

Cooperative Games

Lecture 2: The core

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- **A problem:** Imagine you can do a project alone, or with friends. **Which** friend to choose?
 - Talented, hard working, easy to work with, etc.
 - But your friends are having the same reasoning.
- A condition for a coalition to form:
 - **all** agents prefer to be in it. i.e., none of the participants wishes she were in a different coalition or by herself \Rightarrow **Stability**
- Stability is a necessary but not sufficient condition, (e.g., there may be multiple stable coalitions).
- The **core** is a stability concepts where no agents prefer to deviate to form a different coalition.
- For simplicity, we will only consider the problem of the stability of the grand coalition:
 - \Rightarrow Is the grand coalition stable \Leftrightarrow Is the core non-empty

Today

- Definition of the core
- Some examples of computing the core for games with up to three agents
- Convex games and the core

Definition (valuation or characteristic function)

A *valuation function* v associates a real number $v(S)$ to any subset S , i.e., $v: 2^N \rightarrow \mathbb{R}$

Definition (TU game)

A TU game is a pair (N, v) where N is a set of agents and where v is a valuation function.

Definition (Imputation)

An **imputation** is a payoff distribution x that is efficient and individually rational, i.e.:

- $\sum_{i \in N} x_i = v(N)$ (efficiency)
- for all $i \in N$, $x_i \geq v(\{i\})$ (individual rationality)

Definition (Group rationality)

$\forall C \subseteq N$, $\sum_{i \in C} x_i \geq v(C)$

The core relates to the stability of the grand coalition:
No group of agents has any incentive to change coalition.

Definition (core of a game (N, v))

Let (N, v) be a TU game, and assume we form the grand coalition N .

The **core** of (N, v) is the set:

$$\text{Core}(N, v) = \{x \in \mathbb{R}^n \mid x \text{ is a group rational imputation}\}$$

Equivalently,

$$\text{Core}(N, v) = \{x \in \mathbb{R}^n \mid x(N) \leq v(N) \wedge x(C) \geq v(C) \forall C \subseteq N\}$$

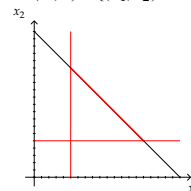
Example

$$N = \{1, 2\}$$

$$v(\{1\}) = 5, v(\{2\}) = 5$$

$$v(\{1, 2\}) = 20$$

$$\text{core}(N, v) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 5, x_2 \geq 5, x_1 + x_2 = 20\}$$



The core may not be fair: the core only considers stability.

three-player majority game

$$N = \{1, 2, 3\}$$

$$v(\{i\}) = 0$$

$$v(\{C\}) = \alpha \text{ for } |C| = 2$$

$$v(N) = 1$$

$$(x_1, x_2, x_3) \in \text{Core}(N, v) \Leftrightarrow \begin{cases} \forall i \in N, x_i \geq 0 \\ \forall (i, j) \in N^2, i \neq j, x_i + x_j \geq \alpha \\ \sum_{i \in N} x_i = 1 \end{cases}$$

$$\Leftrightarrow \begin{cases} \forall i \in N, 0 \leq x_i \leq 1 - \alpha & (1) \\ \sum_{i \in N} x_i = 1 & (2) \end{cases}$$

$\text{Core}(N, v)$ is nonempty iff $\alpha \leq \frac{2}{3}$
(by summing (1) for all $i \in N$ and using (2))

what happens when $\alpha > \frac{2}{3}$ and the core is empty?

Example with barycentric coordinate

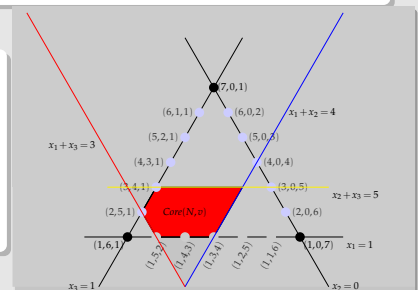
$$v(\emptyset) = 0 \quad v(\{1\}) = 1 \quad v(\{1, 2\}) = 4$$

$$v(\{2\}) = 0 \quad v(\{1, 3\}) = 3 \quad v(\{1, 2, 3\}) = 8$$

$$v(\{3\}) = 1 \quad v(\{2, 3\}) = 5$$

$$\text{set of imputations } J = \left\{ \sum_{i=1}^3 x_i = 8, x_1 \geq 1, x_2 \geq 0, x_3 \geq 1 \right\}$$

J is a triangle with vertices: $(7, 0, 1)$, $(1, 0, 7)$, $(1, 6, 1)$.
On the plane:
 $x_1 + x_2 + x_3 = 8$



Issues with the core

- The core may not always be non-empty
- When the core is not empty, it may not be 'fair'
- It may not be easy to compute
- ⇒ Are there classes of games that have a non-empty core?
- ⇒ Is it possible to characterize the games with non-empty core?

Definition (Convex games)

A game (N, v) is **convex** iff

$$\forall \mathcal{C} \subseteq \mathcal{J} \text{ and } i \notin \mathcal{J}, v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \leq v(\mathcal{J} \cup \{i\}) - v(\mathcal{J}).$$

TU-game is convex if the marginal contribution of each player increases with the size of the coalition he joins.

Bankruptcy game (E, c) $E \geq 0$ is the estate, there are n claimants and $c \in \mathbb{R}_+^n$ is the claim vector (c_i is the claim of the i^{th} claimant). $v(\mathcal{C}) = \max\{0, E - \sum_{i \in N \setminus \mathcal{C}} c_i\}$

Theorem

Each bankruptcy game is convex

Theorem

A TU game (N, v) is convex iff for all coalition S and T
 $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$

Theorem

A convex game has a non-empty core

Proof for convexity of a bankruptcy market

Proof for characterization of a convex game

Proof for non-emptiness of the core for convex games

Summary

- We introduced the core: a stability solution concept.
- We looked at some examples.
- We saw that the core can be empty.
- We proved that convex games have a non-empty core.

Coming next

- Characterization of games with non-empty core (Bondareva Shapley theorem), informal introduction to linear programming.
- Application of Bondareva-Shapley to market games.
- Other games with non-empty core.
- Computational complexity of the core.