Cooperative Games

Lecture 2: The core

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- **A problem:** Imagine you can do a project alone, or with friends. **Which** friend to choose?
  - Talented, hard working, easy to work with, etc.
  - But your friends are having the same reasoning.
- A condition for a coalition to form:

  all agents prefer to be in it.

  i.e., none of the participants wishes she were in a different coalition or by herself ➯ **Stability**
- Stability is a necessary but not sufficient condition, (e.g., there may be multiple stable coalitions).
- The **core** is a stability concepts where no agents prefer to deviate to form a different coalition.
- For simplicity, we will only consider the problem of the stability of the grand coalition:

➯ Is the grand coalition stable ⇔ Is the core non-empty

Today

- Definition of the core
- Some examples of computing the core for games with up to three agents
- Convex games and the core

Definition (valuation or characteristic function)

A *valuation function* v associates a real number $v(S)$ to any subset S, i.e., $v : 2^N \to \mathbb{R}$

Definition (TU game)

A TU game is a pair (N, v) where N is a set of agents and where v is a valuation function.

Definition (Imputation)

An **imputation** is a payoff distribution x that is efficient and individually rational, i.e.:

- $\sum_{i \in N} x_i = v(N)$ (efficiency)
- for all $i \in N$, $x_i \geqslant v(\{i\})$ (individual rationality)

Definition (Group rationality)

$\forall \mathcal{C} \subseteq N, \sum_{i \in \mathcal{C}} x(i) \geqslant v(\mathcal{C})$

The core relates to the stability of the grand coalition:
No group of agents has any incentive to change coalition.

Definition (*core* of a game (N,v))

Let (N,v) be a TU game, and assume we form the grand coalition N.
The **core** of (N,v) is the set:

$$Core(N,v) = \{x \in \mathbb{R}^n \mid x \text{ is a group rational imputation}\}$$

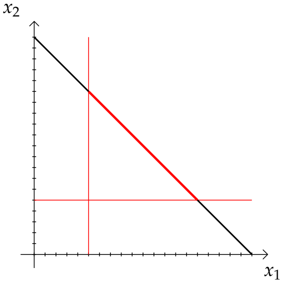
Equivalently,

$$Core(N,v) = \{x \in \mathbb{R}^n \mid x(N) \leqslant v(N) \wedge x(\mathcal{C}) \geqslant v(\mathcal{C}) \; \forall \mathcal{C} \subseteq N\}$$

Example

$$N = \{1,2\}$$
$$v(\{1\}) = 5,\ v(\{2\}) = 5$$
$$v(\{1,2\}) = 20$$

$$core(N,v) = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \geqslant 5, x_2 \geqslant 5, x_1 + x_2 = 20\}$$



The core may not be fair: the core only considers stability.

three-player majority game

$$N = \{1,2,3\}$$
$$v(\{i\}) = 0$$
$$v(\{\mathcal{C}\}) = \alpha \text{ for } |\mathcal{C}| = 2$$
$$v(N) = 1$$

$$(x_1, x_2, x_3) \in Core(N,v) \Leftrightarrow \begin{cases} \forall i \in N,\ x_i \geqslant 0 \\ \forall (i,j) \in N^2\ i \neq j,\ x_i + x_j \geqslant \alpha \\ \sum_{i \in N} x_i = 1 \end{cases}$$
$$\Leftrightarrow \begin{cases} \forall i \in N\ 0 \leqslant x_i \leqslant 1 - \alpha & (1) \\ \sum_{i \in N} x_i = 1 & (2) \end{cases}$$

$Core(N,v)$ is nonempty iff $\alpha \leqslant \frac{2}{3}$
(by summing (1) for all $i \in N$ and using (2))

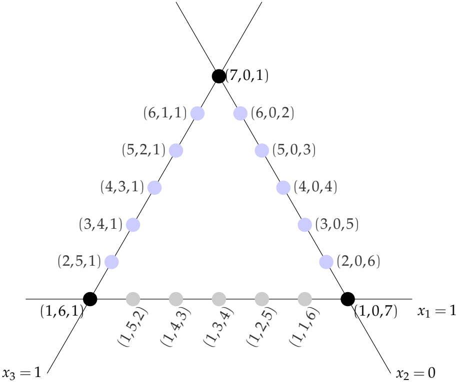
what happens when $\alpha > \frac{2}{3}$ and the core is empty?

Example with barycentric coordinate

$$
\begin{array}{llll}
 & v(\{1\}) = 1 & v(\{1,2\}) = 4 & \\
v(\emptyset) = 0 & v(\{2\}) = 0 & v(\{1,3\}) = 3 & v(\{1,2,3\}) = 8 \\
 & v(\{3\}) = 1 & v(\{2,3\}) = 5 &
\end{array}
$$

set of imputations $\mathcal{I} = \left\{ \sum_{i=1}^{3} x_i = 8, x_1 \geqslant 1, x_2 \geqslant 0, x_3 \geqslant 1 \right\}$

$\mathcal{I}$ is a triangle with vertices: $(7,0,1)$, $(1,6,1)$, $(1,0,7)$.
On the plane:
$x_1 + x_2 + x_3 = 8$



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$$v(\emptyset)=0 \quad \begin{array}{l} v(\{1\})=1 \\ v(\{2\})=0 \\ v(\{3\})=1 \end{array} \quad \begin{array}{l} v(\{1,2\})=4 \\ v(\{1,3\})=3 \\ v(\{2,3\})=5 \end{array} \quad v(\{1,2,3\})=8$$

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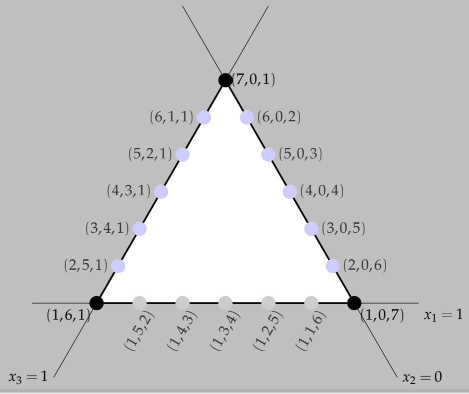

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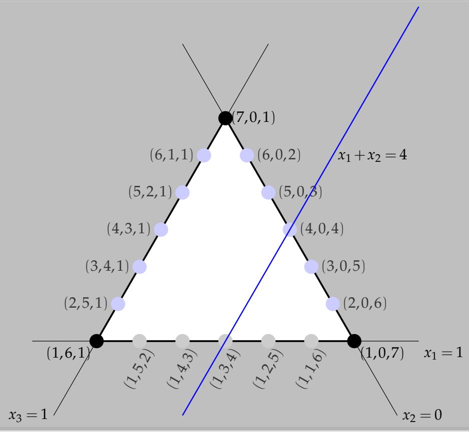

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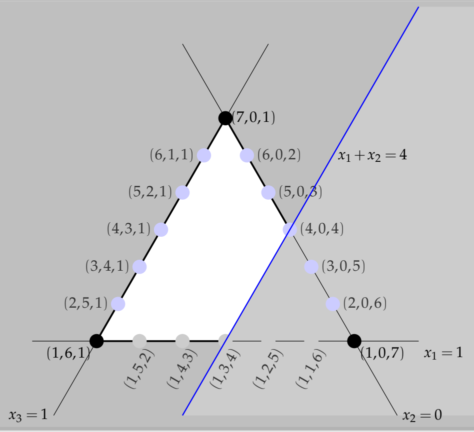

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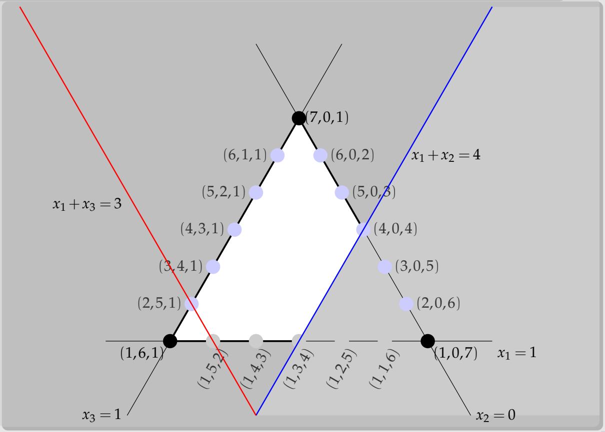

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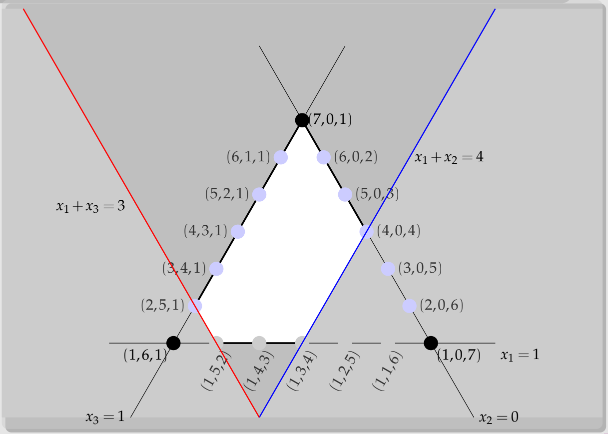

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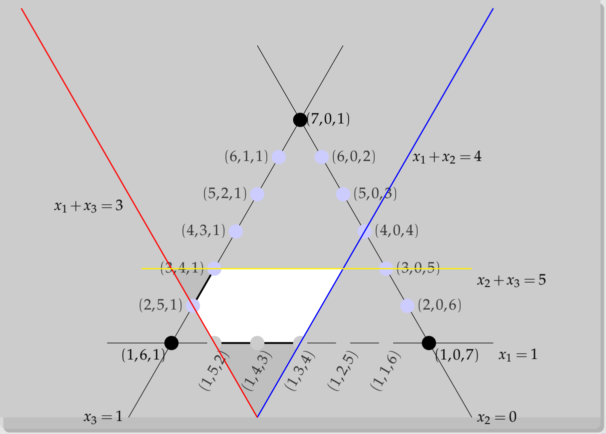

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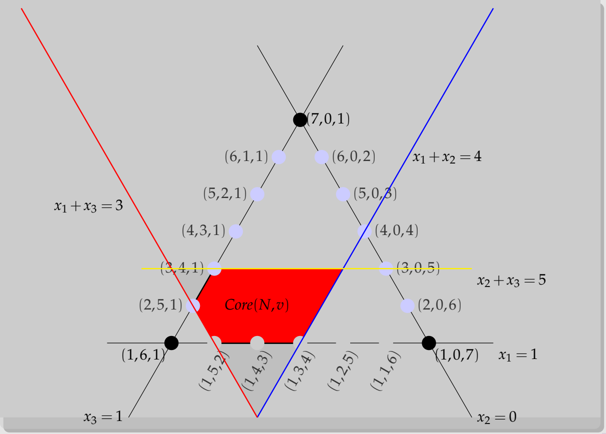

Issues with the core

- The core may not always be non-empty
- When the core is not empty, it may not be 'fair'
- It may not be easy to compute

➫ Are there classes of games that have a non-empty core?

➫ Is it possible to characterize the games with non-empty core?

Definition (Convex games)

A game (N,v) is **convex** iff
$\forall \mathcal{C} \subseteq \mathcal{T}$ and $i \notin \mathcal{T}$, $v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \leqslant v(\mathcal{T} \cup \{i\}) - v(\mathcal{T})$.

TU-game is convex if the marginal contribution of each player increases with the size of the coalition he joins.

Bankruptcy game (E,c) $E \geqslant 0$ is the estate, there are n claimants and $c \in \mathbb{R}_+^n$ is the claim vector (c_i is the claim of the i^{th} claimant). $v(\mathcal{C}) = \max\{0, E - \sum_{i \in N \setminus \mathcal{C}} c_i\}$

Theorem

Each bankruptcy game is convex

Theorem

A TU game (N,v) is convex iff for all coalition S and T
$v(S) + v(T) \leqslant v(S \cup T) + v(S \cap T)$

Theorem

A convex game has a non-empty core

Proof for convexity of a bankruptcy market

Proof for characterization of a convex game

Proof for non-emptyness of the core for convex games

Summary

- We introduced the core: a stability solution concept.
- We looked at some examples.
- We saw that the core can be empty.
- We proved that convex games have a non-empty core.

Coming next

- Characterization of games with non-empty core (Bondareva Shapley theorem), informal introduction to linear programming.
- Application of Bondareva-Shapley to market games.
- Other games with non-empty core.
- Computational complexity of the core.