Deriving cooperative games from non-cooperative ones

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April 20, 2010

Consider a set $N = \{1, \ldots, n\}$ of players with $n > 1$. A ***strategic game*** (or ***non-cooperative game***) for n players consists of

- a non-empty finite set C_i of ***strategies***,
- a ***payoff function*** $p_i : C_1 \times \ldots \times C_n \rightarrow \mathbb{R}$

for each player i.

We write then a strategic game as a sequence

$$(C_1, \ldots, C_n, p_1, \ldots, p_n).$$

The idea is that the players simultaneously choose a strategy and subsequently each player receives a payoff from the resulting joint strategy.

Given $s \in C_1 \times \ldots \times C_n$ we denote the ith element of s by s_i and given a subset $I := \{i_1, \ldots, i_m\}$ of N we abbreviate the sequence $(s_{i_1}, \ldots, s_{i_m})$ to s_I and $C_{i_1} \times \ldots \times C_{i_m}$ to C_I. Occasionally we write then $(s_I, s_{N \setminus I})$ instead of s.

As an example of a strategic game consider the well-known game called Scissors, Stone and Paper. In this game, often played by children, two players simultaneously make a sign with a hand that identifies one of these three objects. If both players make the same sign, the game is a draw. Otherwise one player wins 1 Euro from the other player according to the following rules:

- scissors defeat (cut) the paper,
- the paper defeats (wraps) the stone,

- the stone defeats (breaks) scissors.

This game is represented by the following payoff bimatrix:

		Two		
		Stone	Paper	Scissors
	Stone	0, 0	-1, 1	1, -1
One	Paper	1, -1	0, 0	-1, 1
	Scissors	-1, 1	1, -1	0, 0

So $p_{\texttt{One}}(\texttt{Stone}, \texttt{Paper}) = -1$, $p_{\texttt{Two}}(\texttt{Stone}, \texttt{Paper}) = 1$, etc.

Fix now a strategic game $G := (C_1, \ldots, C_n, p_1, \ldots, p_n)$. We first explain two natural ways that a TU-game can be derived from a strategic game. To start with, given a joint strategy s and a coalition $S \subseteq N = \{1, \ldots, n\}$ we define

$$p_S(s) := \sum_{i \in S} p_i(s).$$

So $p_S(s)$ is the aggregate payoff coalition S gets when players $1, \ldots, n$ respectively choose strategies $s_1, \ldots, s_n$.

Suppose now that the players in the coalition S chose the collective strategy s_S. Then the coalition S is guaranteed the aggregate payoff

$$\min_{s_{N \setminus S} \in C_{N \setminus S}} p_S(s_S, s_{N \setminus S}).$$

Having this in mind we define a TU-game (N, v^α) by putting for a coalition S:

$$v^\alpha(S) := \max_{s_S \in C_S} \min_{s_{N \setminus S} \in C_{N \setminus S}} p_S(s_S, s_{N \setminus S}).$$

Intuitively this means that if the players in the coalition S are allowed to choose their collective strategy first, then the coalition is guaranteed to achieve together $v^\alpha(S)$. Note that this definition adopts a pessimistic approach in that it is assumed that the coalition $N \setminus S$ will always try to choose a joint strategy that minimizes the collective payoff to coalition S.

Suppose now that given the coalition S, the players in the coalition $N \setminus S$ chose the collective strategy $s_{N \setminus S}$. Then the coalition S is guaranteed the aggregate payoff $\max_{s_S \in C_S} p_S(s_S, s_{N \setminus S})$. Having this in mind we define a TU-game (N, v^β) by putting for a coalition S:

$$v^\beta(S) := \min_{s_{N\setminus S} \in C_{N\setminus S}} \max_{s_S \in C_S} p_S(s_S, s_{N\setminus S}).$$

Intuitively this means that if the players in the coalition $N \setminus S$ are allowed to choose their collective strategy first, then the coalition S is guaranteed to achieve together $v^\beta(S)$.

Note that

$$v^\alpha(N) = v^\beta(N) = \max_{s \in C_N} p_N(s).$$

To compare these two definitions note first the following general result.

Lemma 1 *Consider a function $f : X \times Y \to \mathbb{R}$, where X and Y are finite sets. Then*

$$\max_{x \in X} \min_{y \in Y} f(x, y) \leq \min_{y \in Y} \max_{x \in X} f(x, y).$$

Proof. We have for all $x' \in X, y' \in Y$

$$\min_{y \in Y} f(x', y) \leq f(x', y') \leq \max_{x \in X} f(x, y').$$

So for all $y' \in Y$

$$\max_{x \in X} \min_{y \in Y} f(x, y) \leq \max_{x \in X} f(x, y')$$

and consequently

$$\max_{x \in X} \min_{y \in Y} f(x, y) \leq \min_{y \in Y} \max_{x \in X} f(x, y).$$

□

Theorem 2 *For all coalitions S we have $v^\alpha(S) \leq v^\beta(S)$.*

Proof. By Lemma 1. □

To see that the (N, v^α) and (N, v^β) TU-games can differ consider the following simple example.

Example 1 Take the following 2-persons game:

	L	R
T	1,0	0,1
B	0,1	1,0

Let us focus first on the singleton coalition consisting of player 1. If he moves first, he can guarantee at most payoff 0 to himself. Indeed, if he chooses T, then player 2 can choose R and if he chooses B, then player 2 can choose L. In both cases player 1 gets only 0. So $v^{\alpha}(\{1\}) = 0$. Analogously $v^{\alpha}(\{2\}) = 0$. Also $v^{\alpha}(\{1,2\}) = 1$.

On the other hand, if player 2 moves first, then player 1 can always guarantee payoff 1 to himself, by choosing T in response to L and B in response to R. So $v^{\beta}(\{1\}) = 1$. Analogously $v^{\beta}(\{2\}) = 1$ and $v^{\beta}(\{1,2\}) = 1$. □

The following general result will be useful in a moment.

Lemma 3 *Consider a function $f : X_1 \times X_2 \times X_3 \rightarrow \mathbb{R}$, where X_1, X_2 and X_3 are finite sets. Then*

$$\max_{x_1 \in X_1} \min_{(x_2,x_3) \in X_2 \times X_3} f(x_1, x_2, x_3) \leq \max_{(x_1,x_2) \in X_1 \times X_2} \min_{x_3 \in X_3} f(x_1, x_2, x_3).$$

Proof. Straightforward and omitted. □

Theorem 4 *The (N, v^{α}) TU-game is superadditive.*

Proof. By Lemma 3. □

In contrast, the (N, v^{β}) TU-game is not superadditive. Indeed, it suffices to take the v^{β} game from the above example.

Next, we discuss two analogous ways that an NTU-game can be derived from a strategic game.

We begin by repeating the choices made when modelling TU-games as NTU-games. So as the set of outcomes X we take the set of all allocations $\mathbb{R}^n$ and put for $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$

$$\mathbf{x} \succeq_i \mathbf{y} \text{ iff } \mathbf{x}_i \geq \mathbf{y}_i.$$

Consider now a coalition S of players. We say that $\mathbf{x} \in \mathbb{R}^n$ is ***assurable*** for S in the strategic game G if

$$\exists s_S \in C_S \; \forall s_{N \setminus S} \in C_{N \setminus S} \; \forall i \in S \; p_i(s_S, s_{N \setminus S}) \geq \mathbf{x}_i.$$

Intuitively this means that if the players in S are allowed to choose their strategies first, then they can always achieve in G the payoff at least as large as in the allocation $\mathbf{x}$.

Then we put

$$V^{\alpha}(S) := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \text{ is assurable for } S \text{ in } G\}.$$

So

$$V^{\alpha}(S) = \bigcup_{s_S \in C_S} \bigcap_{s_{N\setminus S} \in C_{N\setminus S}} \{\mathbf{x} \in \mathbb{R}^n \mid \forall i \in S \ p_i(s_S, s_{N\setminus S}) \geq \mathbf{x}_i\}.$$

Next, we say that $\mathbf{x} \in \mathbb{R}^n$ is ***unpreventable*** for S in G if

$$\forall s_{N\setminus S} \in C_{N\setminus S} \ \exists s_S \in C_S \ \forall i \in S \ p_i(s_S, s_{N\setminus S}) \geq \mathbf{x}_i.$$

Intuitively it means that if the players in $N \setminus S$ are allowed to choose their strategies first, then players in S can achieve in G the payoff at least as large as those in the allocation $\mathbf{x}$.

Then we put

$$V^{\beta}(S) := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \text{ is unpreventable for } S \text{ in } G\}.$$

So

$$V^{\beta}(S) = \bigcap_{s_{N\setminus S} \in C_{N\setminus S}} \bigcup_{s_S \in C_S} \{\mathbf{x} \in \mathbb{R}^n \mid \forall i \in S \ p_i(s_S, s_{N\setminus S}) \geq \mathbf{x}_i\}.$$

Note 5 *For all coalitions S, $V^{\alpha}(S) \subseteq V^{\beta}(S)$.*

Proof. By the fact that for each formula ϕ the implication

$$\exists x \forall y \ \phi(x, y) \rightarrow \forall y \exists x \ \phi(x, y)$$

holds. □

The (N, V^{α}) and (N, V^{β}) NTU-games can differ.

Example 2 Reconsider the 2-persons game from Example 1:

	L	R
T	1,0	0,1
B	0,1	1,0

We noticed already that if player 1 moves first, then he can guarantee at most payoff 0 to himself. So if $(x_1, x_2) \in V^\alpha(\{1\})$, then $x_1 \leq 0$. On the other hand, if player 2 moves first, then player 1 can always guarantee payoff 1 to himself, so $(1, 0) \in V^\beta(\{1\})$. □

To analyze so defined NTU-games we introduce the following adaptation of the notion of superadditivity to NTU-games.

We say that an NTU-game $(N, X, V, (\succeq_i)_{i \in N})$ is ***superadditive*** if for all disjoint coalitions S, T

$$V(S) \cap V(T) \subseteq V(S \cup T).$$

The following observation shows that this notion indeed generalizes it from the class of TU-games to NTU-games.

Note 6 *Consider a TU-game (N, v) and the corresponding NTU-game $(N, X, V, (\succeq_i)_{i \in N})$. Then (N, v) is superadditive iff $(N, X, V, (\succeq_i)_{i \in N})$ is superadditive.*

Proof.
$(\Rightarrow)$ Suppose (N, v) is superadditive. Take two disjoint coalitions S, T and $\mathbf{x} \in V(S) \cap V(T)$. Then $\sum_{i \in S} \mathbf{x}_i \leq v(S)$ and $\sum_{i \in T} \mathbf{x}_i \leq v(T)$, so $\sum_{i \in S \cup T} \mathbf{x}_i \leq v(S) + v(T)$. But by superadditivity $v(S) + v(T) \leq v(S \cup T)$. Hence $\mathbf{x} \in V(S \cup T)$.

$(\Leftarrow)$ Suppose $(N, X, V, (\succeq_i)_{i \in N})$ is superadditive. Take two disjoint coalitions S, T and $\mathbf{x} \in \mathbb{R}^n$ such that $\sum_{i \in S} \mathbf{x}_i = v(S)$ and $\sum_{i \in T} \mathbf{x}_i = v(T)$. Then $\mathbf{x} \in V(S) \cap V(T)$, so by superadditivity $\mathbf{x} \in V(S \cup T)$. So by definition $\sum_{i \in S \cup T} \mathbf{x}_i \leq v(S \cup T)$, i.e. $v(S) + v(T) \leq v(S \cup T)$. □

The following result then clarifies the status of the (N, V^α) NTU-game.

Theorem 7 *The NTU-game $(N, X, V^\alpha, (\succeq_i)_{i \in N})$ is superadditive.*

Proof. Given a coalition U and $\mathbf{x} \in \mathbb{R}^n$ we say that $s_U \in C_U$ *assures* $\mathbf{x}$ if

$$\forall s_{N \setminus U} \in C_{N \setminus U} \; \forall i \in U \; p_i(s_U, s_{N \setminus U}) \geq \mathbf{x}_i.$$

Consider two disjoint coalitions S, T and $\mathbf{x} \in V(S) \cap V(T)$. Choose $s_S \in C_S$ that assures $\mathbf{x}$ and $s_T \in C_T$ that assures $\mathbf{x}$. Then, since S and T are disjoint, $s_{S \cup T} \in C_{S \cup T}$ and $s_{S \cup T}$ assures $\mathbf{x}$ as well, so $\mathbf{x} \in V(S \cup T)$. □

Analogous result for the V^β NTU-game holds only for specific strategic games. We do not discuss the details here.