concordance indices that can be used to reconstruct the category limits. All proofs are given in Appendix.

[1] Formally, we should write $\phi_{jh}(g_j(a_k))$ instead of $\phi_{jh}(a_k)$.

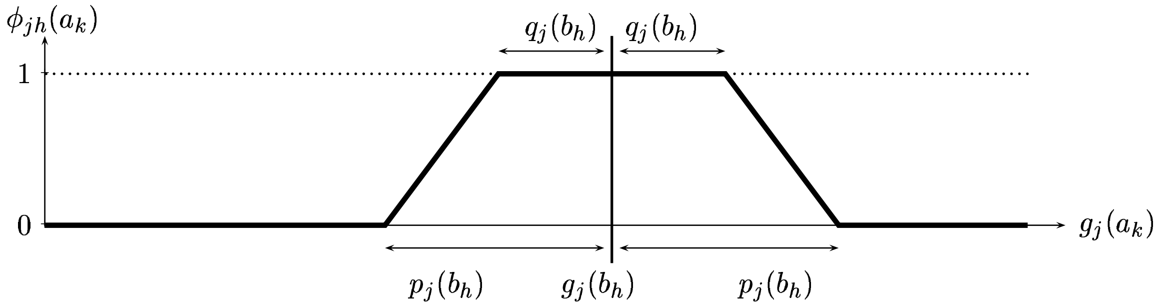

FIG. 3 – Partial concordance indices and category limits characterized by $\phi_{jh}(a_k)$

Proposition 3.1. *The category limit characterization function $\phi_{jh}(a_k)$ is symmetrical through the vertical line $x = g_j(b_h)$ (symmetry condition)*

To ensure the consistency of the categories, we base on the condition "No alternative should be indifferent to more than one profiles" (see (Mousseau *et al.*, 2000) and (Yu, 1992)). We have to express this condition by constraints concerning partial concordance indices in order to introduce it into our program. We have the following results :

Proposition 3.2. *When using the ELECTRE TRI pessimistic procedure, if $a_k \to C_{h_k}$ then :*
(i) - $\forall h \geq h_k, \neg a_k I b_h$
(ii) - $\forall h < h_k - 1, b_h S a_k \Rightarrow b_h I a_k$
i.e., The indifference between a_k and b_h appears only for $h \leq h_k - 1$ when $b_h S a_k$ takes place.

Proposition 3.3. *If $\exists h_0$ s.t. $\neg b_{h_0} S a_k$ then $\forall h \leq h_0, \neg b_h S a_k$*

Proposition 3.4. *Conditions (i) and (ii) are equivalent using the pessimistic assignment procedure :*
(i) No alternative in A^* is indifferent to more than one profiles.
(ii) $\forall a_k \in A^*, a_k \to C_{h_k}$ then $\neg b_{h_k - 2} S a_k$

Remark : This condition ensures only the consistency in the set A^*, not in A. It is weaker than the following condition :

$$g_j(b_{h+1}) \geq g_j(b_h) + p_j(b_h) + p_j(b_{h+1}) \tag{10}$$

which is too strong (sufficient but not necessary) to ensure the consistency in A. Furthermore, this condition is impossible to express using the partial concordance indices.

3.3 Variables of the problem

In ELECTRE TRI pessimistic assignment procedure, an alternative a_k is assigned to category C_h ($a_k \to C_h$) iff $c(a_k, b_{h-1}) \geq \lambda$ and $c(a_k, b_h) < \lambda$. To ensure the consistency of the profiles, we need the condition $c(b_{h_k-2}, a_k) < \lambda$. Let us suppose that the DM has assigned the alternative $a_k \in A^*$ to category C_{h_k} ($a_k \to_{DM} C_{h_k}$). Let us define the slack variables x_k, y_k and z_k unrestricted in sign such that $c(a_k, b_{h_k-1}) - x_k = \lambda, c(a_k, b_{h_k}) + y_k = \lambda$ and $c(b_{h_k-2}, a_k) + z_k = \lambda$.

These slack variables are used only as an aid to construct the objective function. Then, we can eliminate them easily by using $\beta = \min_{a_k \in A^*}\{x_k, y_k, z_k\}$. So they are not introduced explicitly in the program. Therefore, the optimization problem will include the following variables $(2mnp + 2)$:

$c_j(a_k, b_h), c_j(b_h, a_k), \forall j \in F, \forall k \in K, \forall h \in B$	partial concordance indices ($2mnp$)
λ	cutting level (1)
β	(1)

For technical reasons (see 3.6), we introduce two fictitious alternatives in A^* : a^* (ideal alternative) and a_* (anti-ideal) defined as follows

$$\left.\begin{array}{l} a_* : \forall j \in F, g_j(a_*) < \min_{a_k \in A^*}\{g_j(a_k)\} \\ a^* : \forall j \in F, g_j(a^*) > \max_{a_k \in A^*}\{g_j(a_k)\} \end{array}\right\} \forall j \in F, \forall h \in B \quad (11)$$

a^* is obviously assigned to the best category (C_{p+1}) and an a_* to the worst category (C_1). These two alternatives ensure that there is always a transition between 0 and 1 in the set of the partial concordance indices within each criterion. Indeed, from the definition of a^* and a_*, it is obvious that :

$$\left.\begin{array}{l} c_j(a_*, b_h) = 0 \text{ and } c_j(b_h, a_*) = 1 \\ c_j(a^*, b_h) = 1 \text{ and } c_j(b_h, a^*) = 0 \end{array}\right\} \forall j \in F, \forall h \in B \quad (12)$$

3.4 Accuracy criterion

If the values of the slack variables x_k, y_k and z_k are all positive, then ELECTRE TRI pessimistic assignment procedure will assign alternative a_k to the "correct" category and the consistency of categories is respected. If, however, x_k or y_k is negative, the ELECTRE TRI pessimistic assignment procedure will assign alternative a_k to a "wrong" category. If z_k is negative, the consistency of categories is not respected. The lower the minimum of these values, the less adapted is the ELECTRE TRI model to give an account of the assignment of a_k made by the DM. Moreover, if x_k, y_k and z_k are all positive, then a_k is assigned consistently with the DM's statement, and the consistency is respected for all $\lambda' \in [\lambda - \min\{y_k, z_k\}, \lambda + x_k]$.

Let us consider now the set of alternatives $A^* = \{a_1, a_2, ..., a_k, ..., a_n\}$ and suppose that the DM has assigned the alternative a_k to the category C_{h_k}, $\forall a_k \in A^*$. The ELECTRE TRI model will be consistent with the DM's assignments iff $x_k \geq 0, y_k \geq 0$ and $z_k \geq 0$, $\forall a_k \in A^*$. Consistently with the preceding argument, an accuracy criterion to be maximized can be defined as : $\beta = \min_{a_k \in A^*}\{x_k, y_k, z_k\}$.

3.5 Optimization problem to be solved

In order to avoid strict inequalities, we introduce an arbitrary small positive constant ε. From the results provided in sections 3.1 and 3.2, we obtain the following constraints :

Bounds of variables

$0.5 \leq \lambda \leq 1$

$0 \leq c_j(a_k, b_h) \leq 1, \forall j \in F, \forall k \in K, h \in B$

$0 \leq c_j(b_h, a_k) \leq 1, \forall j \in F, \forall k \in K, h \in B$

Other constraints

$$\max\{c_j(a_k, b_h), c_j(a_h, b_k)\} = 1, \forall j \in F, \forall k \in K, h \in B \qquad (mnp)$$

$$\left.\begin{array}{l} c_j(a_k, b_h) \leq c_j(a_l, b_h) \text{ if } g_j(a_k) < g_j(a_l) \\ c_j(a_k, b_h) = c_j(a_l, b_h) \text{ if } g_j(a_k) = g_j(a_l) \end{array}\right\} \forall j \in F, \forall k, l \in K, h \in B \qquad (m(n-1)p)$$

$$\left.\begin{array}{l} c_j(b_h, a_k) \geq c_j(b_h, a_l) \text{ if } g_j(a_k) < g_j(a_l) \\ c_j(b_h, a_k) = c_j(b_h, a_l) \text{ if } g_j(a_k) = g_j(a_l) \end{array}\right\} \forall j \in F, \forall k, l \in K, h \in B \qquad (m(n-1)p)$$

$$c_j(a_k, b_{h+1}) \leq c_j(a_k, b_h), \forall j \in F, \forall k \in K, h = 1, 2, ..., p-1 \qquad (mn(p-1))$$

$$c_j(b_h, a_k) \leq c_j(b_{h+1}, a_k), \forall j \in F, \forall k \in K, h = 1, 2, ..., p-1 \qquad (mn(p-1))$$

$$\beta \leq \textstyle\sum_{j \in F} k_j c_j(a_k, b_{h_k-1}) - \lambda, \forall k \in K \qquad (n-1)$$

$$\beta + \epsilon \leq \lambda - \textstyle\sum_{j \in F} k_j c_j(a_k, b_{h_k}), \forall k \in K \qquad (n)$$

$$\beta + \epsilon \leq \lambda - \textstyle\sum_{j \in F} k_j c_j(b_{h_k-2}, a_k), \forall k \in K \qquad (n-2)$$

These constraints suffer from two limitations :

- they do not take into account the symmetric condition (see proposition 3.1),
- condition $\max\{c_j(a_k, b_h), c_j(a_h, b_k)\} = 1$ is non-linear.

Here, (see (Naux, 1996)), by observing that most of the values of $c_j(a_k, b_h), c_j(b_h, a_k)$ are 0 or 1, we accept the hypothesis of integrity to obtain a rough preliminary solution of the problem. Under this hypothesis, we replace $c_j(a_k, b_h), c_j(b_h, a_k) \in [0, 1]$ by $c_j(a_k, b_h), c_j(b_h, a_k) \in \{0, 1\}$. The constraint $max\{c_j(a_k, b_h), c_j(b_h, a_k)\} = 1$ becomes $c_j(a_k, b_h) + c_j(b_h, a_k) \geq 1$. This hypothesis helps to overcome these two limitations as the program turns out to be a linear one and the verification of the condition of symmetric can be postponed to the next phase which determines the profiles and the thresholds. As a consequence, the optimization problem P_1 to be solved is the following :

$$\max \beta \qquad . \quad (13)$$

$$\text{s.t.} \quad \beta \leq \sum_{j \in F} k_j c_j(a_k, b_{h_k-1}) - \lambda, \forall k \in K \qquad (14)$$

$$\beta + \epsilon \leq \lambda - \sum_{j \in F} k_j c_j(a_k, b_{h_k}), \forall k \in K \qquad (15)$$

$$\beta + \epsilon \leq \lambda - \sum_{j \in F} k_j c_j(b_{h_k-2}, a_k), \forall k \in K \qquad (16)$$

$$1 \leq c_j(a_k, b_h) + c_j(b_h, a_k), \forall j \in F, \forall k \in K, h \in B \qquad (17)$$

$$c_j(a_k, b_{h+1}) \leq c_j(a_k, b_h), \forall j \in F, \forall k \in K, h = 1, 2, ..., p-1 \qquad (18)$$

$$c_j(b_{h+1}, a_k) \geq c_j(b_h, a_k), \forall j \in F, \forall k \in K, h = 1, 2, ..., p-1 \qquad (19)$$

$$c_j(a_k, b_h) \leq c_j(a_l, b_h), \forall j \in F, \forall k, l \in K, h \in B, \text{ if } g_j(a_k) < g_j(a_l) \qquad (20)$$

$$c_j(a_k, b_h) = c_j(a_l, b_h), \forall j \in F, \forall k, l \in K, h \in B, \text{ if } g_j(a_k) = g_j(a_l) \qquad (21)$$

$$c_j(b_h, a_k) \geq c_j(b_h, a_l), \forall j \in F, \forall k, l \in K, h \in B, \text{ if } g_j(a_k) < g_j(a_l) \qquad (22)$$

$$c_j(b_h, a_k) = c_j(b_h, a_l), \forall j \in F, \forall k, l \in K, h \in B, \text{ if } g_j(a_k) = g_j(a_l) \qquad (23)$$

$$0.5 \leq \lambda \leq 1 \qquad (24)$$

$$c_j(a_k, b_h) \in \{0, 1\}, \forall j \in F, \forall k \in K, h \in B \qquad (25)$$

$$c_j(b_h, a_k) \in \{0, 1\}, \forall j \in F, \forall k \in K, h \in B \qquad (26)$$

The program obtained is an MIP (Mixed Integer Program) which contains $2mnp+2$ variables and $4n+3mp+2$ constraints. As we mentioned previously, the slack variables x_k, y_k, z_k can be eliminated from the problem formulation since they are defined by the constraints (14), (15) and (16).

3.6 Refinement of the result

We introduced the integrity hypothesis to simplify the problem. It is important to check whether it is possible to improve the result by relaxing the integrity condition for some values $c_j(a_k, b_h)$ or $c_j(b_h, a_k)$.

We solve the same problem, except the integrity condition $c_j(a_k, b_h), c_j(b_h, a_k) \in \{0,1\}$ is replaced by the initial condition $c_j(a_k, b_h), c_j(b_h, a_k) \in [0,1]$ for values to relax, other values become constants (already determined by the program $(P1)$). It is quite natural to consider the points to relax in the neighborhood of the transition between 0 and 1. For each criterion g_j and each profile b_h, we define the following values :

$$\begin{cases} z_{2jh} = \max\{g_j(a_k|a_k \in A^*, c_j(a_k, b_h) = 0\} \\ z_{1jh} = \max\{g_j(a_k|a_k \in A^*, g_j(a_k) < z_{2jh}\} \text{ or } -\infty \text{ if the set is empty} \\ z_{3jh} = \min\{g_j(a_k|a_k \in A^*, c_j(a_k, b_h) = 1\} \\ t_{2jh} = \min\{g_j(a_k|a_k \in A^*, c_j(b_h, a_k) = 0\} \\ z_{1jh} = \max\{g_j(a_k|a_k \in A^*, c_j(b_h, a_k) = 1\} \\ z_{3jh} = \min\{g_j(a_k|a_k \in A^*, g_j(a_k) > t_{2jh}\} \text{ or } +\infty \text{ if the set is empty.} \end{cases} \quad (27)$$

With the insertion of the ideal alternative a^* and the anti-ideal alternative a_*, the existence of the values $z_{2jh}, z_{3jh}, t_{1jh}, t_{2jh}$ is ensured (see Figure 4).

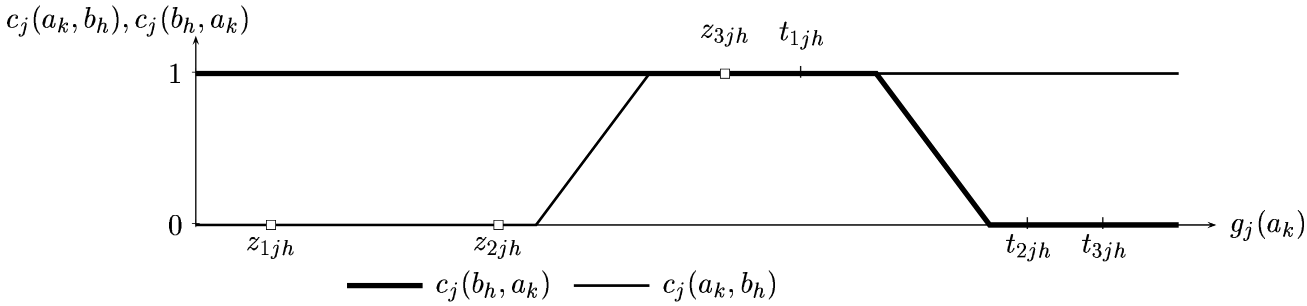

FIG. 4 – Values around the transition between 0 and 1

The values z_{2jh} and t_{2jh} are to be relaxed according to the following rules :
- if $z_{1jh} > -\infty$ and $z_{3jh} < t_{1jh}$ then z_{2jh} will be relaxed.
- if $t_{3jh} < +\infty$ and $z_{3jh} < t_{1jh}$ then t_{2jh} will be relaxed.

This choice ensures the symmetric condition and is rather technically complicated, see (Ngo The, 1998) for more details.

The indices $c_j(a_k, b_h), c_j(b_h, a_k)$ corresponding to relaxed values z_{2jh} and t_{2jh} are variables to determine of the new program, and will be denoted as cz_{jh}, ct_{jh}. Other values for $c_j(a_k, b_h), c_j(b_h, a_k)$ become constants (0 or 1) determined in the previous program. However, this refinement fails to show any improvement in the experiments realized.

4 Phase 2 : determination of category limits from partial concordance indices

Once all partial concordance indices $c_j(a_k, b_h)$ and $c_j(b_h, a_k)$ are determined, so are x_k, y_k, z_k as well as β. All values of $g_j(b_h), p_j(b_h), q_j(b_h)$ satisfying the following conditions can be accepted.

$c_j(a_k, b_h) = 0 \Rightarrow g_j(b_h) - p_j(b_h) \geq g_j(a_k)$	(D1)
$c_j(a_k, b_h) = 1 \Rightarrow g_j(b_h) - q_j(b_h) \leq g_j(a_k)$	(D2)
Otherwise, $c_j(a_k, b_h) = \frac{g_j(a_k)+p_j(b_h)-g_j(b_h)}{p_j(b_h)-q_j(b_h)}$	(D3)
$c_j(b_h, a_k) = 0 \Rightarrow g_j(b_h) + p_j(b_h) \leq g_j(a_k)$	(D4)
$c_j(b_h, a_k) = 1 \Rightarrow g_j(b_h) + q_j(b_h) \geq g_j(a_k)$	(D5)
Otherwise, $c_j(b_h, a_k) = \frac{g_j(b_h)+p_j(b_h)-g_j(a_k)}{p_j(b_h)-q_j(b_h)}$	(D6)
$g_j(b_{h+1}) \geq g_j(b_h)$	(D7)
$p_j(b_h) \geq q_j(b_h)$	(D8)
$q_j(b_h) \geq 0$	(D9).

Under the integrity hypothesis, the conditions (D3) and (D6) are not considered, there exists a certain degree of freedom in the determination of $g_j(b_h), p_j(b_h), q_j(b_h)$. To determine these values, intuitively, we consider an ideal solution as one that haves the following two characteristics :
- the profile characterization function $\phi_{jh}(a_k)$ has a reasonable form which depends on the ratio $\frac{q_j(b_h)}{p_j(b_h)}$.
- for each criterion g_j, the profiles $g_j(b_h), h \in K$ are well "distributed" along the scale.

These two characteristics can be used as a guideline for an multi-objective optimization program, or least, an optimization program with an objective function which aggregates these two characteristics. In this paper, we propose a direct computation of these values in which we try to position $g_j(b_h)$ as close as possible to the center of the "plateau" of the profile characterization function $\phi_{jh}(g_j(a_k))$, and then establish $p_j(b_h)$ the largest possible. Finally, $q_j(b_h)$ will be fixed "approximatively" to $\frac{p_j(b_h)}{2}$.

It is obvious that the determination of these values also concerns the transitions between 0 and 1 of the the partial concordance indices. Therefore, we will make use of the values $z_{ijh}, t_{ijh}, i = 1, 2, 3, j \in F, h \in B$ defined in (27). For each $j \in F$, we proceed by decreasing order of the categories ($h = p, p-1, ...$) assuming that $g_j(b_{p+1}) = +\infty$.

Algorithm 4.1.
For $j = 1..m$ *do*
  For $h = p..1$ *do*
    $g_j(b_h) = \min\{\frac{t_{2jh}+z_{2jh}}{2}, g_j(b_{h+1})\}$
    $p_j(b_h) = \min\{t_{2jh} - g_j(b_h), g_j(b_h) - z_{2jh}\}$
    $q_j(b_h) = \max\{\frac{p_j(b_h)}{2}, t_{1jh} - g_j(b_h), g_j(b_h) - z_{3jh}\}$

Let us now prove that the conditions (D1)-(D9) are satisfied. To simplify the notation, we use $c_j(z_{2jh}, b_h)$ instead of $c_j(a_k, b_h)$ where $g_j(a_k) = z_{2jh}$.

Proposition 4.1. $\forall j, h$ *it holds*
(i) $z_{3jh} > z_{2jh} > z_{1jh}$
(ii) $t_{3jh} > t_{2jh} > t_{1jh}$
(iii) $t_{1jh} \geq z_{2jh}$

(iv) $t_{2jh} \geq z_{3jh}$
(v) $\neg \exists a_k$ *such that* $g_j(a_k)$ *is in the intervals limited by* $z_{1jh}, z_{2jh}, z_{3jh}$ *or* $t_{1jh}, t_{2jh}, t_{3jh}$.
(vi) - $z_{rjh} \leq z_{rj(h+1)}, t_{rjh} \leq t_{rj(h+1)}, r = 1, 2, 3$

Proposition 4.2. $\forall h \in B, z_{2jh} \leq \frac{t_{1jh}+z_{2jh}}{2} < \frac{t_{2jh}+z_{3jh}}{2} \leq t_{2jh}$

All the conditions (D1)-(D9) can be verified easily from propositions 4.1 and 4.2.

5 How to deal with additional information

In the course of the interactive process, the DM may want to add information concerning the category limits (which can take the form of upper and/or lower bounds for $g_j(b_h), q_j(b_h), p_j(b_h)$) as well as the nature of the criteria. While such information can be taken into account directly in the second phase, it is not the case with the first phase as $g_j(b_h), p_j(b_h), q_j(b_h)$ do not intervene explicitly (through variables). We will discuss hereafter how to integrate these constraints in the first phase.

5.1 Constraints on the profiles and the thresholds

In order to integrate constraints on the profiles and the thresholds in the first phase, we construct rules generating constraints on $c_j(a_k, b_h), c_j(b_h, a_k)$ from given constraints on $g_j(b_h), p_j(b_h), q_j(b_h)$. These rules are resumed in the proposition 5.1.

Proposition 5.1. *We have the following generating rules which hold* $\forall j \in F, \forall k, l \in K, \forall h \in B$.

Original constraints	Rules generating constraints	Rule #
$b_{jh} \leq g_j(b_h)$	if $g_j(a_k) \leq b_{jh}$ then $c_j(b_h, a_k) = 1$	R1
$g_j(b_h) \leq B_{jh}$	if $g_j(a_k) \geq B_{jh}$ then $c_j(a_k, b_h) = 1$	R2
$q_{jh} \leq q_j(b_h)$	if $\lvert g_j(a_l) - g_j(a_k)\rvert < 2q_{jh}$ then $c_j(a_k, b_h) + c_j(b_h, a_l) \geq 1$	R3
$q_j(b_h) \leq Q_{jh}$	if $g_j(a_l) - g_j(a_k) > 2Q_{jh}$ then $c_j(a_k, b_h) + c_j(b_h, a_l) \leq 1$	R4
$p_j(b_h) \leq P_{jh}$	if $g_j(a_l) - g_j(a_k) > 2P_{jh}$ then $c_j(a_k, b_h) + c_j(b_h, a_l) \leq 1$	R5
$p_{jh} \leq p_j(b_h)$	if $g_j(a_l) - g_j(a_k) < 2p_{jh}$ then $c_j(a_k, b_h) + c_j(b_h, a_l) \geq 1$	R6
$b_{jh} \leq g_j(b_h) \leq B_{jh}$ $q_{jh} \leq q_j(b_h) \leq Q_{jh}$ $p_{jh} \leq p_j(b_h) \leq P_{jh}$	$g_j(a_k) \leq b_{jh} - P_{jh} \Rightarrow c_j(a_k, b_h) = 0$ $g_j(a_k) \geq B_{jh} - q_{jh} \Rightarrow c_j(a_k, b_h) = 1$ $g_j(a_k) \leq b_{jh} + q_{jh} \Rightarrow c_j(b_h, a_k) = 1$ $g_j(a_k) \geq B_{jh} + P_{jh} \Rightarrow c_j(b_h, a_k) = 0$	R7.1 R7.2 R7.3 R7.4

Whenever we have an additional constraint, we add the corresponding generated constraints into the program. However, it should be noticed that the generated constraints are not equivalent (necessary but not sufficient) to the original constraints as we can see in the demonstration.

5.2 Constraints on the nature of criteria

The DM may want to build a model in which the nature of the criteria is specified, i.e., less general than the pseudo-criterion considered in our general. This information leads to some more constraints to add into the program.

- **Quasi-criterion** $p = q$
  We have $c_j(a_k, b_h), c_j(a_h, b_k) \in \{0, 1\}$, i.e., we don't have to introduce the integrity hypothesis as it is already satisfied.
- **Pre-criterion** $q = 0$
  In this case, we can introduce these constraints into the program $P1$:
  $c_j(a_k, b_h) + c_j(b_h, a_k) = 1, \forall j \in F, \forall h \in B, \forall k, l \in K$
  This is a special case of $q_{jh} \leq q_j(b_h) \leq Q_{jh}$ where $q_{jh} = Q_{jh} = 0$.

- **True-criterion** $p = q = 0$
  The same as with pre-criterion and quasi-criterion, i.e. :
  - there is no need of the integrity hypothesis
  - the constraints $c_j(a_k, b_h) + c_j(b_h, a_k) = 1, \forall j \in F, \forall h \in B, \forall k, l \in K$ will be inserted into the program.

5.3 How to get a strong consistency

If we want to always ensure the consistency of the categories in A, independently of the set A^*, we must base on the following condition (see (Mousseau *et al.*, 2000) and (Yu, 1992)) :
$g_j(b_{h+1}) \geq g_j(b_h) + p_j(b_h) + p_j(b_{h+1}), \forall j \in F, h \in B$
This condition is indeed a sufficient condition to ensure the consistency in A (but not necessary). To introduce this condition into the program $P1$, we have to represent it by means of the partial concordance indices. As we know, in the reconstruction of a continuous function ($c_j(x, b_h)$ for example, here x is concretized by $g_j(a_k)$) from a set of discrete points ($c_j(a_k, b_h)$), a loss of information is unavoidable. In our case, we do not have an equivalence condition but only either a necessary condition (stated in proposition 5.2.(i)) or a sufficient condition (proposition 5.2.(ii)).

For each $j \in F$, consider a permutation $\sigma_j(k), k \in K$ such that $g_j(a_{\sigma_j(k)}) \leq g_j(a_{\sigma_j(k+1)})$.

Proposition 5.2. *it holds :*
(i) if $g_j(b_{h+1}) \geq g_j(b_h) + p_j(b_h) + p_j(b_{h+1})$ *then* $\min\{c_j(b_h, a_k), c_j(a_k, b_{h+1})\} = 0$.
(i) if $\min\{c_j(b_h, a_{\sigma(k)}), c_j(a_{\sigma(k+1)}, b_{h+1})\} = 0$ *then* $g_j(b_{h+1}) \geq h_j(b_h) + p_j(b_h) + p_j(b_{h+1})$

To ensure the consistency of categories in A despite the set A^* given, we have to introduce the constraints given by proposition 5.2.(ii) into the program $P1$.

6 Empirical validation of the inference procedure

The experimental issues in which we are interested are the following :
- Are the assignments of alternatives from A^* more stable when using the results of the program than when considering the profiles given by the DM ? The term stable is used as unsensitive of the assignments to changes of the profiles. In other words, is the tool able to increase the stability of assignments of alternatives in a set A^* ?

– The results depend on the information given as input, i.e., on the set A^* of assignment examples. How large should A^* be in order to derive the profiles in a reliable manner ?
– In practical decision situations, real DMs do not always provide reliable information. The tool should be able to highlight the assignment examples that are contradictory or not representable through the ELECTRE TRI preference model. Therefore, a question to consider concerns the reliability of the optimization procedure to identify inconsistencies in the DM's judgments ?

It should be highlighted that the empirical work presented is based on a single dataset and should be extended (varying the number of criteria, of categories, ...). Such extended empirical work hardly fits in this paper and should be considered as another paper.

6.1 Experimental Design

In this experiment, we consider only the program $P1$ without additional conditions, and the construction of the profiles. This experiment is a laboratory work, i.e., it takes its material in a past real world case study to perform a posteriori computation in order to test the operational validity of the optimization model proposed. The data considered are taken from (Mousseau *et al.*, 2001) which, in turn, comes from the real world application described in (Yu, 1992).

This application considers the problem of assigning a set A of 100 alternatives to three ordered categories C_1, C_2 and C_3 on the basis of seven criteria (preferences on all criteria are decreasing with the evaluations, i.e., the lower the better).

As no interaction with the DM is possible, we consider the assignment of ELECTRE TRI pessimistic procedure (with the parameters given in (Yu, 1992)) as assignment examples expressed by a "fictitious" DM. Concerning the importance coefficients, we take the mean value of the coefficients inferred for the group of A_{48}^j (the one with the largest size) which are in the results of the procedure of inference of importance coefficients presented in (Mousseau *et al.*, 2001).

We randomly generate 80 subsets of A, the cardinality of these subsets being respectively 6, 12, 18, 24, 30, 36, 42, 48 (10 sets of each size) denoted by A_i^j the j^{th} subset of size i. Each of these subsets is conceived so that the alternatives are assigned uniformly on the three categories.

Let us define the stability of assignments as the variation of the cutting level λ leaving the assignments of alternatives unchanged. The ability to improve the "stability" of the assignment of the alternatives is observed through the value $\beta_0(i) - \beta_d(i)$ where $i \in \{6, ..., 48\}$ is the size of the subsets A^* chosen. $\beta_o(i), \beta_d(i)$ are respectively the mean stability (for all subset A_i^j) of assignments resulting from our procedure and that computed from the initial model given in (Yu, 1992) (representing the "fictitious" DM).

To determine the minimum required size of A^*, we compute Err_{100}, the percentage of assignment errors resulting from the use of the obtained profiles on the whole set A.

To estimate the capacity to identify the inconsistencies in the assertions of the DM, we intentionally introduce in A^* an "assignment error". Let β the resulting stability of assignment (usually < 0), an inconsistency is identified if the alternative under consideration is found in the set E of alternatives which are the most difficult to assign ($E = \{a_k, \min\{x_k, y_k, z_k\} = \beta\}$). We will observe $\beta(i)$ (mean stability of the subsets having size i), $n(i)$ (mean cardinalities of E). For this question, we consider 48 subsets E_i^j ($i = \{6, ..., 48\}, j = 1..6$, each with one assignment error of which the type is j.

Type j	Initial Cat.	Erronous Cat.
1	C_1	C_2
2	C_2	C_1
3	C_2	C_3
4	C_3	C_2
5	C_1	C_3
6	C_3	C_1

TAB. 1 – Types of errors introduced

6.2 Results

6.2.1 Ability to improve the stability of the assignment

The results of the test are summarized in Table 2. Considering these results, we can observe :

– Firstly, the results show that the larger the set of assignment examples, the less stable the assignments, i.e., the more sensitive are these assignments to a change in profiles. This is evident as each assignment example adds $3 + 5mp$ constraints to the program.

– Within this set of data, there is a considerable improvement of the stability of the assignments whatever the size of the set of examples.

Size : i	$\beta_o(i)$	$\beta_d(i)$	$\beta_o(i) - \beta_d(i)$
6	0.2692	0.0723	0.1969
12	0.2288	0.0059	0.2229
18	0.2019	0.0636	0.1383
24	0.2019	0.0616	0.1403
30	0.1966	0.0563	0.1403
36	0.1966	0.0611	0.1355
42	0.2019	0.0298	0.1721
48	0.1913	0.0490	0.1423
		mean	0.1611

TAB. 2 – Improvement of the stability of the solution

6.2.2 The amount of information necessary

We observe now the means of assignment errors in A when different sizes of A^* are considered.The parameters to be inferred, $g_j(b_h), p_j(b_h), q_j(b_h)$, (there are $3mp$ parameters) depend on the number of criteria as well as the number of categories. Considering the above results, it seems that $2mp$ (28=2x7x2 in this example) is a reasonable balance for the estimation of the number of assignment examples to infer weights in a reliable way (see Figure 5). However, it is important to notice that, in this example, we have to accept a certain tolerance of errors, approximately 1.5%.

6.2.3 Ability to identify inconsistencies in assignments

The results of the test are summarized in Table 3. In all the tests, the "wrongly" assigned alternative is found in the alternatives being the most difficult to assign ($min\{x_k, y_k, z_k\} = \beta$). However, within

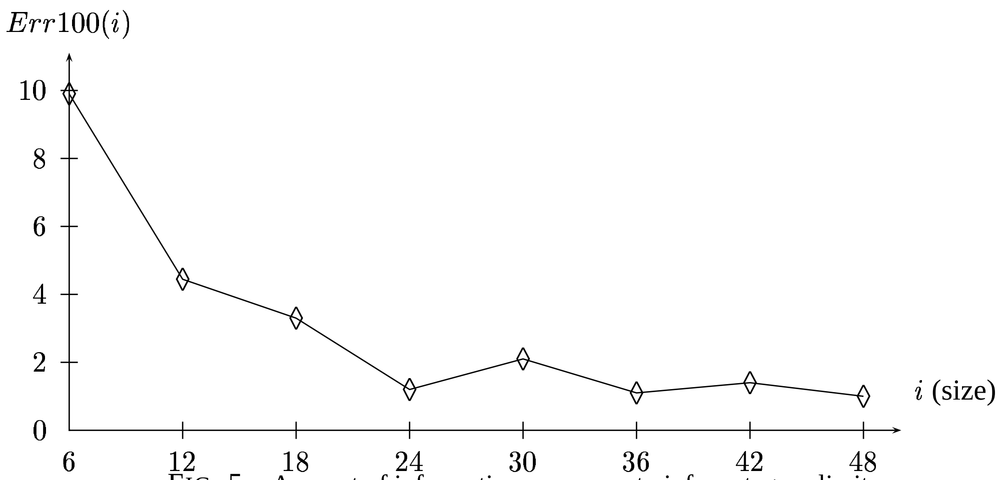

FIG. 5 – Amount of information necessary to infer category limits

this experiment, the model does not seem to be very efficient in identifying errors as the most difficult examples represent a large proportion of $A^* \; \frac{n(i)}{i}$.

Size : i	$\beta(i)$	$n(i)$	$n(i)/i$
6	0.10	3.36	0.56
12	0.13	7.68	0.64
18	0.04	6.30	0.35
24	0.03	7.68	0.32
30	0.04	13.80	0.46
36	0.01	12.96	0.36
42	0.02	11.34	0.27
48	0.00	25.92	0.54

TAB. 3 – Identification of “errors”

7 Conclusion

This paper presents an inference procedure aiming at inferring the category limits of the ELECTRE TRI method on the basis of assignment examples. This procedure is grounded on a mathematical programming formulation and is validated through a laboratory experiment. This inference procedure is intended to be used interactively in an aggregation-disaggregation process. Moreover, this procedure complements previous results on partial inference models, namely the weight inference procedure (Mousseau *et al.*, 2001) and veto inference procedure (Dias & Mousseau, 2002). Such partial inference procedures can be used in conjunction (*e.g.*, fixing weights and veto so as to infer category limits, and then fixing category limits and veto so as to infer weights).

The proposed inference procedure is suitable for a DM to define category limits of ELECTRE TRI method providing assignment examples. Moreover, we believe that such procedure is helpful in order to provide a formal framework for the DM to learn about the relation between the category limits and his/her preference in a constructive learning process.

References

Bana e Costa, C. 1996. Les problématiques de l'aide à la décision : vers l'enrichissement de la trilogie Choix-Tri-Rangement. *RAIRO/Operations Research*, **30**(2), 191–216.

Dias, L.C., & Mousseau, V. 2002. *Inferring Electre's veto-related parameters from outranking examples.* Research report n^{o}5/2002. INESC Coimbra.

Jacquet-Lagrèze, E., & Siskos, J. 1982. Assessing a set of additive utility functions for multicriteria decision-making, the UTA method. *European Journal of Operational Research*, **10**(2), 151–164.

Jacquet-Lagrèze, E., & Siskos, Y. 2001. Preference disaggregation : 20 years of MCDA experience. *European Journal of Operational Research*, **130**(2), 233–245.

Mousseau, V., & Slowinski, R. 1998. Inferring an ELECTRE TRI model from assignment examples. *Journal of Global Optimization*, **12**(2), 157–174.

Mousseau, V., Slowinski, R., & Zielniewicz, P. 2000. A User-oriented Implementation of the ELECTRE TRI Method Integrating Preference Elicitation Support. *Computers & Operations Research*, **27**(7-8), 757–777.

Mousseau, V., Figueira, J., & Naux, J.-Ph. 2001. Using assignment examples to infer weights for ELECTRE TRI method : Some experimental results. *European Journal of Operational Research*, **130**(2), 263–275.

Naux, J.-Ph. 1996. *Calage d'un modèle Electre Tri à l'aide d'actions exemples.* Mémoire de DEA. Université Paris-Dauphine.

Ngo The, A. 1998. *Inférence des profils limites dans le calage d'un modèle de type Electre Tri.* Mémoire de DEA. Université Paris-Dauphine.

Roy, B. 1985. *Méthodologie multicritère d'aide à la décision.* Paris : Economica.

Roy, B., & Bouyssou, D. 1993. *Aide multicritère à la décision : Méthodes et cas.* Paris : Economica.

Roy, B., & Vincke, Ph. 1984. Relational systems of preferences with one or more pseudo-criteria : some new concepts and results. *Management Science*, **30**(11), 1323–1334.

Yu, W. 1992. *Aide multicritère à la décision dans le cadre de la problématique du tri : concepts, méthodes et applications.* Ph.D. thesis, Université Paris-Dauphine.

Appendix : Proofs

Proof of proposition 3.2

$a_k \to C_{h_k} \Leftrightarrow [c(a_k, b_{h_k}) < \lambda$ and $c(a_k, b_{h_k-1}) \geq \lambda]$

(i) $c(a_k, b_{h_k}) < \lambda \Rightarrow \neg a_k S b_{h_k}$

$\Rightarrow \forall h > h_k, c(a_k, b_h) \leq c(a_k, b_{h_k}) < \lambda \Rightarrow \neg a_k S b_h$

$\Rightarrow \forall h > h_k, \neg a_k I b_h$

(ii) $c(a_k, b_{h_k-1}) \geq \lambda \Rightarrow a_k S b_{h_k-1}$

$\Rightarrow \forall h \leq h_k - 1, c(a_k, b_h) \geq c(a_k, b_{h_k-1}) \geq \lambda \Rightarrow a_k S b_h$

$\Rightarrow \forall h \leq h_k - 1, b_h S a_k \Rightarrow b_h I a_k$ ■

Proof of proposition 3.3

$\neg b_{h_0} S a_k \Rightarrow c(b_{h_0}, a_k) < \lambda$

$\Rightarrow \forall h \leq h_0, c(b_h, a_k) \leq c(b_{h_0}, a_k) < \lambda \Rightarrow \neg b_h S a_k$ ■

Proof of proposition 3.4
$(i) \Rightarrow (ii)$
Let b_{h_1} the profile (unique if exist) s.t. $a_k I b_{h_1}$
By 3.2, $h \geq h_k \Rightarrow \neg a_k I b_h$, therefore $a_k I b_{h_1} \Rightarrow h_1 \leq h_k - 1$
By definition, $a_k I b_{h_1} \Rightarrow a_k S b_{h_1}$ and $b_{h_1} S a_k$
If $h_1 < h_k - 1$ then $c(b_{h_k-1}, a_k) \geq c(b_{h_1}, a_k) \geq \lambda \Rightarrow b_{h_k-1} S a_k \Rightarrow a_k I b_{h_k-1}$, contradictory to the unicity of h_1.
Therefore, $h_1 = h_k - 1$. As $h_k - 2 < h_k - 1 \Rightarrow [\neg a_k I b_{h_k-2}$ and $a_k S b_{h_k-2}] \Rightarrow \neg b_{h_k-2} S a_k$.
$(ii) \Rightarrow (i)$
$\neg b_{h_k-2} S a_k \Rightarrow \forall h \leq h_k - 2, \neg b_h S a_k$
Let h_1 the profile s.t. $a_k I b_{h_1}$. Then, $h_1 > h_k - 2$ and $h_1 \leq h_k - 1 \Rightarrow h_1 = h_k - 1$ ■

Proof of proposition 4.1
(i),(ii) Obvious from definitions.
(iii) By definition, $c_j(z_{2jh}, b_h) = 0 \Rightarrow c_j(b_h, z_{2jh}) = 1$
$\Rightarrow z_{2jh} \leq t_{1jh} = \max\{g_j(a_k) | a_k \in A^*, c_j(b_h, a_k) = 1\}$.
(iv) Similar to (iii).
(v) Obvious from definition.
(vi) We demonstrate only for z_{2jh}, the other cases are similar.
By definition, $0 = c_j(z_{2jh}, b_h) \geq c_j(z_{2jh}, b_{h+1}) \Rightarrow c_j(z_{2jh}, b_{h+1}) = 0$
$\Rightarrow z_{2jh} \leq z_{2j(h+1)} = \max\{g_j(a_k) | a_k \in A^*, c_j(a_k, b_{h+1}) = 0\}$. ■

Proof of proposition 4.2
From proposition 4.1, we have :
$z_{2jh} \leq \frac{t_{1jh}+z_{jh}}{2} < \frac{t_{2jh}+z_{3jh}}{2} \leq t_{2jh}$
And from the procedure : $g_j(b_h) \leq \frac{t_{2jh}+z_{2jh}}{2}$
$z_{2jh} < z_{3jh} \Rightarrow g_j(b_h) < \frac{t_{2jh}+z_{3jh}}{2}$
For the first category b_p, we have $g_j(b_{(p+1)}) = +\infty$ then
$g_j(b_p) = \frac{t_{2jp}+z_{2jp}}{2}$.
$t_{2jp} > t_{1jp} \Rightarrow g_j(b_p) > \frac{t_{1jp}+z_{2jp}}{2}$.
Suppose that the inequality $g_j(b_i) > \frac{t_{1ji}+z_{2ji}}{2}$ holds $i = h+1$.
$g_j(b_h) = \min\{\frac{t_{2jh}+z_{2jh}}{2}, g_j(b_{h+1})\}$
$\frac{t_{2jh}+z_{2jh}}{2} > \frac{t_{1jh}+z_{2jh}}{2}$ and $g_j(b_{h+1}) > \frac{t_{1j(h+1)}+z_{2j(h+1)}}{2} \geq \frac{t_{1jh}+z_{2jh}}{2}$
$\Rightarrow g_j(b_h) > \frac{t_{1jh}+z_{2jh}}{2}$. Therefore, the inequality holds for $i = h$.
By induction, the inequality holds for $\forall m \in B$. ■

Proof of proposition 5.1

- (R1) $g_j(a_k) \leq b_{jh} \Rightarrow g_j(a_k) \leq g_j(b_h) \Rightarrow c_j(b_h, a_k) = 1$.
- (R2) $g_j(a_k) \geq B_{jh} \Rightarrow g_j(a_k) \geq g_j(b_h) \Rightarrow c_j(a_k, b_h) = 1$.
- (R3) $[(c_j(a_k, b_h) < 1)$ and $(c_j(b_h, a_l) < 1)] \Rightarrow [(g_j(a_k) < g_j(b_h) - q_j(b_h))$ and $(g_j(a_l) > g_j(b_h) + q_j(b_h))]$

$\Rightarrow g_j(a_l) - g_j(a_k) \geq 2g_j(b_h) \geq 2q_{jh}$.
We have
$|g_j(a_l) - g_j(a_k)| < 2q_{jh} \Rightarrow$ not $[c_j(a_k,b_h) < 1$ and $c_j(b_h,a_l) < 1] \Rightarrow c_j(a_k,b_h) + c_j(b_h,a_l) \geq 1$.

- (R4) $[(c_j(a_k,b_h) = 1)$ and $(c_j(b_h,a_l) = 1)] \Rightarrow [(g_j(a_k) > g_j(b_h) - q_j(b_h))$ and $(g_j(a_l) < g_j(b_h) + q_j(b_h))]$
  $\Rightarrow g_j(a_l) - g_j(a_k) \leq 2q_j(b_h) \leq 2Q_{jh}$
  We have
  $g_j(a_l) - g_j(a_k) > 2Q_{jh} \Rightarrow$ not $[(c_j(a_k,b_h) = 1)$ and $(c_j(b_h,a_l) = 1)] \Rightarrow c_j(a_k,b_h) + c_j(b_h,a_l) \leq 1$ (under integrity hypothesis).

- (R5) $[(c_j(a_k,b_h) > 0)$ and $(c_j(b_h,a_l) > 0)] \Rightarrow [(g_j(a_k) > g_j(b_h) - p_j(b_h))$ and $(g_j(a_l) < g_j(b_h) + p_j(b_h))]$
  $\Rightarrow g_j(a_l) - g_j(a_k) \leq 2p_j(b_h) \leq 2P_{jh}$
  So we have
  $g_j(a_l) - g_j(a_k) > 2P_{jh} \Rightarrow$ not $[(c_j(a_k,b_h) > 0)$ and $(c_j(b_h,a_l) > 0)] \Rightarrow c_j(a_k,b_h) + c_j(b_h,a_l) \leq 1$

- (R6) $[(c_j(a_k,b_h) = 0)$ and $(c_j(b_h,a_l) = 0)] \Rightarrow [(g_j(a_k) < g_j(b_h) - p_j(b_h))$ and $(g_j(a_l) > g_j(b_h) + p_j(b_h))]$
  $\Rightarrow g_j(a_l) - g_j(a_k) \geq 2p_j(b_h) \geq 2p_{jh}$
  So we have
  $g_j(a_l) - g_j(a_k) < 2p_{jh} \Rightarrow$ not $[(c_j(a_k,b_h) = 0)$ and $(c_j(b_h,a_l) = 0)] \Rightarrow c_j(a_k,b_h) + c_j(b_h,a_l) \geq 1$ (under integrity hypothesis).

- (R7)
  1/ $g_j(a_k) \leq b_{jh} - P_{jh} \leq g_j(b_h) - p_j(b_h) \Rightarrow c_j(a_k,b_h) = 0$
  2/ $g_j(a_k) \geq B_{jh} - q_{jh} \geq g_j(b_h) - q_j(b_h) \Rightarrow c_j(a_k,b_h) = 1$
  3/ $g_j(a_k) \leq b_{jh} + q_{jh} \leq g_j(b_h) + q_j(b_h) \Rightarrow c_j(b_h,a_k) = 1$
  4/ $g_j(a_k) \geq B_{jh} + P_{jh} \geq g_j(b_h) + p_j(b_h) \Rightarrow c_j(b_h,a_k) = 0$ ■

Proof of proposition 5.2
(i) $c_j(b_h,a_k) > 0 \Rightarrow g_j(a_k) < g_j(b_h) + p_j(b_h) \leq g_j(b_{h+1}) - p_j(b_{h+1}) \Rightarrow c_j(a_k,b_{h+1}) = 0$
(ii) Let $a_{\sigma(k*)}$ the first alternative such that $c_j(b_h,a_{\sigma(k^*)}) = 0$ then $c_j(b_h,a_{\sigma(k^*-1)}) > 0$, the existence of these two alternatives is ensured by the two additional fictious alternatives a_*, a^*.
Replace k by $k^* - 1$ in the condition $\min\{c_j(b_h,a_{\sigma(k)}), c_j(a_{\sigma(k+1)},b_{h+1})\} = 0$
we obtain $c_j(b_h,a_{\sigma(k^*-1)}) > 0 \Rightarrow c_j(a_{\sigma(k^*},b_{h+1}) = 0$
We have
$c_j(b_h,a_{\sigma(k^*)}) = 0 \Rightarrow g_j(a_{\sigma(k^*)}) \geq g_j(b_h) + p_j(b_h)$
$c_j(a_{\sigma(k^*)},b_{h+1}) = 0 \Rightarrow g_j(a_{\sigma(k^*)} < g_j(b_{h+1}) - p_j(b_{h+1})$
Therefore,
$g_j(b_h) + p_j(b_h) \leq g_j(a_{\sigma(k^*)}) < g_j(b_{h+1}) - p_j(b_{h+1}) \Rightarrow g_j(b_{h+1}) \geq g_j(b_h) + p_j(b_h) + p_j(b_{h+1})$ ■