A novel integer linear programming approach for global L_0 minimization

Diego Delle Donne

joint work with **Leo Liberti** and **Matthieu Kowalski**

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Sparse Approximation Problems

Goals: Given a linear system $Hx = y$, find $\hat{x}$ such that:

- $||y - H\hat{x}|| \leq \alpha$
- $||\hat{x}||_0 := \#\{j \mid \hat{x}_j \neq 0\} \leq k$.

Applications:

- Data compression
- Image recovery
- Signal processing
- Machine learning
- Etc.

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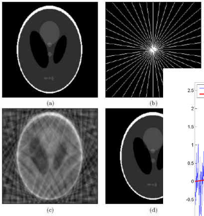

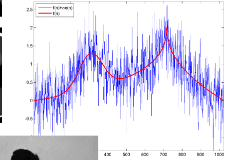

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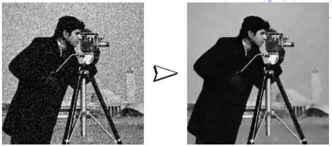

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Several problems may be defined:

1. $\mathcal{P}_{0/p}$: minimize $||x||_0$ s.t. $||y - Hx||_p \leq \alpha$,
2. $\mathcal{P}_{p/0}$: minimize $||y - Hx||_p$ s.t. $||x||_0 \leq k$,
3. $\mathcal{P}_{0+p}$: minimize $\lambda_1 ||y - Hx||_p + \lambda_2 ||x||_0$ for some $\lambda_1, \lambda_2 \in \mathbb{R}$.

We'll focus on Problem 1...

... with a particular interest in the norm ℓ_2

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Outline

MIP formulations

Initial polyhedral study

An IP reformulation

Computational results

MIP formulations: ℓ_1 and ℓ_∞

(Bourguignon et al. 2016)

- Binary variable b_j states whether $x_j \neq 0$ or not
- Continuous variables w collect the *misfit error*

$\text{MIP}_{0/1}$	$\text{MIP}_{0/\infty}$
$\min \sum_{j\in[m]} b_j$	$\min \sum_{j\in[m]} b_j$
$-Mb_j \leq x_j \leq Mb_j \quad \forall j \in [m]$	$-Mb_j \leq x_j \leq Mb_j \quad \forall j \in [m]$
$-w_i \leq y_i - \sum_{j\in[m]} h_{ij}x_j \leq w_i \quad \forall i \in [n]$	$-w \leq y_i - \sum_{j\in[m]} h_{ij}x_j \leq w \quad \forall i \in [n]$
$\sum_{i\in[n]} w_i \leq \alpha$	$w \leq \alpha$
$x_j \in \mathbb{R},\ b_j \in \{0,1\} \quad \forall j \in [m]$	$x_j \in \mathbb{R},\ b_j \in \{0,1\} \quad \forall j \in [m]$
$w_i \in \mathbb{R} \quad \forall i \in [n]$	$w \in \mathbb{R}$

Observation:
M is a sufficiently big constant, which is necessary to properly formulate this MIPs

MIP formulations: ℓ_2

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$$\text{MIP}_{0/2}$$

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\begin{aligned}
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& -w_i \leq y_i - \sum_{j \in [m]} h_{ij} x_j \leq w_i & \forall i \in [n] \\
& \sum_{i \in [n]} w_i^2 \leq \alpha^2 & \\
& x_j \in \mathbb{R},\ b_j \in \{0, 1\} & \forall j \in [m] \\
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MIP formulations: Dealing with the big-M approach

(Bourguignon et al. 2016)

1. Set an initial value for M
2. Solve the MIP and get an optimal solution $\hat{x}$
3. if some $\hat{x}_j = M$, then increase M and repeat from Step 2
4. else, then STOP with "optimal" solution $\hat{x}$

Problem: This algorithm may stop with a suboptimal solution!

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MIP formulations: Dealing with the big-M approach

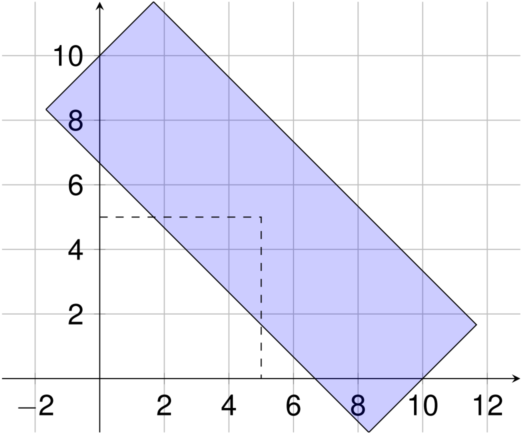
Some small examples:

$$H = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$y = \begin{pmatrix} 15 \\ 15 \end{pmatrix}$$

$$\alpha = 10$$

norm: $p = 1$



If $M = 5$ all feasible solutions have $||x||_0 = 2$, so we may get an "optimal" not tight on M. However, the real optimums lie on the axis.

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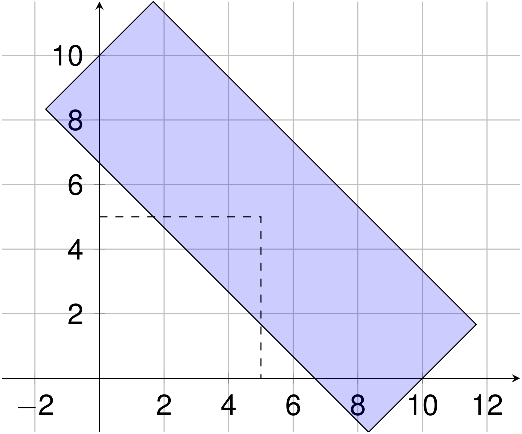

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MIP formulations: Dealing with the big-M approach

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$$H = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

$$y = \begin{pmatrix} 7 \\ 15 \end{pmatrix}$$

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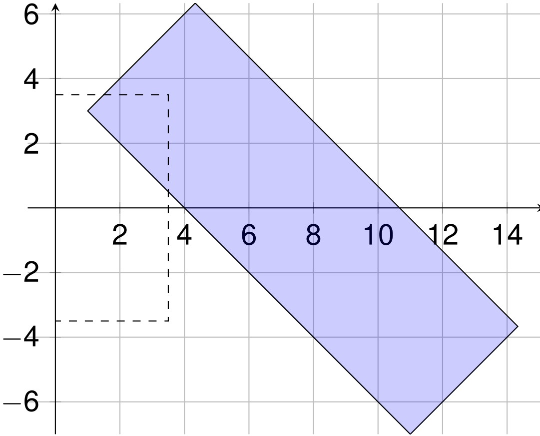

If $M = 3.5$ we have the same problem...

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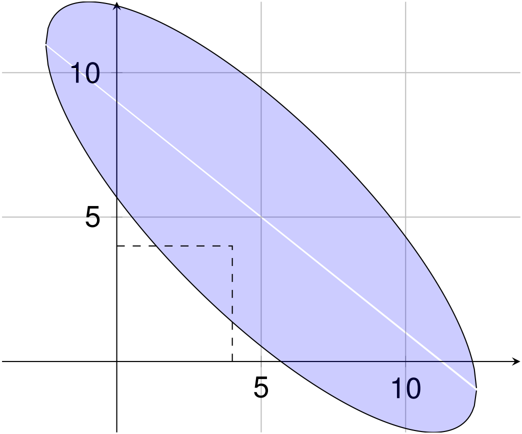

With $M = 4$ we have the same problem...

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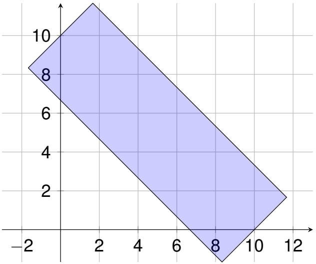
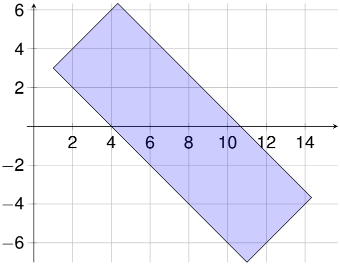
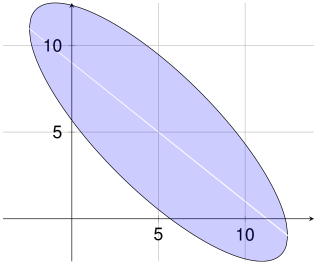




Can we find a proper M?



MIP formulations: Dealing with the big-M approach

Some small examples:

$$H = \begin{pmatrix} 1 & 1 \end{pmatrix}$$

$$y = 10$$

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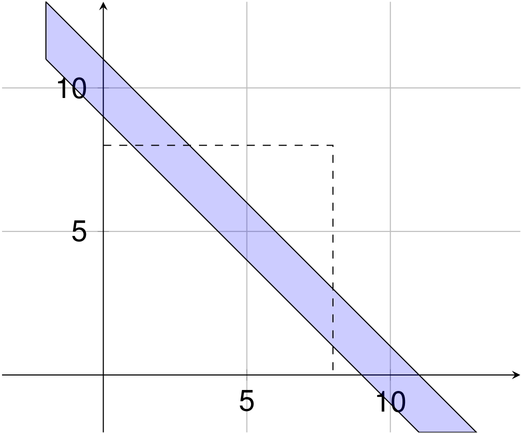
norm: $p \in \{1, 2, \infty\}$



With $M = 8$ we have the same problem...

MIP formulations: Dealing with the big-M approach

Observation: We may get rid of the big-M constraints by adding some non-convexities. For example,

$$\sout{-Mb_j \leq x_j \leq Mb_j}$$

$$x_j = x_j b_j, \qquad \forall j \in [m]$$

Summing up:

- No big-M $\implies$ non-convexities
- Big-M $\implies$ Not clear which M to use?
- Known bound for x $\implies$ Known Big-M issues...

In any case... we don't have an "elegant" solution...

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Polyhedral study

Definition (Forbidden support)

A set of columns J is a *forbidden support* for $\mathcal{P}_{0/p}$ if there exist no solutions with support J. Equivalently, if $\min_{x \in \mathbb{R}^m}\{||y - H^J x^J||_p\} > \alpha$.

Proposition

If $J \subseteq [m]$ is a forbidden support for $\mathcal{P}_{0/p}$, then the forbidden support inequality is valid for $\mathrm{MIP}_{0/p}$.

$$\sum_{j \in [m] \setminus J} b_j \geq 1 \quad (1)$$

Proposition

For $p \in \{1, 2, \infty\}$, we can efficiently test if a set J is a forbidden support by finding $\min_{x \in \mathbb{R}^m}\{||y - H^J x^J||_p\}$.

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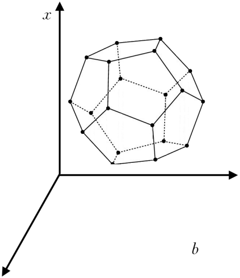
An interesting remark about Forbidden Support Inequalities:



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$$\mathcal{P} = \begin{cases} -Mb \leq x \leq Mb \\ ||y - Hx||_p \leq \alpha \\ x \in \mathbb{R}^m, \; b \in \{0,1\}^m \end{cases}$$

$$\mathcal{P}_{proj} = \{ b \in \{0,1\}^m \mid \text{ exists a solution } (x, b) \in \mathcal{P} \}$$

Proposition

Given a support $b \in \mathcal{P}_{proj}$, we can efficiently find a solution $(x, b) \in \mathcal{P}$.

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$\mathcal{P}_{proj} = \{ b \in \{0,1\}^m \mid$ *b satisfies all Forbidden Support ineq.* $(1) \}$

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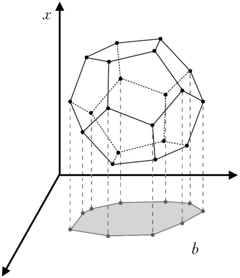

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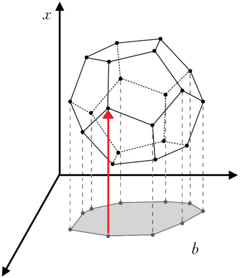

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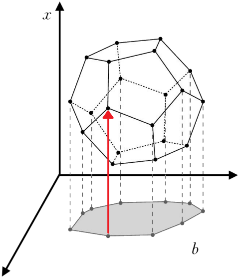

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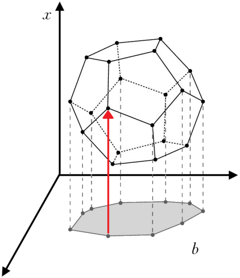

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A novel IP reformulation

Therefore, we can obtain an optimum support by solving:

$$[\mathrm{IP}^{cov}_{0/p}] \quad \min \sum_{j \in [m]} b_j$$

$$\sum_{j \in [m] \setminus J} b_j \geq 1, \quad \forall \text{ forbidden support } J \subseteq [m]$$

$$b \in \{0,1\}^m$$

- **Pros:**
  - Linear formulation (even for ℓ_2)
  - It does not need the big M!
    (neither to obtain x afterwards...)
  - Is a Minimum Set Covering problem!
  - It is a well-known pure combinatorial problem
- **Cons:**
  - Exponentially-many constraints...
    (but we can deal with this...)

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$$\sum_{j \in [m] \setminus J} b_j \geq 1, \quad \forall \text{ forbidden support } J \subseteq [m]$$

$$b \in \{0,1\}^m$$

- **Pros:**
  - Linear formulation (even for ℓ_2)
  - It does not need the big M!
    (neither to obtain x afterwards...)
  - Is a Minimum Set Covering problem!
  - It is a well-known pure combinatorial problem

- **Cons:**
  - Exponentially-many constraints...
    (but we can deal with this...)

A novel IP reformulation

Therefore, we can obtain an optimum support by solving:

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Solving $\mathrm{IP}^{cov}_{0/p}$

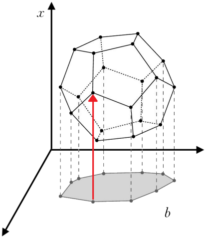
Sketch of the algorithm:

1. Solve a combinatorial relaxation of $\mathrm{IP}^{cov}_{0/p}$ with just a few constraints and obtain a "minimum" support $b \in \{0,1\}^m$
2. **If** b is a forbidden support,
   - Add to the formulation the constraint associated to b and repeat from Step 1.
3. Else, b is a feasible support, so
   - then finish with a proper solution $x \in \mathbb{R}^m$.

$$\min \sum_{j \in [m]} b_j$$

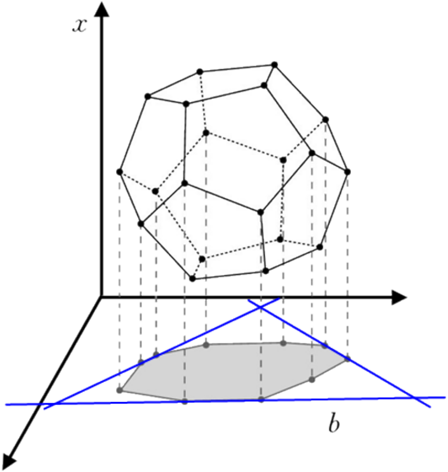
$$\sum_{j \in [m] \setminus J} b_j \geq 1, \quad \forall \text{ FS } J \subseteq [m]$$

$$b \in \{0,1\}^m$$



Solving $\text{IP}_{0/p}^{cov}$

Graphical sketch:



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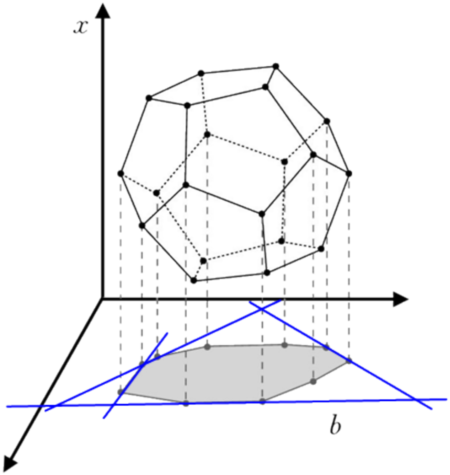

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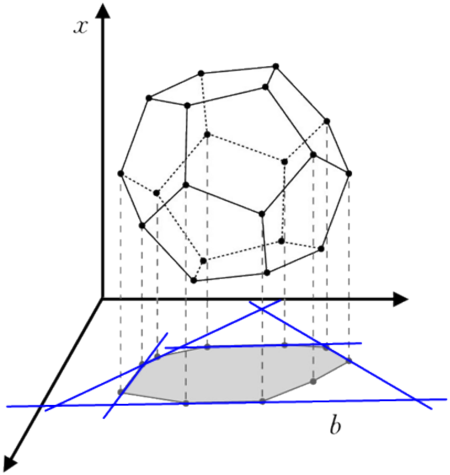

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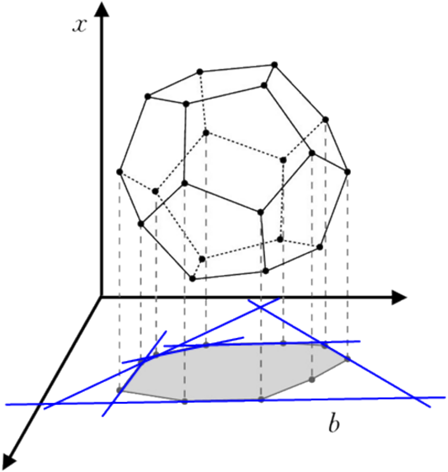

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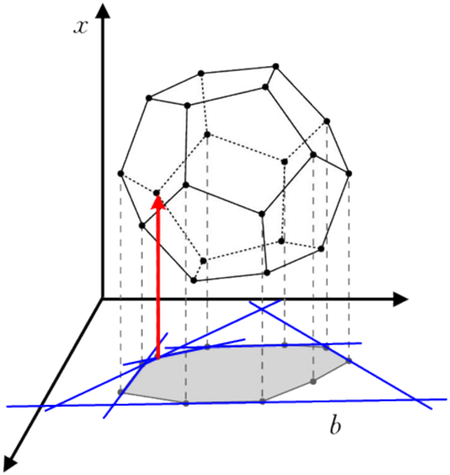
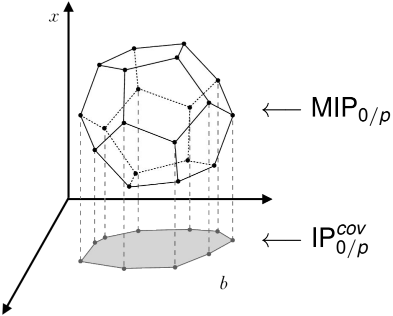


Back to some more polyhedral stuff...

An interesting remark



- Many valid inequalities for Set Covering polytopes are known
- We can apply these cuts to $\mathrm{MIP}_{0/p}$!

Valid inequalities arising from Set Covering

Proposition

Let $\mathcal{J} \subseteq 2^{[m]}$ be a family of forbidden supports for $\mathcal{P}_{0/p}$, and define $J^{none} := [m] \setminus \bigcup_{J \in \mathcal{J}} J$, and $J^{some} := [m] \setminus (J^{none} \cup \bigcap_{J \in \mathcal{J}} J)$. Then the forbidden support family inequality

$$\sum_{j \in J^{none}} 2b_j + \sum_{j \in J^{some}} b_j \geq 2 \qquad (2)$$

is valid for $\mathrm{MIP}_{0/p}$.

These inequalities (as many other known inequalities) may be used as cuts in a cutting plane approach to solve $\mathrm{MIP}_{0/p}$ and $\mathrm{IP}_{0/p}^{cov}$.

Checkpoint

Another quick checkpoint:

- Existing big-M formulation
- New valid inequalities
- Description of feasible supports
- New IP approach for $\mathcal{P}_{0/p}$ (the first, to our knowledge)
- Set covering polytope

Solution approaches

We will evaluate 3 approaches:

1. $\text{MIP}_{0/p}$: iterative algorithm from the literature to solve the big-M formulation.

2. $\text{BC}_{0/p}$: same as above but solving the formulation with a simple branch & cut algorithm based on forbidden support cuts.

3. $\text{IP}^{cov}_{0/p}$: novel IP formulation (with dynamically added constraints)

A simple branch & Cut approach

- Forbidden support inequalities as cuts
- Rounding primal heuristic procedure

Separation problem: Given a fractional solution $(\hat{x}, \hat{b}, \hat{w})$, find a forbidden support J such that

$$\sum_{j \in [m] \setminus J} \hat{b}_j < 1$$

It is not clear at all how to **efficiently** solve this problem (if possible!)

... we resort to a heuristic separation routine.

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A simple branch & Cut approach

Heuristic separation routine for Forbidden Support cuts

Sketch of the algorithm:

1. Get as many b_j as possible while keeping $\sum_j \hat{b}_j < 1$
2. Take J as the complement of those indexes
3. **If** J is a forbidden support we already have a valid cut!
   - This cut may be weak if $|J|$ is too small
   - Try to expand $|J|$ to a wider forbidden support if possible
4. **else**, we failed to get a forbidden support cut for $(\hat{x}, \hat{b}, \hat{w})$

Obs: We shall use this cuts (and the rounding heuristic) also for $\mathrm{IP}_{0/p}^{cov}$

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Computational results

We will evaluate 3 approaches:

1. $\text{MIP}_{0/p}$: iterative algorithm from the literature to solve the big-M formulation.
2. $\text{BC}_{0/p}$: same as above but solving the formulation with a simple branch & cut algorithm based on forbidden support cuts.
3. $\text{IP}_{0/p}^{cov}$: novel IP formulation (with dynamically added constraints and branch & cut)

Two goals:

- Solution times: One-dimensional deconvolution problems (from Bourguignon et al., 2016).
- Solution quality: Submatrices of a “pathological” case (from Mairal and Yu, 2012).

Obs: We use CPLEX 12.6 callback framework from it’s Java API.

Computational results

Deconvolution instances for $\mathcal{P}_{0/2}$

			$MIP_{0/2}$				$BC_{0/2}$				$IP^{cov}_{0/2}$			
SNR	K	# ins	solv	time	uns	supp	solv	time	uns	supp	solv	time	uns	supp
10	5	50	49	352	1	5.0	46	274	4	5.8	50	26	0	-
	7	48	8	949	40	7.9	4	612	44	7.9	37	576	11	7.6
	9	16	0	-	16	10.2	0	-	16	10.3	2	898	14	11.1
20	5	50	50	105	0	-	50	24	0	-	50	4	0	-
	7	49	29	700	20	7.9	48	190	1	9.0	49	19	0	-
	9	41	4	673	37	10.6	22	729	19	11.2	41	99	0	-
30	5	50	50	62	0	-	50	9	0	-	50	2	0	-
	7	50	48	529	2	15.0	49	31	1	23.0	50	5	0	-
	9	50	12	1119	38	10.3	50	235	0	-	50	22	0	-
Instances solved...														
... by all:		242	360 sec.				104 sec.				12 sec.			
... by none:		25	9.04				9.36				9.56			

Computational results

Deconvolution instances for $\mathcal{P}_{0/2}$

			$MIP_{0/2}$				$BC_{0/2}$				$IP_{0/2}^{cov}$			
SNR	K	# ins	solv	time	uns	supp	solv	time	uns	supp	solv	time	uns	supp
10	5	50	49	352	1	5.0	46	274	4	5.8	50	26	0	-
	7	48	8	949	40	7.9	4	612	44	7.9	37	576	11	7.6
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	9	50	12	1119	38	10.3	50	235	0	-	50	22	0	-
Instances solved...														
... by all:		242		360 sec.				104 sec.				12 sec.		
... by none:		25		9.04				9.36				9.56		

Computational results

Deconvolution instances for $\mathcal{P}_{0/\infty}$

			$MIP_{0/\infty}$				$BC_{0/\infty}$				$IP^{cov}_{0/\infty}$			
SNR	K	# ins	solv	time	uns	supp	solv	time	uns	supp	solv	time	uns	supp
10	5	50	33	340	17	7.4	40	375	10	7.7	32	1012	18	8.4
	7	48	5	1033	43	8.3	10	941	38	8.7	2	1465	46	11.2
	9	16	0	-	16	10.0	0	-	16	10.2	0	-	16	15.1
20	5	50	38	331	12	7.3	45	356	5	7.4	41	964	9	9.7
	7	49	11	717	38	7.9	30	878	19	8.7	11	1340	38	11.6
	9	41	1	1144	40	9.6	2	1095	39	9.9	0	-	41	15.2
30	5	50	44	330	6	7.3	48	290	2	9.0	45	810	5	9.2
	7	50	17	591	33	8.0	37	758	13	8.7	26	1343	24	13.4
	9	50	1	1142	49	9.7	9	1110	41	10.0	3	1443	47	16.2
Instances solved...														
... by all:		129	366 sec.				290 sec.				954 sec.			
... by none:		179	9.04				9.32				13.57			

Computational results

Pathological instances

		$MIP_{0/2}$				$BC_{0/2}$				$IP^{cov}_{0/2}$			
Size	K	tl	supp	top	best	tl	supp	top	best	tl	supp	top	best
20×40	4	0	16.2	2	0	0	15.5	2	0	0	4.0	10	8
	6	0	14.7	2	0	1	16.2	2	0	0	5.8	10	8
	8	0	20.5	1	0	1	19.5	1	0	9	8.7	9	9
30×60	4	0	9.8	4	0	2	9.2	5	0	0	4.0	10	5
	6	2	15.8	3	0	4	16.6	3	0	0	5.7	10	7
	8	2	11.9	6	1	5	12.1	5	0	10	9.8	7	4
40×80	4	5	11.0	4	0	5	11.1	4	0	0	4.0	10	6
	6	2	14.6	6	0	5	16.4	6	0	7	7.6	8	4
	8	4	17.2	6	0	6	19.1	5	0	9	10.1	8	4

Computational results

Pathological instances - The 40 cases solved by the three methods

	$IP^{cov}_{0/2}$	$MIP_{0/2}$		$BC_{0/2}$	
Instance	supp	supp	err	supp	err
20.40.4.1	4	13	225%	12	200%
20.40.4.2	4	4	0%	4	0%
20.40.4.3	4	26	550%	26	550%
20.40.4.4	4	29	625%	29	625%
20.40.4.5	4	7	75%	6	50%
20.40.4.6	4	13	225%	12	200%
20.40.4.7	4	18	350%	16	300%
20.40.4.8	4	30	650%	30	650%
20.40.4.9	4	18	350%	16	300%
20.40.4.10	4	4	0%	4	0%
20.40.6.1	6	6	0%	6	0%
20.40.6.2	6	11	83%	11	83%
20.40.6.3	6	31	417%	31	417%
20.40.6.4	6	17	183%	31	417%
20.40.6.5	6	21	250%	24	300%
20.40.6.6	6	17	183%	17	183%
20.40.6.8	5	10	100%	10	100%
20.40.6.9	5	14	180%	12	140%
20.40.6.10	6	6	0%	6	0%
20.40.8.8	7	34	386%	34	386%

	$IP^{cov}_{0/2}$	$MIP_{0/2}$		$BC_{0/2}$	
Instance	supp	supp	err	supp	err
30.60.4.1	4	5	25%	4	0%
30.60.4.2	4	5	25%	5	25%
30.60.4.4	4	4	0%	4	0%
30.60.4.5	4	8	100%	8	100%
30.60.4.6	4	4	0%	4	0%
30.60.4.7	4	4	0%	4	0%
30.60.4.8	4	4	0%	4	0%
30.60.4.9	4	40	900%	31	675%
30.60.6.1	6	6	0%	6	0%
30.60.6.2	5	33	560%	42	740%
30.60.6.7	6	32	433%	28	367%
30.60.6.8	6	6	0%	6	0%
30.60.6.9	5	5	0%	5	0%
30.60.6.10	5	10	100%	10	100%
40.80.4.1	4	4	0%	4	0%
40.80.4.2	4	4	0%	4	0%
40.80.4.5	4	4	0%	4	0%
40.80.4.6	4	6	50%	6	50%
40.80.4.10	4	4	0%	4	0%
40.80.6.3	6	6	0%	6	0%

Summing up...

- Previous MI**L**P approaches are not exact.
- We presented a new (the first, to our knowledge) exact I**L**P approach with interesting results.
- We showed how a B&C approach may help to speed-up computation (although big-M related issues would hold).

What's next? Exploit the Set Covering structure!

- Implement known cut families for $\mathrm{IP}_{0/p}^{cov}$
- Profit from known algorithms, heuristics, etc.

Thanks for your atention!