Cooperative Games

Lecture 3: The core

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Today

- Minimum cost spanning tree games have a non-empty core.
- Characterize the set of games with non-empty core (Bondareva Shapley theorem), and we will informally introduce linear programming.
- Application of the Bondareva Shapley theorem to market games.

Minimum cost spanning tree games

- N be the set of customers
- 0 be the supplier
- $N_* = N \cup \{0\}$
- $c_{i,j}$ is the cost of connecting i and j by the edge e_{ij} for $(i,j) \in N_*^2$, $i \neq j$
- for a coalition $\mathcal{C}$, $T_\mathcal{C} = (\mathcal{C}, E_\mathcal{C})$ is the minimum cost spanning tree spanning over the set of edges $\mathcal{C} \cup \{0\}$.
- the cost function is $c(S) = \sum_{(i,j) \in E_\mathcal{C}} c_{ij}$
- A **minimum cost spanning tree game** is the associated cost game

Theorem

Every minimum cost spanning tree game has a non-empty core.

The Bondareva Shapley theorem:
a characterization of games with non-empty core.

The theorem was proven independently by
O. Bondareva (1963) and L. Shapley (1967).

Let $\mathcal{C} \subseteq N$. The **characteristic vector** $\chi_{\mathcal{C}}$ of $\mathcal{C}$ is the member of $\mathbb{R}^N$ defined by $\chi^i_{\mathcal{C}} = \begin{cases} 1 \text{ if } i \in \mathcal{C} \\ 0 \text{ if } i \in N \setminus \mathcal{C} \end{cases}$

A **map** is a function $2^N \setminus \emptyset \to \mathbb{R}_+$ that gives a positive weight to each coalition.

Definition (Balanced map)

A function $\lambda : 2^N \setminus \emptyset \to \mathbb{R}_+$ is a **balanced map** iff $\sum_{\mathcal{C} \subseteq N} \lambda(\mathcal{C}) \chi_{\mathcal{C}} = \chi_N$

A map is balanced when the amount received over all the coalitions containing an agent i sums up to 1.

Example: $n = 3$, $\lambda(\mathcal{C}) = \begin{cases} \frac{1}{2} \text{ if } |\mathcal{C}| = 2 \\ 0 \text{ otherwise} \end{cases}$

	1	2	3
{1,2}	$\frac{1}{2}$	$\frac{1}{2}$	0
{1,3}	$\frac{1}{2}$	0	$\frac{1}{2}$
{2,3}	0	$\frac{1}{2}$	$\frac{1}{2}$

Each of the column sums up to 1.
$\frac{1}{2}\chi_{\{1,2\}} + \frac{1}{2}\chi_{\{1,3\}} + \frac{1}{2}\chi_{\{2,3\}} = \chi_{\{1,2,3\}}$

Characterization of games with non-empty core

Definition (Balanced game)

A game is **balanced** iff for each balanced map λ we have $\sum_{\mathcal{C} \subseteq N, \mathcal{C} \neq \emptyset} \lambda(\mathcal{C}) v(\mathcal{C}) \leqslant v(N)$.

Theorem (Bondareva Shapley)

A TU game has a non-empty core iff it is balanced.

Some idea of the proof

Notations:

- Let $\mathcal{V}(N) = \mathcal{V}$ the set of all coalition functions on 2^N.
- Let $\mathcal{V}_{Core} = \{v \in \mathcal{V} \mid Core(N,v) \neq \emptyset\}$.

Can we characterize $\mathcal{V}_{Core}$?

$Core(N,v) = \{x \in \mathbb{R}^n \mid x(\mathcal{C}) \geqslant v(\mathcal{C}) \text{ for all } \mathcal{C} \subseteq N\}$

The core is defined by a set of linear constraints.

➾ The idea is to use results from linear optimization.

Linear programming

A linear program has the following form:

$$\begin{cases} \max c^T x \\ \text{subject to} \begin{cases} Ax \leqslant b, \\ x \geqslant 0 \end{cases} \end{cases}$$

- x is a vector of n variables
- c is the objective function
- A is a $m \times n$ matrix
- b is a vector of size n
- A and b represent the linear constraints

example: maximize $8x_1 + 10x_2 + 5x_3$

$$\text{subject to} \begin{cases} 3x_1 & + & 4x_2 & + & 2x_3 & \leqslant & 7 & (1) \\ x_1 & + & x_2 & + & x_3 & \leqslant & 2 & (2) \end{cases}$$

$$A = \begin{pmatrix} 3 & 4 & 2 \\ 1 & 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}, c = \begin{pmatrix} 8 \\ 10 \\ 5 \end{pmatrix}.$$

A **feasible solution** is a solution that satisfies the constraints.

Example: maximize $8x_1 + 10x_2 + 5x_3$

$$\text{subject to} \begin{cases} 3x_1 + 4x_2 + 2x_3 \leqslant 7 & (1) \\ x_1 + x_2 + x_3 \leqslant 2 & (2) \end{cases}$$

- $\langle 0,1,1 \rangle$ is feasible, with objective function value 15.
- $\langle 1,1,0 \rangle$ is feasible, with objective function value 18.

The **dual** of a LP: finding an upper bound to the objective function of the LP.

$(1) \times 1 + (2) \times 6$ ➯ $9x_1 + 10x_2 + 8x_3 \leqslant 19$

$(1) \times 2 + (2) \times 2$ ➯ $8x_1 + 10x_2 + 6x_3 \leqslant 18$

The coefficients are as large as in the obective function,

➯ the bound is an upper bound for the objective function.

Hence, the solution cannot be better than 18, and we found one, Problem solved! ✔

Primal	Dual
$\begin{cases} \max c^T x \\ \text{subject to} \begin{cases} Ax \leqslant b, \\ x \geqslant 0 \end{cases} \end{cases}$	$\begin{cases} \min y^T b \\ \text{subject to} \begin{cases} y^T A \geqslant c^T, \\ y \geqslant 0 \end{cases} \end{cases}$

Theorem (Duality theorem)

When the primal and the dual are feasible, they have optimal solutions with equal value of their objective function.

Linear Programming and the core

We consider the following **linear programming** problem:

$$(LP) \begin{cases} \min x(N) \\ \text{subject to } x(\mathcal{C}) \geqslant v(\mathcal{C}) \text{ for all } \mathcal{C} \subseteq N, S \neq \emptyset \end{cases}$$

$v \in \mathcal{V}_{core}$ iff the value of (LP) is $v(N)$.

The dual of (LP):

$$(DLP) \begin{cases} \max \sum_{\mathcal{C} \subseteq N} y_\mathcal{C} v(\mathcal{C}) \\ \text{subject to } \begin{cases} \sum_{\mathcal{C} \subseteq N} y_\mathcal{C} \chi_\mathcal{C} = \chi_N \text{ and,} \\ y_\mathcal{C} \geqslant 0 \text{ for all } \mathcal{C} \subseteq N, \mathcal{C} \neq \emptyset. \end{cases} \end{cases}$$

It follows from the duality theorem of linear programming: (N,v) has a non empty core iff $v(N) \geqslant \sum_{\mathcal{C} \subseteq N} y_\mathcal{C} v(\mathcal{C})$ for all feasible vector $(y_\mathcal{C})_{\mathcal{C} \subseteq N}$ of (DLP).

Recognize the balance map in the constraint of (DLP)

Application to Market Games

A **market** is a quadruple (N, M, A, F) where

- N is a set of traders
- M is a set of m continuous good
- $A = (a_i)_{i \in N}$ is the initial endowment vector
- $F = (f_i)_{i \in N}$ is the valuation function vector
- $v(S) = \max \left\{ \sum_{i \in S} f_i(x_i) \;\middle|\; x_i \in \mathbb{R}^m_+, \sum_{i \in S} x_i = \sum_{i \in S} a_i \right\}$
- we further assume that the f_i are continuous and concave.

Theorem

Every Market Game is balanced

Summary

- We introduced a stability solution concept: the core.
- we looked at examples:
  - individual games: some games have an empty core.
  - classes of games have a non-empty core: e.g. convex games and minimum cost spanning tree games.
- We look at a characterization of games with non-empty core: the Shapley Bondareva theorem, which relies on a result from linear programming.
- We Apply the Bondareva-Shapley to market games.

Coming next

- Bargaining sets.