Uniform Covers for Cones and Combinatorial Problems: Results, Connections and Challenges

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A uniform cover is a *family of sets «multi-hypergraph»* $(V, \mathcal{U}), \mathcal{U} \subseteq 2^V$

so that every $v \in V$ is contained in the same number of members of $\mathcal{U}$.

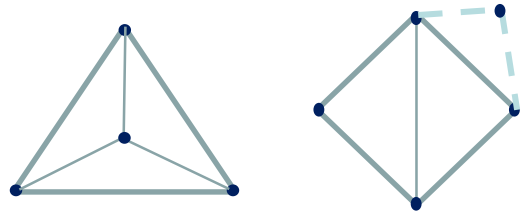
$$\Leftrightarrow \underline{1} \in \text{cone}\,(\chi_U : U \in \mathcal{U})$$

sum of coeffs $=k$ $\Leftrightarrow \underline{1}/k \in \text{conv}\,(\chi_U : U \in \mathcal{U})$

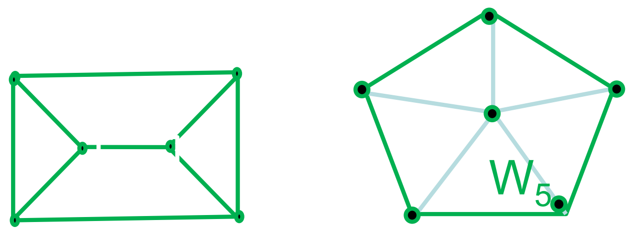
POC, 10 décembre, 2021

Examples of the day

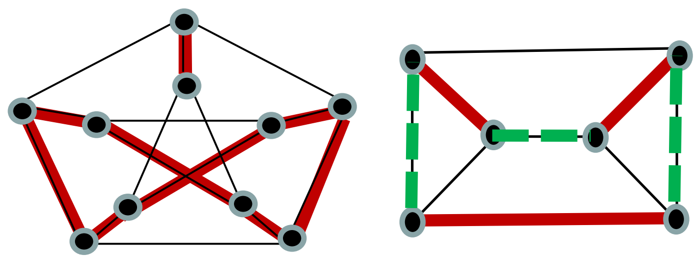
1. Triangles



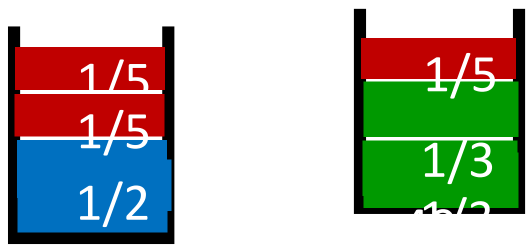
2. Stable sets



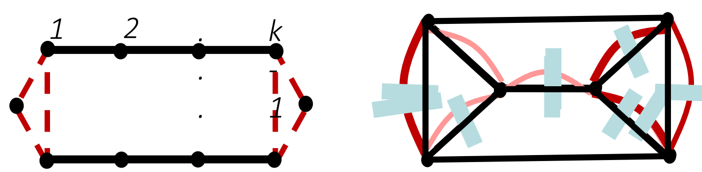
3. Matchings



4. Packed bins



5. Tour



INTEGRALITY PROPERTIES

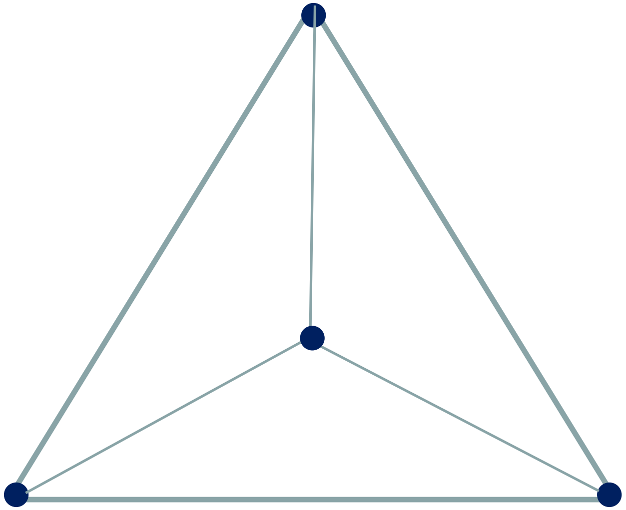
FOR COMBINATORIAL

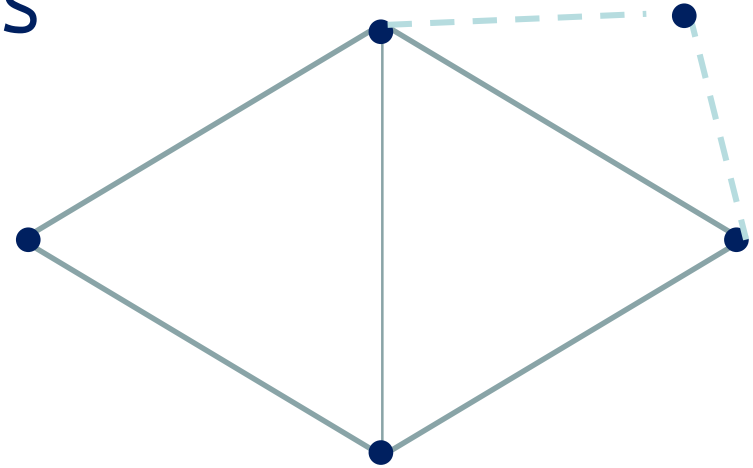
OBJECTS: Modified Integer

Round-UP Property

MIRUP

1. Triangles

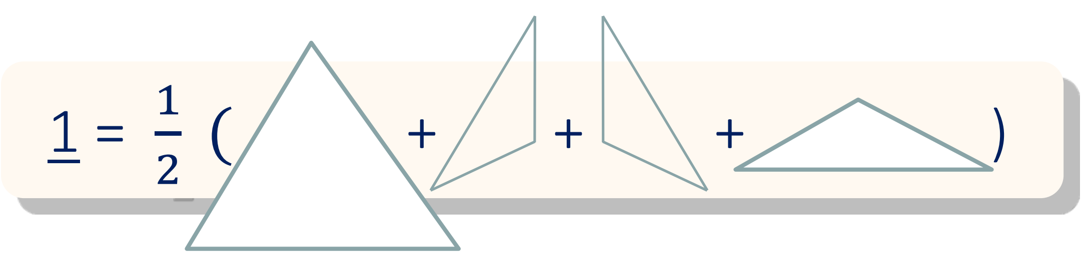




Can the *edges* be uniformly covered ?

$\Leftrightarrow$ Is $\underline{1} \in$ cone (triangles) ?

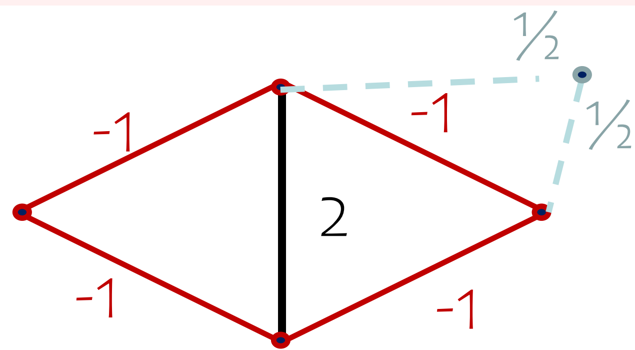
YES



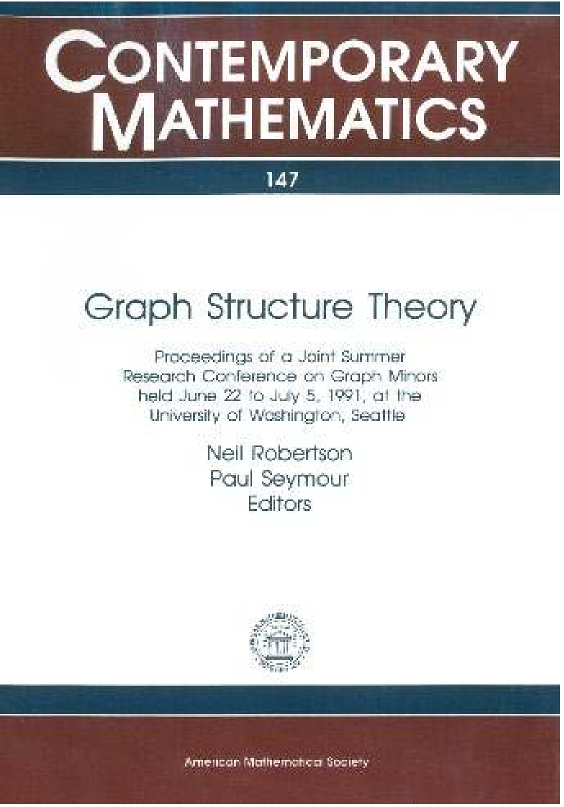
NO

$a^T x \geq 0$ feasible for all triangles is violated:

-1 -1 -1 -1 +2 + ½+½<0



Robertson and Seymour, Graph Minors" (Seattle 1991)



Svatopluk Poljak (1951-1995) asked a problem on "regular covers by triangles", subject studied by Milici and Tuza & als sollicitating Poljak. Traces 1994-2021.

Svetya on the campus of Saint Martin d'Hères, spring 1988

NATHANIEL DEAN

4. Coverings and Integer Flows

4.1. Triangles (Andras Sebő).

uniformly

OPEN PROBLEM 8 (S. POLJAK, A. SEBŐ, P. D. SEYMOUR). Characterize graphs which can be regularly edge-covered by triangles, that is for which there exists a set of triangles covering every edge of the graph the same positive number of times.

Equivalently, when is the all 1's function on the edges of a graph in the cone generated by the edge-characteristic vectors of triangles? More generally, what is the set of linear inequalities describing the cone of triangles (as edge-sets) of a graph? Namely, is the following conjecture true?

30 years old conjecture

Conjecture 9 (S. Poljak, A. Sebő, P. D. Seymour). Let $G = (V, E)$ be a graph. Then the cone generated by the edge-characteristic vectors of triangles is $\{x : v_H^T x \geq 0\}$, where $H = (V', E')$ is a triangle free subgraph of G, and

$$v_H(e) = \begin{cases} -1 & \text{if } e \in E(H) \\ 2 & \text{if } e = xy \in E(G) - E(H) \text{ and } x, y \in V' \\ 1/2 & \text{if } e \text{ has exactly one end in } V' \\ 0 & \text{otherwise.} \end{cases}$$

For a reference on cones of circuits see [35], and for more information on triangle covers see [22]. Svatya Poljak notes (personal communication) that for random graphs Problem 8 has already been settled since in this case the all 1's function is in the cone of cuts if and only if there is a regular triangle cover.

Open Problem 10 (A. Sebő). Let G be a graph. Is it true that for all $R \subseteq E(G)$ satisfying $|E - R| \leq |E \cap R|$ there exist edge-disjoint circuits each containing exactly one edge of R, if and only if there exists no R for which the same inequality holds and in addition the union of cuts for which equality holds contains an odd cut?

2. Stable sets (coloring)

Definition : G is *h-perfect* if

Max clique-size $\omega = 3$: *t-perfect*

STAB(G) = $\{x \in IR_+^{V(G)}$: $x(K) \leq 1\ \forall$ clique K & (*) holds$\}$ }

(*) $x(C) \leq (|C|-1)/2$ for every odd cycle

The **min sum of coeffs for $\underline{1} \in$ cone (stable sets)** = min dual for $\underline{1}$ on antiblocking polytope (P) :

x(S) ≤ 1 for each stable-set S, $x \geq 0$; its vertices: 0, char vectors of cliques, (2k+1)-circuits/k ≤ 2.5

$= \chi_f(G)$: *fractional chrom. number* ***in [2 ,2.5] if triangle-free, else*** $\boldsymbol{\omega}$

$\chi(G)$: *chromatic number = min integer dual for* (P)

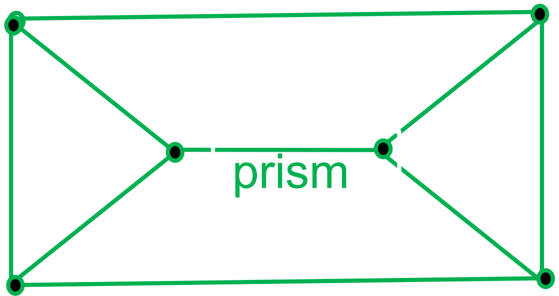
Integer decomp (ID)?
i.e. $\chi = \chi_f$?

Fact : If G is t-perfect, $\chi_f(G) \leq 3$.

Shepherd's (~90) conjecture : If G is t-perfect, $\chi(G) \leq 3$

Laurent and Seymour's counterexample (1994):

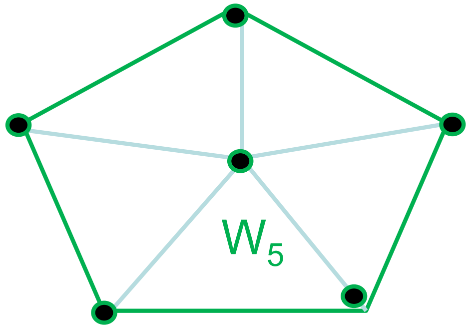
complement (linegraph(prism)):



$\chi_f = 3$, t-perfect, $\chi = 4$

Benchetrit's counterexample (2015) :

complement (linegraph(W_5))



Shepherd's conjecture: If G is t-perfect, $\chi(G) \leq 3$; but what about 4 instead of 3 ?

Conjecture (A.S. 1995) : Every h-perfect graph is $\omega(G)+1$ - colorable,

If this conjecture is true for $\omega(G) = 2$, then it is true in general.

MIRUP

Proof : Like Fulkerson, if $\omega \geq 3$, $\exists$ stable set meeting $\forall$ ω-clique.

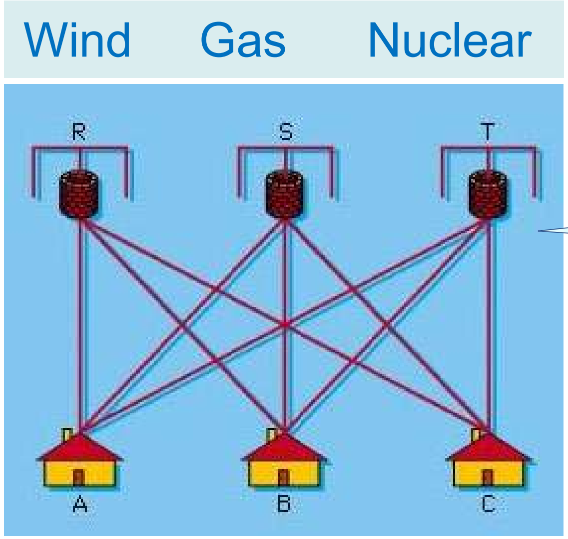
$x(S) \leq 1$ for each stable-set S, $x \geq 0$; its vertices: 0, char vectors of cliques, (2k+1)-circuits/k < 2.5

Related to Int. Decomp ; spec. cases Bruhn, Shaudt, Stein, Benchetrit '10-19)

Chi-boundedness, complementary-boundedness, Gyárfás &als, Conjectures of Ryser, Wegner …

3. Matchings

Set $M \subseteq E$ of pairwise vertex-disjoint edges
Perfect matching: covering all the vertices.



Exercise *: ∃ perfect matching which costs 1/3 of the total costs*

Solution 1: This graph is partitionable into 3 matchings (as any cubic bipartite graph by Kőnig's edge-coloring)

all degrees = 3

BIPARTITE MATCHING POLYTOPE (Birkhoff, 1946) : G=(V,E) bipartite. Then
conv (matchings) = $x \in \mathbb{R}^E$ satisfying $\sum_{e\in\delta(v)} x_e \le 1\ ,\ x \ge 0\ \ \forall v \in V$.

$\delta(v)$:= set of edges incident to v ; **Notation** : $x(\delta(v)) \le 1$

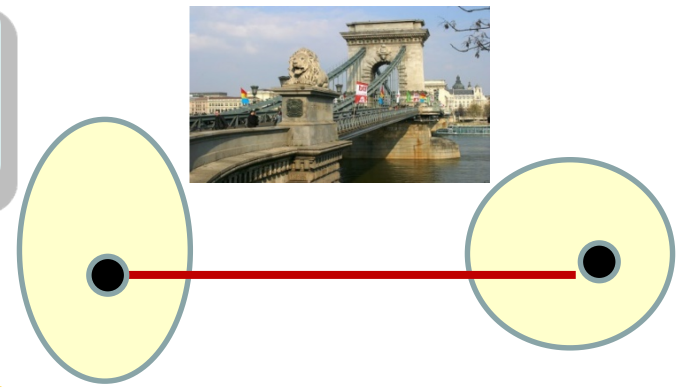
= weighted averages of matchings = mean value of matchings according to the set of probability distributions

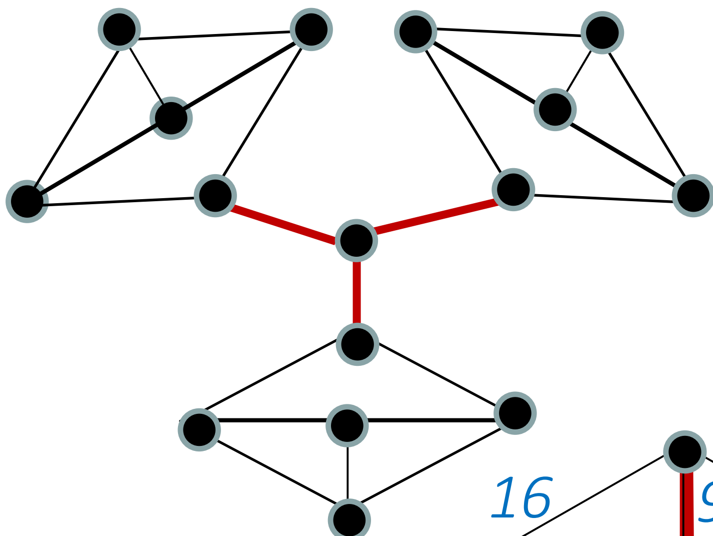
Solution 2: The constant 1/3 function on the edges is in conv (matchings)

Is this true for non-bipartite graphs ?

Theorem (Petersen 1991): G bridgeless, cubic

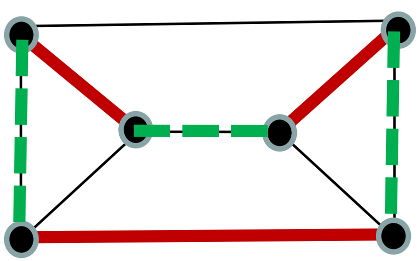
$\Rightarrow$ G has a perfect matching.

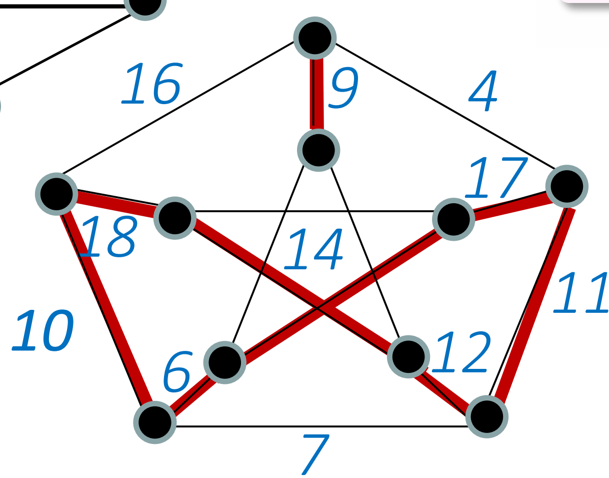




conv (perfect matchings) = $x \geq 0$, $x(\delta(v)) = 1$,
$x(\delta(U)) \geq 1$, $\forall U \subseteq V$, |U| odd (Edmonds, 1965)

So <u>1</u>/3 is still in conv (matchings) !





Generalized Petersen: G=(V,E)

bridgeless, cubic w: E → IR.

$\Rightarrow$ *∃ p.m of weight ≥ 1/3 w(E)*

Similarly: r-regular, $x(\delta(U)) \geq r$ if |U| odd (« r-graph ») $\Rightarrow$ *∃ p.m of weight ≥ 1/r w(E).*

Uniform covers by matchings

Chromatic index: Paritition to a min number of matchings

$\mathcal{M} = \mathcal{M}$ (G) set of all matchings of the graph G.

Incidence vector of M as edge-set

$\chi'(G) := \min \{ \sum_{M \in \mathcal{M}} \lambda_M : \sum_{M \in \mathcal{M}} \lambda_M \chi_M = \underline{1}, \quad \lambda_M \in \mathbb{N} \}$

$\chi'_{frac}(G) := \min \{ \sum_{M \in \mathcal{M}} \lambda_M : \sum_{M \in \mathcal{M}} \lambda_M \chi_M = \underline{1}, \quad \lambda_M \in \mathbb{R}_+ \}$

= min k : $\underline{1}/k$ ∈ MATCHING POLYTOPE

$x(\delta(v)) \leq 1 \ \forall v \in V, \ x(E(U)) \leq \frac{|U|-1}{2} \ \forall U \subseteq V, x \geq 0$

$k \geq \underline{1}(\delta(v))$ = degree of v $\forall v \in V, \quad k \geq \frac{1(E(U))}{\frac{1}{2}(|U|-1)} \quad \forall U \subseteq V, |U| > 1$, odd

The Goldberg-Seymour … *Conjecture* ?

Théorème (Vizing 1964) : *If G is a simple graph,*
$\chi'(G) \lesssim 1 +$ maximum degree of G

Max degree $\leq \chi'_{frac}(G) \leq$

Conjecture 1 : (Goldberg 1973, Seymour 1979) $\chi'(G) \leq \lceil \chi'_{frac}(G) \rceil + 1$,

MIRUP

r-regular and $|\delta(U)| \geq r$ for $|U|$ odd

$\Rightarrow$ *For an r-graphe $\chi'(G) = r+1$.* In 2019, “an alleged proof was announced by Chen, Jing, and Zang “ ; *Not yet fully checked .*

Conjecture 2 (Lovász 1987) : No Petersen minor $\Rightarrow \chi'(G) \leq \lceil \chi'_{frac}(G) \rceil$

For an r-graph without Petersen minor: $\chi'(G) = r$. (Contains the 4-col-thm.)

4. Bin packing – Cutting Stock

Variant Cutting Stock: $s_1, \ldots, s_d$
multiplicities $b_1, \ldots, b_d$

BIN PACKING bins of capacity 1

Input : $0 \le s_1, \ldots, s_n \le 1$ real numbers (object sizes)

Task : Minimise the number of bins

pattern : subset of {1, … , n} that fits into a bin

$\in \mathcal{P}$ with ellipsoids, knapsack separation

Can one hope for an integer y ?

Gilmore-Gomory LP of all « pattern inequalities » (1961-63)

$Px \le 1 \quad (P \in Z_+^{\text{big} \times d})$

$x \ge 0$

$\max \; 1^T x$

$yP \ge \underline{1}$

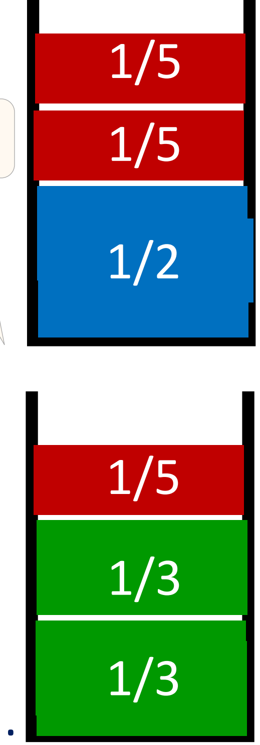
$y \ge 0$

$= \; \min y^T 1$

Examples

Marcotte (1985) : $s = (\frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$

patterns

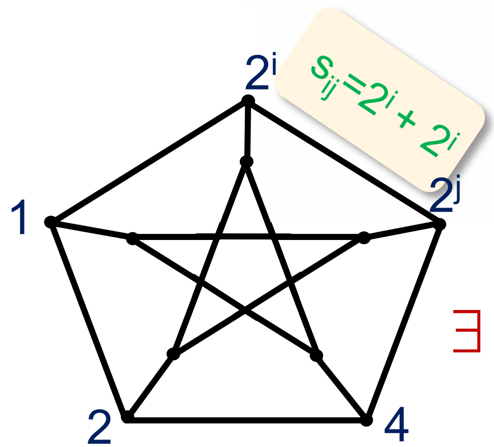


Total size = 59/30   LP = ½ + 2/3 + 4/5 = 59/30

primal   dual

OPT= 3

Rizzi (1997) :



Translates edge coloring:

Bins : 1+ 2+ … + 2^9

∃ k-edge-coloration ⇔ k bins suffice

MIRUP : Modified Integer Round Up

Conjecture : (Scheithauer, Terno, 1997) $\lceil LP \rceil \leq$ **OPT** $= \lceil LP \rceil + 1$

Partial results

MIRUP : Modified Integer Round Up

Conjecture : (Scheithauer, Terno, 1997) $\lceil LP \rceil \leq$ **OPT** $= \lceil LP \rceil + 1$

Karmarkar, Karp (1982) : *ADDITIVE ERROR* $log^2 d$ (AFPTAS)

Jansen, Solis-Oba (2010) : Si on fix d, OPT + 1 $\in \mathcal{P}$

d=2 & : **OPT =** $\lceil LP \rceil$ by McCormick, Smallwood, Spieksma ('93)

d=3,4,5,6 : MIRUP Scheithauer and Terno ('97)

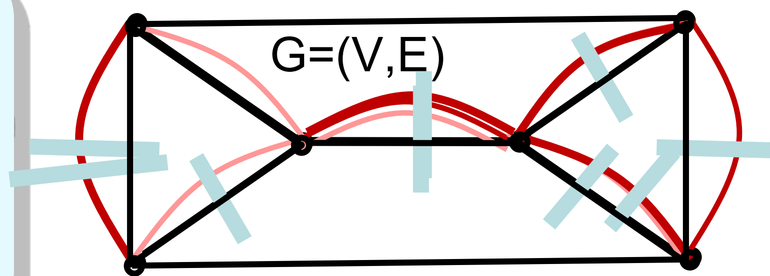
$d \leq 7$: + some general Lemmas for Conj (A.S., Shmonin '06)

For d = 3 $\in$ P ? For d = 8 MIRUP ? Relations between MIRUP problems?

More examples for MIRUP : Common independent sets of 2 matroids, … (Aharoni, Berger)

5. Uniform Covers by tours

Tour : spanning, Eulerian (conn, even deg) subgraph of 2E (0-1-2 vector)



(s,t) - tours : the degree of s, t is odd

Minimise w: E $\rightarrow$ IR_+ on a tour minimiser le poids d'un tour

Euler's theorem and shortcuts → Hamiltonian cycle

Hamiltonian cycle in thematric closure → tour in the original graph

3/2 LP for the TSP

Sharpening the « Christofides »-Serdyukov (1976) 3/2 approx :

Theorem (Wolsey '80, Cunningham'87, Shmoys-Williamson'90)

G=(V,E). If $x \in LP(G)$, then $3/2\, x$ is in the convex hull of tours.

$LP(G) := \{x \in \mathbb{R}_+^E : x(\delta(W)) \geq 2, \quad \text{for all } \emptyset \neq W \subset V\}$

Preuve : Let $x \in LP(G)$. From

x ~ convex combination of trees → random tree F, $E[F]= x$

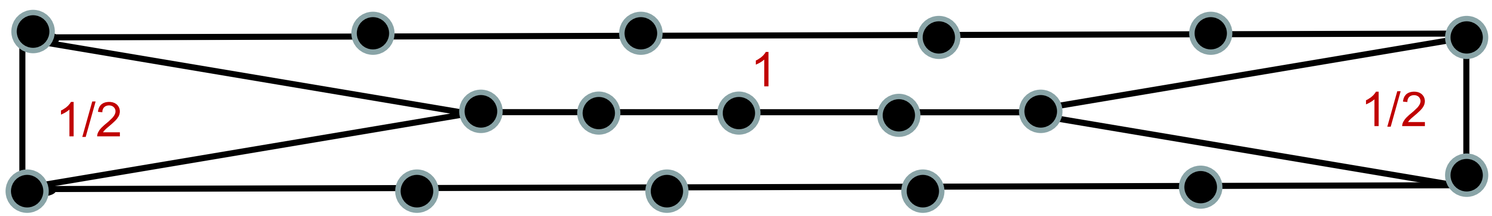
$x/2$ ~ convex combination of «fix-parity subgraphs» J_F, $E[J_F]=x/2$

$$E[F + J_F] = E[F] + E[J_F] = x + x/2 = 3/2\, x$$

2020-2021: 3/2-ε randomized, with $\varepsilon = 10^{-36}$ (Karlin, Klein, Gharan)

Conjectures

Conjecture 1 : (1976) G= (V,E) , x ∈ LP(G) ⇒ *4/3 x ∈ conv (tours}*



Theorem:(A.S. , Vygen, 2014) Pour les métriques distance, OPT ≤ 7/5 LP

Best Results : Haddadan, Newman, R.Ravi ('17-19), some improvement for "fund. Classes" Boyd, A.S. ('17-21)

Conjecture 2 : (A.S. 2015) G= (V,E) 3-edge-connected. Then

<u>8</u>/9 ∈ conv (t : t∈ {0,1,2}^{E} is a tour}

Proof from Conjecture 1 : <u>2</u>/3 *∈ subtour , so from Conjecture 1 :*

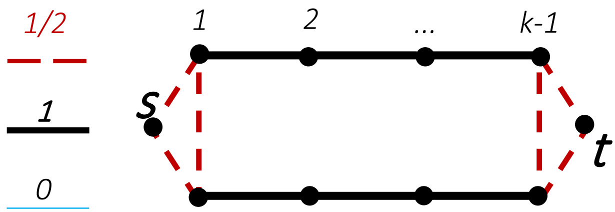
4/3 x <u>2</u>/3 = <u>8</u>/9 ∈ conv (tours}

Analogous s,t conjecture

Conjecture : $G=(V,E)$:
$x \in LP(s,t)$, Then
3/2 x ∈ conv (s,t tours}

$OPT_{LP}=n$

OPT=3/2 n



Theorem 1 (A.S. , Anke van Zuylen 2016) Every bridge separates s and t
$\underline{1} \in$ conv (t : t ∈ $\{0,1,2\}^E$ is the incidence vector of an (s,t)-tour}

Theorem 2 (A.S., Anke van Zuylen 2016) : $x \in P(V,s,t)$,

$$\left(\frac{3}{2}+\frac{1}{34}\right) x \quad \in \text{conv (s,t tours\}}$$

2018 : Rico Zenklusen beautiful 3/2 approximation, ignoring the IP
2019 : Vera Traub, Jens Vygen : slightly improved bound for Theorem 2.
2019 : Rico, Vera, Jens : α-approx for TSP ⇒∀ α'> α : α'-approx for paths

THE HAPPY END:

The talk is closed the problems remain open

Define **Cone (triangles as edge-sets)** with linear inequalities.

MIRUP for coloring h-perfect graphs, for edge-coloring, for bin packing

Conjecture : G= (V,E) 3-edge-connected. Then

$\underline{8}/9 \in \text{conv}\,(t \;:\; t \in \{0,1,2\}^E$ is a tour}.