

# Cooperative Games

## Lecture 3: The core

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Today

- Minimum cost spanning tree games have a non-empty core.
- Characterize the set of games with non-empty core (Bondareva Shapley theorem), and we will informally introduce linear programming.
- Application of the Bondareva Shapley theorem to market games.

### Minimum cost spanning tree games

- $N$  be the set of customers
- $0$  be the supplier
- $N_* = N \cup \{0\}$
- $c_{ij}$  is the cost of connecting  $i$  and  $j$  by the edge  $e_{ij}$  for  $(i,j) \in N_*^2, i \neq j$
- for a coalition  $\mathcal{C}$ ,  $T_{\mathcal{C}} = (\mathcal{C}, E_{\mathcal{C}})$  is the minimum cost spanning tree spanning over the set of edges  $\mathcal{C} \cup \{0\}$ .
- the cost function is  $c(S) = \sum_{(ij) \in E_{\mathcal{C}}} c_{ij}$
- A **minimum cost spanning tree game** is the associated cost game

#### Theorem

Every minimum cost spanning tree game has a non-empty core.

**The Bondareva Shapley theorem:**  
a characterization of games with non-empty core.

The theorem was proven independently by O. Bondareva (1963) and L. Shapley (1967).

Let  $\mathcal{C} \subseteq N$ . The **characteristic vector**  $\chi_{\mathcal{C}}$  of  $\mathcal{C}$  is the member of  $\mathbb{R}^N$  defined by  $\chi_{\mathcal{C}}^i = \begin{cases} 1 & \text{if } i \in \mathcal{C} \\ 0 & \text{if } i \in N \setminus \mathcal{C} \end{cases}$

A **map** is a function  $2^N \setminus \emptyset \rightarrow \mathbb{R}_+$  that gives a positive weight to each coalition.

#### Definition (Balanced map)

A function  $\lambda : 2^N \setminus \emptyset \rightarrow \mathbb{R}_+$  is a **balanced map** iff  $\sum_{\mathcal{C} \subseteq N} \lambda(\mathcal{C}) \chi_{\mathcal{C}} = \chi_N$

A map is balanced when the amount received over all the coalitions containing an agent  $i$  sums up to 1.

**Example:**  $n=3, \lambda(\mathcal{C}) = \begin{cases} \frac{1}{2} & \text{if } |\mathcal{C}|=2 \\ 0 & \text{otherwise} \end{cases}$

	1	2	3
{1,2}	$\frac{1}{2}$	$\frac{1}{2}$	0
{1,3}	$\frac{1}{2}$	0	$\frac{1}{2}$
{2,3}	0	$\frac{1}{2}$	$\frac{1}{2}$

Each of the column sums up to 1.

$$\frac{1}{2}\chi_{\{1,2\}} + \frac{1}{2}\chi_{\{1,3\}} + \frac{1}{2}\chi_{\{2,3\}} = \chi_{\{1,2,3\}}$$

### Characterization of games with non-empty core

#### Definition (Balanced game)

A game is **balanced** iff for each balanced map  $\lambda$  we have  $\sum_{\mathcal{C} \subseteq N, \mathcal{C} \neq \emptyset} \lambda(\mathcal{C}) v(\mathcal{C}) \leq v(N)$ .

#### Theorem (Bondareva Shapley)

A TU game has a non-empty core iff it is balanced.

### Some idea of the proof

#### Notations:

- Let  $\mathcal{V}(N) = \mathcal{V}$  the set of all coalition functions on  $2^N$ .
- Let  $\mathcal{V}_{Core} = \{v \in \mathcal{V} \mid \text{Core}(N, v) \neq \emptyset\}$ .

Can we characterize  $\mathcal{V}_{Core}$ ?

$\text{Core}(N, v) = \{x \in \mathbb{R}^n \mid x(\mathcal{C}) \geq v(\mathcal{C}) \text{ for all } \mathcal{C} \subseteq N\}$

The core is defined by a set of linear constraints.

→ The idea is to use results from linear optimization.

### Linear programming

A linear program has the following form:

$$\begin{cases} \max c^T x \\ \text{subject to } \begin{cases} Ax \leq b \\ x \geq 0 \end{cases} \end{cases}$$

- $x$  is a vector of  $n$  variables
- $c$  is the objective function
- $A$  is a  $m \times n$  matrix
- $b$  is a vector of size  $n$
- $A$  and  $b$  represent the linear constraints

**example:** maximize  $8x_1 + 10x_2 + 5x_3$   
subject to  $\begin{cases} 3x_1 + 4x_2 + 2x_3 \leq 7 & (1) \\ x_1 + x_2 + x_3 \leq 2 & (2) \end{cases}$   
 $A = \begin{pmatrix} 3 & 4 & 2 \\ 1 & 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}, c = \begin{pmatrix} 8 \\ 10 \\ 5 \end{pmatrix}$ .

A **feasible solution** is a solution that satisfies the constraints.

**Example:** maximize  $8x_1 + 10x_2 + 5x_3$

$$\text{subject to } \begin{cases} 3x_1 + 4x_2 + 2x_3 \leq 7 & (1) \\ x_1 + x_2 + x_3 \leq 2 & (2) \end{cases}$$

- $(0,1,1)$  is feasible, with objective function value 15.
- $(1,1,0)$  is feasible, with objective function value 18.

The **dual** of a LP: finding an upper bound to the objective function of the LP.

$$(1) \times 1 + (2) \times 6 \Leftrightarrow 9x_1 + 10x_2 + 8x_3 \leq 19$$

$$(1) \times 2 + (2) \times 2 \Leftrightarrow 8x_1 + 10x_2 + 6x_3 \leq 18$$

The coefficients are as large as in the objective function,

$\Rightarrow$  the bound is an upper bound for the objective function.

Hence, the solution cannot be better than 18, and we found one, Problem solved! ✓

Primal	Dual
$\begin{cases} \max c^T x \\ \text{subject to } \begin{cases} Ax \leq b, \\ x \geq 0 \end{cases} \end{cases}$	$\begin{cases} \min y^T b \\ \text{subject to } \begin{cases} y^T A \geq c^T, \\ y \geq 0 \end{cases} \end{cases}$

#### Theorem (Duality theorem)

When the primal and the dual are feasible, they have optimal solutions with equal value of their objective function.

### Linear Programming and the core

We consider the following **linear programming** problem:

$$(LP) \begin{cases} \min x(N) \\ \text{subject to } x(C) \geq v(C) \text{ for all } C \subseteq N, C \neq \emptyset \end{cases}$$

$v \in \mathcal{V}_{core}$  iff the value of (LP) is  $v(N)$ .

The dual of (LP):

$$(DLP) \begin{cases} \max \sum_{e \subseteq N} y_e v(e) \\ \text{subject to } \begin{cases} \sum_{e \subseteq N} y_e x_e = x_N \text{ and,} \\ y_e \geq 0 \text{ for all } e \subseteq N, e \neq \emptyset. \end{cases} \end{cases}$$

It follows from the duality theorem of linear programming:  $(N, v)$  has a non empty core iff  $v(N) \geq \sum_{e \subseteq N} y_e v(e)$  for all feasible vector  $(y_e)_{e \subseteq N}$  of (DLP).

Recognize the balance map in the constraint of (DLP)

### Application to Market Games

A **market** is a quadruple  $(N, M, A, F)$  where

- $N$  is a set of traders
- $M$  is a set of  $m$  continuous good
- $A = (a_i)_{i \in N}$  is the initial endowment vector
- $F = (f_i)_{i \in N}$  is the valuation function vector
- $v(S) = \max \left\{ \sum_{i \in S} f_i(x_i) \mid x_i \in \mathbb{R}_+^m, \sum_{i \in S} x_i = \sum_{i \in S} a_i \right\}$
- we further assume that the  $f_i$  are continuous and concave.

#### Theorem

Every Market Game is balanced

### Summary

- We introduced a stability solution concept: the core.
- we looked at examples:
  - individual games: some games have an empty core.
  - classes of games have a non-empty core: e.g. convex games and minimum cost spanning tree games.
- We look at a characterization of games with non-empty core: the Shapley Bondareva theorem, which relies on a result from linear programming.
- We Apply the Bondareva-Shapley to market games.

### Coming next

- Bargaining sets.