On the complexity of cooperative solution concepts and Computing Shapley values, manipulating value division schemes, and checking core membership in multi-issue domains

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On the complexity of cooperative solution concepts

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- Graph Games $\subseteq$ TU Games
- Games of this type are completely representable as a weighted graph
- The games that are representable are exactly the games that satisfy the condition that the value of a coalition is the sum of the value of each of the pairs in the coalition

The game is represented as a weighted (undirected, irreflexive) fully connected graph $G = \langle N, E, W \rangle$ with

- N a set of agents
- $E \subseteq N \times N$ a set of pairs of agents
- $W : E \to \mathbb{R}$ the weight function

The players are thus represented as vertices in the graph.

Definition (The coalition value)

The value of a coalition $S \subseteq N$ is represented as

$$v(S) = \sum_{e \in E \cap S^2} W(e)$$

(cf. the exercise regarding this)

Cons:

- The games are highly restrictive on v (e.g.:)
  - The value of a coalition is the sum of the value of each of the pairs in the coalition
  - All games are super additive
  - The agents are worth nothing alone, i.e. every singleton coalition has a value of zero

Pros:

- Size: We get rid of the need for explicitly defining 2^n values for v
  - We no longer have exponential input size (only $O(n^2)$)

Our goal:

- Polynomial time solution concepts?

Theorem

The Shapley value of agent i is

$$\phi(i) = \frac{1}{2} \times \sum_{i \neq j} W(i,j)$$

Proof.

Proven in assignment 3.3. □

Theorem

Computing the Shapley value is $O(n^2)$, where $n = |N|$ is the number of agents.

Proof.

The Shapley value of agent i has been shown equivalent to this representation:

$$\phi(i) = \frac{1}{2} \times \sum_{i \neq j} W(i,j)$$

Computing the value of $\phi(i)$ can be done by iterating over all edges, which yields that computing ϕ is $O(n^2)$, since $|E|$ is $O(|N|^2)$ (in fact $|E|$ is $\Theta(\frac{n^2-n}{2})$). □

The Core of v_G is defined (or can be reformulated) as the set of all imputations x such that $v(S) \leq x(S)$ for all $S \subseteq N$.

Theorem

If the Core is non-empty, the Shapley value is in the Core.

But we can make an even stronger theorem

Theorem

The following are equivalent

- *The Shapley value is in the Core*
- *The graph G has no* **negative cut**
- *The Core is non-empty*

Pf. The Shapley value is in the Core iff G has no negative cut.

- Let $e(S,x) = v(S) - x(S)$ be the excess of coalition S at the imputation x. It is easy to see that x is in the Core if and only if the excess of all coalitions $S \subseteq N$ is non-positive
- Moreover, the excess of S at the Shapley value, $e(S,\phi)$ is $-\frac{1}{2}$ times the weight of the edges going from S to $N \setminus S$
- The excess is thus half the weight of the *cut* $(S, N \setminus S)$
- Hence the Shapley value is in the Core if and only if there is no negative cut $(S, N \setminus S)$

□

Pf. The Core is nonempty iff G has no negative cut.

"$\Leftarrow$": If G has no negative cut, the Shapley value is in the Core (by the previous proof).
"$\Rightarrow$": By expanding definitions. □

NEGATIVE-CUT is NP-complete. We prove this by reducing WEIGHTED-MAX-CUT[1] to NEGATIVE-CUT to prove its NP-hardness. It is polynomial to test whether a candidate indeed is a solution.

Theorem

The following are NP-complete:

- *Given v_G and an imputation x, is it not in the core of v_G?*
- *Given v_G, is the Shapley value of v_G not in the core of v_G?*
- *Given v_G, is the core of v_G empty?*

[1] The weighted maximum cut problem is a well known NP-complete problem stated as follows: Given a graph G and an integer k, determine whether there is a cut of size at least k in G.

We can make explicit a (yet another) restriction on the set of games the gives a guarantee of non-emptiness of the Core and a very nice complexity result.

Theorem

When all weights of G are non-negative, the Shapley value is in the core of v_G, hence the Core is non-empty.

Proof.

When all edge weights are non-negative, there are no negative cuts. □

Theorem

When all weights of G are non-negative, we can test in polynomial time whether an imputation x is in the core of v_G.

Proof.

By reducing the problem to NETWORK-FLOW. □

Computing Shapley values, manipulating value division schemes, and checking core membership in multi-issue domains

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Multi-issue Valuation Functions

Definition (Multi-issue valuation functions)

A decomposition over T *issues of a valuation function* $v : 2^A \rightarrow \mathbb{R}$ *is a vector* $(v_1, v_2 \ldots v_T)$ *of valuation functions* $v_i : 2^N \rightarrow \mathbb{R}$ *such that:*

$$v(S) = \sum_{i=1}^{T} v_i(S) \text{ for all } S \subseteq N$$

Motivation

Often the task that the coalition undertakes consists of independent subtasks — issues — that require specific competences. Every agent has certain skills and can only address issues that match them.
Each issue has its own valuation function.

Simple Results

- If $v = \sum_{i=1}^{T} v_i$ is a decomposition of v and each v_i is monotonic (increasing), than v is monotonic.
- If $v = \sum_{i=1}^{T} v_i$ is a decomposition of v and each v_i is superadditive, than v is superadditive.

Advantages

- The decomposition can lead to a more concise representation if the individual v_i are concisely representable.
- May decrease the computational complexity in some cases.

Concerned Agents

Definition (Agents concerned with a given issue)

We say that v_i concerns only $C_i \subseteq N$ if:

$$v_i(S_1) = v_i(S_2) \text{ whenever } S_1 \cap C_i = S_2 \cap C_i$$

(or equivalently $v_i(S) = v_i(S \cup \{i\})$ for all $S \subseteq N$ and $i \notin C_i$)

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- We only need to specify $\sum_{i=1}^{T} 2^{|C_i|}$ values
- Shapley value is easier to compute

Observations

- If $v = \sum_{i=1}^{T} v_i$ is a decomposition of v, than

  $Sh_a(N, v) = \sum_{i=1}^{T} Sh_a(N, v_i)$ for any agent a.

- $Sh_a(N, v) = \frac{1}{|N|!} \sum_{\pi \in \prod(N)} mc_a(\pi) =$

  $$\sum_{S \subseteq N \setminus \{a\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} (v(S \cup \{a\}) - v(S))$$

- Computable in $O(2^{|N|})$

Lemma

Lemma

$Sh_a(N, v_i) = Sh_a(C_i, v_i)$ *for any agent* $a \in C_i$ *and* $Sh_a(N, v_i) = 0$ *for any agent* $a \notin C_i$

Proof.

$$Sh_a(N, v_i) = \frac{1}{|N|!} \sum_{\pi} mc_a(\pi) = \frac{1}{|N|!} \sum_{\pi_{C_i}} \sum_{\pi \Rightarrow \pi_{C_i}} mc_a(\pi_{C_i}) =$$

$$\frac{1}{|N|!} \sum_{\pi_{C_i}} \frac{|N|!}{|C_i|!} mc_a(\pi_{C_i}) = \frac{1}{|C_i|!} \sum_{\pi_{C_i}} mc_a(\pi_{C_i}) = Sh_a(C_i, v_i)$$

□

Theorem

Theorem

Let $v = \sum_{i=1}^{T} v_i$ *be a decomposition of* v*.*
Assuming that the factorials are precomputed and table look-ups for $v_i(S)$ *take constant time we can compute the Shapley value for a given agent in* $O(\sum_{i=1}^{T} 2^{|C_i|})$ *or less precisely* $O(T \cdot 2^{max|C_i|})$

MAX-MARGINAL-CONTRIBUTION

Let $v = \sum_{i=1}^{T} v_i$ be a decomposition of v and C_i be a set of agents such that for each i v_i concerns only C_i.

For a given agent $a \in N$ and $k \in \mathbb{R}$ is there a coalition $S \subseteq N \setminus \{a\}$ such that $v(S \cup \{a\}) - v(S) \geq k$?

MAX-MARGINAL-CONTRIBUTION is NP-complete

- Obviously NP — given a coalition we can easily compute the marginal contribution of *a*
- NP-complete — we reduce an arbitrary MAX-2-SAT [1] instance to MAX-MARGINAL-CONTRIBUTION with $|C_i| = 3$ and $Rng(v_i) = \{0, 1, 2\}$

[1] MAX-2-SAT is the following NP-hard problem: Given a number $r \in \mathbb{N}$ and a formula in conjunctive normal form such that each conjunct has exactly two literals (variable or negation of variable), is there an assignment for the variables that makes at least r clauses true?

The Reduction

- Take an agent a_v for each variable in the formula
- Take an agent a whose marginal contribution we will maximize
- For each clause we have an issue and C_i consists of the two agents for the variables in the clause and a
- Let for each i P_i be the set of agents for positive variables in clause i and N_i be the set for negated variables
- $k = r$

The Reduction 2

$$v_i(S) = \begin{cases} 0 & \exists n \in N_i(n \notin S) \& a \notin S \\ 1 & \exists n \in N_i(n \notin S) \& a \in S \\ 1 & N_i \subseteq S \wedge \neg\exists p \in P_i(p \in S) \\ 1 & N_i \subseteq S \wedge \exists p \in P_i(p \in S) \wedge a \notin S \\ 2 & N_i \subseteq S \wedge \exists p \in P_i(p \in S) \wedge a \in S \end{cases}$$

Notes

- In case of a convex valuation function it is obvious that the agent would want to be last! So the problem in this case is not NP-hard!
- Heuristics can help approximate it.

CHECK-IF-BLOCKED

Let $v = \sum_{i=1}^{T} v_i$ be a decomposition of v, C_i be a set of agents such that for each i v_i concerns only C_i and x be a payoff vector.

Is there a blocking coalition $S \subseteq N$?
(That is $v(S) > \sum_{a \in S} x(a)$)

CHECK-IF-BLOCKED is NP-complete

- Obviously NP — given a coalition we can easily check if it is blocking
- NP-complete — we reduce an arbitrary VERTEX-COVER [2] instance to CHECK-IF-BLOCKED with $|C_i| = 3$ and $Rng(v_i) = \{0, 1\}$

[2] VERTEX-COVER is the following NP-hard problem: Given a number $r \in \mathbb{N}$ and a graph $G = (V, E)$ is there a set of vetrices W such that $|W| < r$ and W covers all the edges in E?

The Reduction

- For every vertex $v \in V$ we introduce an agent a_v
- We introduce one additional agent a_0
- For every edge $e \in E$ we have an issue
- C_e for a given edge e consists of a_0 and the agents for the two ends of the edge
- $v_e(S) = 1$ if $a_0 \in S$ and at least one of the agents for the ends of the edge e is in S
  and $v_e(S) = 0$ otherwise
- $x(a_0) = T - \frac{1}{2}$ and $x(a_v) = \dfrac{1}{2(r + \frac{1}{2})}$

The Reduction 2

Let W be a solution to the VERTEX-COVER and $S = \{a_0\} \cup \{a_v | v \in W\}$, then $v_e(S) = 1$ for every issue e.

So $\sum_{a \in S} x(a) = T - \frac{1}{2} + |W| \cdot \frac{1}{2(r + \frac{1}{2})} \leq T - \frac{1}{2} + r \cdot \frac{1}{2(r + \frac{1}{2})} < T - \frac{1}{2} + r \frac{1}{2r} = T = v(S)$, so S is blocking.

The Reduction 3

Now let S be a blocking coalition, then $a_0 \in S$. Let $W = \{v | a_v \in S\}$. $\sum_{a \in S} x(a) = T - \frac{1}{2} + |W| \cdot \frac{1}{2(r + \frac{1}{2})}$ and $T \geq v(S) > \sum_{a \in S} x(a)$. Since $|W|$ is an integer, $|W| \leq r$.

Additionally we have $v(S) > x(a_0) = T$, so for every issue e we have $v_e(S) = 1$ and therefore W is a solution to the VERTEX-COVER problem.

Notes

This result implies that the core is maybe an unnecessary strong stability concept — if nobody can find a coalition that would benefit from breaking away, because it is computationally too difficult, then the grand coalition is still stable in practice.
Of course computational complexity is not a significant barrier if the instances are small enough.

Questions?

Thank you!