Cooperative Games

Lecture 1: Introduction

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Why study coalitional games?

Coalitional (or Cooperative) games are a branch of game theory in which **cooperation** or collaboration between agents can be modeled. Coalitional games can also be studied from a computational point of view (e.g., the problem of succint reprensentation and reasoning).

A coalition may represent a set of:

- persons or group of persons (labor unions, towns)
- objectives of an economic project
- artificial agents

We have a population N of n agents.

Definition (Coalition)

A **coalition** $\mathcal{C}$ is a set of agents: $\mathcal{C} \in 2^N$.

Two main classes of games

1- Games with Transferable Utility (TU games)

- Two agents can **compare** their utility
- Two agents can **transfer** some utility

Definition (valuation or characteristic function)

A *valuation function* v associates a real number $v(S)$ to any subset S, i.e., $v : 2^N \to \mathbb{R}$

Definition (TU game)

A TU game is a pair (N, v) where N is a set of agents and where v is a valuation function.

2- Games with Non Transferable Utility (NTU games)

It is **not** always possible to compare the utility of two agents or to transfer utility (e.g., no price tags). Agents have preference over coalitions.

Today

We provide some examples of TU games.

We discuss some desirable solution properties.

We end with a quick overview of the course and practicalities

Informal example: a task allocation problem

- A set of tasks requiring different expertises needs to be performed, tasks may be decomposed.
- Agents do not have enough resource on their own to perform a task.
- Find complementary agents to perform the tasks
  - robots have the ability to move objects in a plant, but multiple robots are required to move a heavy box.
  - transportation domain: agents are trucks, trains, airplanes, ships... a task is a good to be transported.
- **Issues:**
  - What coalition to form?
  - How to reward each each member when a task is completed?

Market games

A **market** is a quadruple (N, M, A, F) where

- N is a set of traders
- M is a set of m continuous good
- $A = (a_i)_{i \in N}$ is the initial endowment vector
- $F = (f_i)_{i \in N}$ is the valuation function vector

Assumptions of the model:

- The **utility** of agent i for possessing $x \in \mathbb{R}^m_+$ and an amount of money $p \in \mathbb{R}$ is $u_i(x, p) = f_i(x) + p$. The money models side payments.
- Initially, agents have **no money**.
- p_i can be **positive** or **negative** (like a bank account).
- Agents can increase their utility by **trading**: after a trade among the members of S, they have an endowment $(b_i)_{i \in S}$ and money $(p_i)_{i \in S}$ such that $\sum_{i \in S} a_i = \sum_{i \in b} b_i$ and $\sum_{i \in S} p_i = 0$.

Market games (cont.)

Definition (Market game)

A game (N,v) is a market game if there exists a market (N,M,A,F) such that, for every $S \subseteq N$,

$$v(S) = \max\left\{ \sum_{i\in S} f_i(x_i) \;\middle|\; x_i \in \mathbb{R}^m_+, \sum_{i\in S} x_i = \sum_{i\in S} a_i \right\}$$

Shapley. The solutions of a symmetric market game, in Contributions to the Theory of Games, Luce and Tuckers editors, 1959

Shapley and Shubik. On market games, *Journal of Economic Theory, 1, 9-25, 1969*

Cost allocation games

Definition (Cost allocation game)

A cost allocation game is a game (N,c) where

- N represents the potential customers of a public service or a public facility.
- $c(S)$ is the cost of serving the members of S

Mathematically speaking, a cost game is a game. The special status comes because of the different intuition (worth of a coalition vs. cost of a coalition).
We can associate a cost game with a "traditional game" using the corresponding saving game (N,v) given by

$$v(S) = \sum_{i \in S} c(\{i\}) - c(S).$$

Examples of cost allocation games

- **Sharing a water supply system:** n towns considers building a common water treatment facility. The cost of a coalition is the minimum cost of supplying the coalition members by the most efficient means.
- **Airport game:** n types of planes can land on a runway. The cost to accommodate a plane of type k is c_k. The cost is defined as $c(S) = \max_{k \in S}\{c_k\}$
- **Minimum cost spanning tree games:** a set H of houses have to be connected to a power plant P. The houses can be linked directly to P or to another house. The cost of connecting two locations $(i,j) \in H \cup \{P\}$ is c_{ij}. Let $S \subseteq H$. $\Gamma(S)$ is the minimum cost spanning tree spanning over the set of edges $S \cup \{P\}$. The cost function is $c(S) = \sum_{\text{all edges of } \Gamma(S)} c_{ij}$.

Simple or Voting games

Definition (voting games)

A game (N, v) is a **voting game** when

the valuation function takes two values

- 1 for a winning coalitions
- 0 for the losing coalitions

v satisfies *unanimity*: $v(N) = 1$

v satisfies *monotonicity*: $S \subset T \Rightarrow v(S) \leqslant v(T)$

Weighted Voting games

Definition (weighted voting games)

A game $(N, w_{i\in N}, q, v)$ is a **weighted voting game** when v satisfies unanimity, monotonicity and the valuation function is defined as

$$v(S) = \begin{cases} 1 \text{ when } \sum_{i\in S} w_i \geqslant q \\ 0 \text{ otherwise} \end{cases}$$

Example: 1958 European Economic Community: Belgium, Italy, France, Germany, Luxembourg and the Netherlands. Each country gets the following number of votes:

- Italy, France, Germany: 4
- Belgium, the Netherlands: 2
- Luxembourg: 1

The threshold of the game is $q = 12$.

Some types of TU games

$\forall \mathcal{C}_1, \mathcal{C}_2 \subseteq N \mid \mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$, $i \in N$, $i \notin \mathcal{C}_1$

- **additive (or inessential):** $v(\mathcal{C}_1 \cup \mathcal{C}_2) = v(\mathcal{C}_1) + v(\mathcal{C}_2)$
  trivial from the game theoretic point of view
- **superadditive:** $v(\mathcal{C}_1 \cup \mathcal{C}_2) \geqslant v(\mathcal{C}_1) + v(\mathcal{C}_2)$ satisfied in many applications: it is better to form larger coalitions.
- **weakly superadditive:** $v(\mathcal{C}_1 \cup \{i\}) \geqslant v(\mathcal{C}_1) + v(\{i\})$
- **subadditive:** $v(\mathcal{C}_1 \cup \mathcal{C}_2) \leqslant v(\mathcal{C}_1) + v(\mathcal{C}_2)$
- **convex:** $\forall \mathcal{C} \subseteq \mathcal{T}$ and $i \notin \mathcal{T}$,
  $v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \leqslant v(\mathcal{T} \cup \{i\}) - v(\mathcal{T})$.
  Convex game appears in some applications in game theory and have nice properties.
- **monotonic:** $\forall \mathcal{C} \subseteq \mathcal{T} \subseteq N$ $v(\mathcal{C}) \leqslant v(\mathcal{T})$.

The main problem

In the game (N,v) we want to form the **grand coalition**.

Each agent i will get a **personal payoff** x_i.

What are the interesting **properties** that x should satisfy?

How to **determine** the payoff vector x?

problem: a game (N,v) in which v is a worth of a coalition

solution: a payoff vector $x \in \mathbb{R}^n$

An example

$$N = \{1,2,3\}$$
$$v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0$$
$$v(\{1,2\}) = 90$$
$$v(\{1,3\}) = 80$$
$$v(\{2,3\}) = 70$$
$$v(\{1,2,3\}) = 105$$

What should we do?

- form $\{1,2,3\}$ and share equally $\langle 35,35,35\rangle$?
- 3 can say to 1 “let's form $\{1,3\}$ and share $\langle 40,0,40\rangle$”.
- 2 can say to 1 “let's form $\{1,2\}$ and share $\langle 45,45,0\rangle$”.
- 3 can say to 2 “OK, let's form $\{2,3\}$ and share $\langle 0,46,24\rangle$”.
- 1 can say to 2 and 3, “fine! $\{1,2,3\}$ and $\langle 33,47,25\rangle$
- ... is there a “good” solution?

Some properties

Let $x \in \mathbb{R}^n$ be a solution of the TU game (N, v)

Feasible solution: $\sum_{i \in N} x(i) \leqslant v(N)$

Anonymity: a solution is independent of the names of the player.

Definition (marginal contribution)

The **marginal contribution** of agent i for a coalition $\mathcal{C} \subseteq N \setminus \{i\}$ is $mc_i(\mathcal{C}) = v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})$.

Let mc_i^{min} and mc_i^{max} denote the minimal and maximal marginal contribution.

x is **reasonable from above** if $\forall i \in N \; x^i < mc_i^{max}$

➪ mc_i^{max} is the strongest **threat** that an agent can use against a coalition.

x is **reasonable from below** if $\forall i \in N \; x^i > mc_i^{min}$

➪ mc_i^{min} is a minimum acceptable reward.

Some properties

Let x, y be two solutions of a TU-game (N,v).

Efficiency: $x(N) = v(N)$

- ➪ the payoff distribution is an allocation of the entire worth of the grand coalition to all agents.

Individual rationality: $\forall i \in N, x(i) \geqslant v(\{i\})$

- ➪ agent obtains at least its self-value as payoff.

Group rationality: $\forall \mathcal{C} \subseteq N$, $\sum_{i \in \mathcal{C}} x(i) = v(\mathcal{C})$

- ➪ if $\sum_{i \in \mathcal{C}} x(i) < v(\mathcal{C})$ some utility is lost
- ➪ if $\sum_{i \in \mathcal{C}} x(i) > v(\mathcal{C})$ is not possible

Pareto Optimal: $\sum_{i \in N} x(i) = v(N)$

- ➪ no agent can improve its payoff without lowering the payoff of another agent.

An **imputation** is a payoff distribution x that is efficient and individually rational.

Summary

- Two main classes of games: TU games and NTU games
- Examples of TU games: market games, cost allocation games, voting games
- Some classes of TU games: superadditive, convex, etc.
- Some desirable properties of a solution

Coming next

A first solution concept to ensure stable coalitions: the **core**.

Definition (Core for superadditive games)

The **core** of a game (N, v) is the set:
$\{x \in \mathbb{R}^n \mid x(S) \geqslant v(S) \text{ for all } S \subseteq N\}$

Course overview

- Game theory stability concepts: the core, the kernel, the nucleolus, the Shapley value.
- Voting games and power indices
- Games with externalities
- NTU games
- Additional topics:
  - Coalitional Logic (ATL)
  - Issues raised by multiagent systems (seach for optimal CS, uncertainty, safety, manipulation, overlapping coalition).

Practicalities

- Webpage: `http://staff.science.uva.nl/~stephane/Teaching/CoopGames/2010/`
  It will contain the slides, posted shortly before class, and some handouts.
- Evaluation: 6ECTS $= 6 \cdot 28$h $= 168$h.
  - some homeworks (̃biweekly) ̃40% of the grade. LaTeXis preferred, but you can hand-write your solution.
  - final paper 50% of the grade (more details at the end of the first block)
  - final presentation 10% of the grade
- Attendance: not part of the grade.
- Announcement: MoL defense: Frank Nebel
  title: Shortest Path Games - Computational Complexity of Solution Concepts
  Date and time: Friday 5 February 2010, 14.00 hrs
  Location: Room A1.10