Cooperative Games

Lecture 4: Game with Coalition structures, Core and Bargaining Set

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Today

- If agents desire the kind of stability offered by the core, they will be unable to reach an agreement.
- ➩ they have no choice but to **relax** their stability requirements.
- Need a solution that allows agents to reach an agreement, but maintain some stability.
- First we will consider the problem of stability of coalition structure, i.e., a partition of the set of agents is formed first, and then, members of a coalition negotiate their payoff.
- Then, we will consider the bargaining set, which relaxes the requirements of the core.

Coalition Structure

Definition (Coalition Structure)

A **coalition structure (CS)** is a partition of the grand coalition into coalitions.
$\mathcal{S} = \{\mathcal{C}_1, \dots, \mathcal{C}_k\}$ where $\cup_{i \in \{1..k\}} \mathcal{C}_i = N$ and $i \neq j \Rightarrow \mathcal{C}_i \cap \mathcal{C}_j = \emptyset$.
We note $\mathscr{S}_N$ the set of all coalition structures over the set N.

ex: $\{\{1,3,4\}\{2,7\}\{5\}\{6,8\}\}$ is a coalition structure for $n = 8$ agents.

We will study three solution concepts: the **bargaining set**, the **nucleolus** and the **kernel**. They form the **"bargaining set family"** and we will see later why. In addition, the definition of each of these solution concepts uses a CS.

We start by defining a game with coalition structure, and see how we can define the core of such a game. Then, we'll start studying the bargaining set family.

Game with Coalition Structure

Definition (TU game)

A TU game is a pair (N, v) where N is a set of agents and where v is a valuation function.

Definition (Game with Coalition Structures)

A **TU-game with coalition structure** $(N, v, \mathcal{S})$ consists of a TU game (N, v) and a CS $\mathcal{S} \in \mathscr{S}_N$.

- We assume that the players agreed upon the formation of $\mathcal{S}$ and only the payoff distribution choice is left open.
- The CS may model affinities among agents, may be due to external causes (e.g. affinities based on locations).
- The agents may refer to the value of coalitions with agents outside their coalition (i.e. opportunities they would have outside of their coalition).
- (N, v) and $(N, v, \{N\})$ represent the same game.

Definition (*core* of a game (N,v))

The core of a TU game (N,v) is defined as
$Core(N,v) = \{x \in \mathbb{R}^n \mid x(N) \leqslant v(N) \wedge x(\mathcal{C}) \geqslant v(\mathcal{C})\ \forall \mathcal{C} \subseteq N\}$

The set of **feasible** payoff vectors for $(N,v,\mathcal{S})$ is
$X_{(N,v,\mathcal{S})} = \{x \in \mathbb{R}^n \mid \text{ for every } \mathcal{C} \in \mathcal{S}\ \ x(\mathcal{C}) \leqslant v(\mathcal{C})\}$.

Definition (Core of a game with CS)

The **core** $Core(N,v,\mathcal{S})$ of $(N,v,\mathcal{S})$ is defined by
$\{x \in \mathbb{R}^n \mid (\forall \mathcal{C} \in \mathcal{S}, x(\mathcal{C}) \leqslant v(\mathcal{C})) \text{ and } (\forall \mathcal{C} \subseteq N, x(\mathcal{C}) \geqslant v(\mathcal{C}))\}$

We have $Core(N,v,\{N\}) = Core(N,v)$.

The next theorems are due to Aumann and Drèze.

R.J. Aumann and J.H. Drèze. **Cooperative games with coalition structures**, *International Journal of Game Theory*, 1974

Definition (Superadditive cover)

The **superadditive cover** of (N,v) is the game $(N,\hat{v})$ defined by

$$\begin{cases} \hat{v}(\mathcal{C}) = \max_{\mathcal{P} \in \mathscr{S}_{\mathcal{C}}} \left\{ \sum_{T \in \mathcal{P}} v(T) \right\} \forall \mathcal{C} \subseteq N \setminus \emptyset \\ \hat{v}(\emptyset) = 0 \end{cases}$$

- We have $\hat{v}(N) = \max_{\mathcal{P} \in \mathscr{S}_N} \left\{ \sum_{T \in \mathcal{P}} v(T) \right\}$, i.e., $\hat{v}(N)$ is the maximum value that can be produced by N. We call it the **value of the optimal coalition structure**.
- The superadditive cover is a superadditive game (**why?**).

Theorem

Let $(N,v,\mathcal{S})$ be a game with coalition structure. Then

a) $Core(N,v,\mathcal{S}) \neq \emptyset$ iff $Core(N,\hat{v}) \neq \emptyset \wedge \hat{v}(N) = \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C})$

b) if $Core(N,v,\mathcal{S}) \neq \emptyset$, then $Core(N,v,\mathcal{S}) = Core(N,\hat{v})$

Proof of part **a)**

$\Rightarrow$ Let $x \in Core(N,v,\mathcal{S})$. We show that $x \in Core(N,\hat{v})$ as well.
Let $\mathcal{C} \subseteq N \setminus \emptyset$ and $P_{\mathcal{C}} \in \mathscr{S}_{\mathcal{C}}$ be a partition of $\mathcal{C}$.
By definition of the core, for every $S \subseteq N$ $x(S) \geqslant v(S)$.
$x(\mathcal{C}) = \sum_{i \in \mathcal{C}} x_i = \sum_{S \in P_{\mathcal{C}}} x(S) \geqslant \sum_{S \in P_{\mathcal{C}}} v(S)$, which is valid for all partitions of $\mathcal{C}$. Hence, $x(\mathcal{C}) \geqslant \max_{\mathcal{P}_{\mathcal{C}} \in \mathscr{S}_{\mathcal{C}}} \sum_{S \in \mathcal{P}_{\mathcal{C}}} v(S) = \hat{v}(\mathcal{C})$.

We have just proved $\forall \mathcal{C} \subseteq N \setminus \emptyset$, $x(\mathcal{C}) \geqslant \hat{v}(\mathcal{C})$, and so x is **group rational**. ✔

We now need to prove that $\hat{v}(N) = \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C})$.

$x(N) = \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C})$ since x is in the core of $(N,v,\mathcal{S})$ (efficient).
Applying the inequality above, we have
$x(N) = \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C}) \geqslant \hat{v}(N)$.
Applying the definition of the valuation function $\hat{v}$, we have $\hat{v}(N) \geqslant \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C})$.Consequently, $\hat{v}(N) = \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C})$ and it follows that x is **efficient** for the game $(N,\hat{v})$✔

Hence $x \in Core(N,\hat{v})$.

Proof of part **a)**

$\Leftarrow$ Let's assume $x \in Core(N, \hat{v})$ and $\hat{v}(N) = \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C})$.

We need to prove that $x \in Core(N, v, \mathcal{S})$.
For every $\mathcal{C} \subseteq N$, $x(\mathcal{C}) \geqslant \hat{v}(\mathcal{C})$ since x is in the core of $Core(N, \hat{v})$.Then $x(\mathcal{C}) \geqslant \max_{\mathcal{P}_\mathcal{C} \in \mathscr{S}_\mathcal{C}} \sum_{S \in \mathcal{P}_\mathcal{C}} v(S) \geqslant v(\mathcal{C})$ using $\{\mathcal{C}\}$ as a partition of $\mathcal{C}$.
This proves x is group rational. ✔
$x(N) = \hat{v}(N) = \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C})$ since x is efficient.
It follows that $\forall \mathcal{C} \in \mathcal{S}$, we must have $x(\mathcal{C}) = v(\mathcal{C})$,which proves x is feasible for the CS $\mathcal{S}$, and that x is efficient.✔
Hence, $x \in Core(N, v, \mathcal{S})$. ✔

- proof of part **b)**: we have just proved that $x \in Core(N, \hat{v})$ implies that $x \in Core(N, v, \mathcal{S})$ and $x \in Core(N, v, \mathcal{S})$ implies that $x \in Core(N, \hat{v})$. This proves that if $Core(N, v, \mathcal{S}) \neq \emptyset$, $Core(N, \hat{v}) = Core(N, v, \mathcal{S})$.

Definition (Substitutes)

Let (N,v) be a game and $(i,j) \in N^2$. Agents i and j are **substitutes** iff $\forall \mathcal{C} \subseteq N \setminus \{i,j\}$, $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$.

A nice property of the core related to fairness:

Theorem

Let $(N,v,\mathcal{S})$ be a game with coalition structure,
let i and j be substitutes, and let $x \in Core(N,v,\mathcal{S})$.
If i and j belong to different members of $\mathcal{S}$, then $x_i = x_j$.

Proof

Let $(i,j) \in N^2$ be substitutes, $\mathcal{C} \in \mathcal{S}$ such that $i \in \mathcal{C}$ and $j \notin \mathcal{C}$.
Let $x \in Core(N,v,\mathcal{S})$.
Since i and j are substitutes, we have

$$v((\mathcal{C} \setminus \{i\}) \cup \{j\}) = v((\mathcal{C} \setminus \{i\}) \cup \{i\}) = v(\mathcal{C}).$$

Since $x \in Core(N,v,\mathcal{S})$, we have $\forall \mathcal{C} \subseteq N$, $x(\mathcal{C}) \geqslant v(\mathcal{C})$, we apply this to $(\mathcal{C} \setminus \{i\}) \cup \{j\}$:
$0 \geqslant v((\mathcal{C} \setminus \{i\}) \cup \{j\}) - x((\mathcal{C} \setminus \{i\}) \cup \{j\}) = v(\mathcal{C}) - x(\mathcal{C}) + x_i - x_j$, since $\mathcal{C} \in \mathcal{S}$ and $x \in Core(N,v,\mathcal{S})$, $x(\mathcal{C}) = v(\mathcal{C})$ we have $x_j \geqslant x_i$. Since i and j play symmetric roles, we have $x_i = x_j$. ✔ □

A second solution concept:

The bargaining set.

R.J. Aumann and M. Maschler. **The bargaining set for cooperative games**, in *Advances in game theory (Annals of mathematics study)*, 1964.

M. Davis and M. Maschler. **Existence of stable payoff configurations for cooperative games**, *Bulletin of of the American mathematical society*, 1963.

Let $(N, v, \mathcal{S})$ be a game with coalition structure and x an imputation.
The bargaining set models stability in the following sense:

Any **argument** from an agent i against a payoff distribution x is of the following form:

> *I get too little in the imputation x, and agent j gets too much! I can form a coalition that excludes j in which some members benefit and all members are at least as well off as in x.*

The argument is **ineffective** for the bargaining set if agent j can answer the following:

> *I can form a coalition that excludes agent i in which all agents are at least as well off as in x, and as well off as in the payoff proposed by i for those who were offered to join i in the argument.*

Definition (Objection)

Let $(N,v,\mathcal{S})$ be a game with coalition structure, $x \in X_{(N,v,\mathcal{S})}$ (the set of all feasible payoff vectors for $(N,v,\mathcal{S})$), $\mathcal{C} \in \mathcal{S}$ be a coalition, and i and j two distinct members of $\mathcal{C}$ ($(i,j) \in \mathcal{C}^2$, $i \neq j$).

An **objection of i against** j is a pair (P,y) where

- $P \subseteq N$ is a coalition such that $i \in P$ and $j \notin P$.
- $y \in \mathbb{R}^p$ where p is the size of P
- $y(P) \leqslant v(P)$ (y is a feasible payoff distribution for the agents in P)
- $\forall k \in P$, $y_k \geqslant x_k$ and $y_i > x_i$ (agent i strictly benefits from y, and the other members of P do not do worse in y than in x.)

An objection (P,y) of i against j is a **potential threat** by coalition P, which contains i but not j, to deviate from x. The goal is not to change $\mathcal{S}$, but to obtain a side payment from j to i, i.e., to modify x within $X_{(N,v,\mathcal{S})}$.

Definition (Counter-objection)

An **counter-objection to** (P, y) is a pair (Q, z) where

- $Q \subseteq N$ is a coalition such that $j \in Q$ and $i \notin Q$.
- $z \in \mathbb{R}^q$ where q is the size of Q
- $z(Q) \leqslant v(Q)$ (z is a feasible payoff distribution for the agents in Q)
- $\forall k \in Q$, $z_k \geqslant x_k$ (the members of Q get at least the value in x)
- $\forall k \in Q \cap P$ $z_k \geqslant y_k$ (the members of Q which are also members of P get at least the value promised in the objection)

In a counter-objection, agent j must show that it can protect its payoff x_j in spite of the existing objection of i.

Definition (Stability)

Let $(N, v, \mathcal{S})$ a game with coalition structure. A vector $x \in X_{(N,v,\mathcal{S})}$ is **stable** iff for each objection at x there is a counter-objection.

Definition (Pre-bargaining set)

The **pre-bargaining set (*preBS*)** is the set of all stable members of $X_{(N,v,\mathcal{S})}$.

Lemma

Let $(N, v, \mathcal{S})$ a game with coalition structure, we have

$$Core(N, v, \mathcal{S}) \subseteq preBS(N, v, \mathcal{S}).$$

This is true since, if $x \in Core(N, v, \mathcal{S})$, no agent i has any objection against any other agent j.

Example

Let (N,v) be a 7-player simple majority game, i.e.

$$v(\mathcal{C}) = \begin{cases} 1 \text{ if } |\mathcal{C}| \geqslant 4 \\ 0 \text{ otherwise} \end{cases}.$$

Let us consider $x = \langle -\frac{1}{5}, \frac{1}{5}, \ldots, \frac{1}{5} \rangle$. It is clear that $x(N) = 1$.

Let us prove that **x is in the pre-bargaining set** of the game $(N,v,\{N\})$.

Objections within members of $\{2,3,4,5,6,7\}$ will have a counter-objection by symmetry. ✔

Let us consider the objections (P,y) of 1 against another member of $\{2,3,4,5,6,7\}$. Since the players $\{2,\ldots,7\}$ play symmetric roles, we consider an objection of 1 against 7 using successively $P = \{1,2,3,4,5,6\}$, $P = \{1,2,3,4,5\}$, $P = \{1,2,3,4\}$, $P = \{1,2,3\}$, $P = \{1,2\}$ and $P = \{1\}$.. We will look for a counter-objection of player 7 using (Q,z).

- We consider that $P = \{1,2,3,4,5,6\}$. We need to find the payoff vector $y \in \mathbb{R}^6$ so that (P, y) is an objection.
  $y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \ldots, \frac{1}{5} + \alpha_6 \rangle$,
  The conditions for (P, y) to be an objection are the following:
  - each agent is as well off as in x: $\alpha > -\frac{1}{5}$, $\alpha_i \geqslant 0$
  - y is feasible for coalition P: $\sum_{i=2}^{6} \left(\alpha_i + \frac{1}{5}\right) + \alpha \leqslant 1$.

  w.l.o.g $0 \leqslant \alpha_2 \leqslant \alpha_3 \leqslant \alpha_4 \leqslant \alpha_5 \leqslant \alpha_6$.

  Then $\displaystyle\sum_{i=2}^{6} \left(\frac{1}{5} + \alpha_i\right) + \alpha = \frac{5}{5} + \sum_{i=2}^{6} \alpha_i + \alpha = 1 + \sum_{i=2}^{6} \alpha_i + \alpha \leqslant 1.$

  Then $\displaystyle\sum_{i=2}^{6} \alpha_i \leqslant -\alpha < \frac{1}{5}.$

- ➯ We need to find a counter-objection for (P, y).
  claim: we can choose $Q = \{2,3,4,7\}$ and
  $z = \langle \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5} + \alpha_4, \frac{1}{5} + \alpha_5 \rangle$
  $z(Q) = \frac{1}{5} + \alpha_2 + \frac{1}{5} + \alpha_3 + \frac{1}{5} + \alpha_4 + \frac{1}{5} + \alpha_5 = \frac{4}{5} + \sum_{i=2}^{5} \alpha_i \leqslant 1$ since $\sum_{i=2}^{5} \alpha_i \leqslant \sum_{i=2}^{6} \alpha_i < \frac{1}{5}$ so z is feasible.
  It is clear that $\forall i \in Q$, $z_i \geqslant x_i$ ✔and that $\forall i \in Q \cap P$, $z_i \geqslant y_i$ ✔
  Hence, (Q, z) is a counter-objection. ✔

- Now, let us consider that $P = \{1,2,3,4,5\}$. The vector $y = \langle \alpha, \frac{1}{5}+\alpha_2, \frac{1}{5}+\alpha_3, \frac{1}{5}+\alpha_4, \frac{1}{5}+\alpha_5 \rangle$ is an objection when

  $\alpha > -\frac{1}{5}$, $\alpha_i \geqslant 0$, $\sum_{i=2}^{5}(\frac{1}{5}+\alpha_i)+\alpha \leqslant 1$

  This time, we have $\sum_{i=2}^{5}(\frac{1}{5}+\alpha_i)+\alpha = \frac{4}{5}+\sum_{i=2}^{5}\alpha_i+\alpha \leqslant 1$

  then $\sum_{i=2}^{5}\alpha_i \leqslant 1-\frac{4}{5}-\alpha = \frac{1}{5}-\alpha$ and finally $\sum_{i=2}^{5}\alpha_i \leqslant \frac{1}{5}-\alpha < \frac{2}{5}$.

➪ We need to find a counter-objection to (P,y)

claim: we can choose $Q = \{2,3,6,7\}$, $z = \langle \frac{1}{5}+\alpha_2, \frac{1}{5}+\alpha_3, \frac{1}{5}, \frac{1}{5} \rangle$

It is clear that $\forall i \in Q$, $z_i \geqslant x_i$ ✔and $\forall i \in P \cap Q$ $z_i \geqslant y_i$ (for agent 2 and 3).

$z(Q) = \frac{1}{5}+\alpha_2+\frac{1}{5}+\alpha_3+\frac{1}{5}+\frac{1}{5} = \frac{4}{5}+\alpha_2+\alpha_3$. We have $\alpha_2+\alpha_3 < \frac{1}{5}$, otherwise, we would have $\alpha_2+\alpha_3 \geqslant \frac{1}{5}$ and since the α_i are ordered, we would then have $\sum_{i=2}^{5}\alpha_i \geqslant \frac{2}{5}$, which is not possible.

Hence $z(Q) \leqslant 1$ which proves z is feasible ✔

Using similar arguments, we find a counter-objection for each other objections (you might want to fill in the details at home).

- $P = \{1,2,3,4\}$, $y = \langle \alpha, \frac{1}{5}+\alpha_1, \frac{1}{5}+\alpha_2, \frac{1}{5}+\alpha_3 \rangle$, $\alpha > -\frac{1}{5}$, $\alpha_i \geqslant 0$, $\sum_{i=2}^{4} \alpha_i + \alpha \leqslant \frac{2}{5} \Rightarrow \sum_{i=2}^{4} \alpha_i \leqslant \frac{2}{5} - \alpha < \frac{3}{5}$.

➡ $Q = \{2,5,6,7\}$, $z = \langle \frac{1}{5}+\alpha_2, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \rangle$ since $\alpha_2 \leqslant \frac{1}{5}$

- $|P| \leqslant 3$ $P = \{1,2,3\}$, $v(P) = 0$, $y = \langle \alpha, \alpha_1, \alpha_2 \rangle$, $\alpha > -\frac{1}{5}$, $\alpha_i \geqslant 0$, $\alpha_1 + \alpha_2 \leqslant -\alpha < \frac{1}{5}$

➡ $Q = \{4,5,6,7\}$, $z = \langle \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \rangle$ will be a counter argument (1 cannot provide more than $\frac{1}{5}$ to any other agent).

- For each possible objection of 1, we found a counter-objection. Using similar arguments, we can find a counter-objection to any objection of player 7 against player 1.

➡ $x \in preBS(N, v, \mathcal{S})$.✔

Bargaining set

In the example, agent 1 gets $-\frac{1}{5}$ when $v(\mathcal{C}) \geqslant 0$ for all coalition $\mathcal{C} \subseteq N$! This shows that the pre-bargaining set may **not** be individually rational.

Let $I(N,v,\mathcal{S}) = \left\{x \in X_{(N,v,\mathcal{S})} \mid x_i \geqslant v(\{i\}) \forall i \in N\right\}$ be the **set of individually rational payoff vector** in $X_{(N,v,\mathcal{S})}$.

Lemma

If a game is weakly superadditive, $I(N,v,\mathcal{S}) \neq \emptyset$.

Definition (Bargaining set)

Let $(N,v,\mathcal{S})$ a game in coalition structure.
The **bargaining set** (BS) is defined by

$$BS(N,v,\mathcal{S}) = I(N,v,\mathcal{S}) \cap preBS(N,v,\mathcal{S}).$$

Lemma

We have $Core(N,v,\mathcal{S}) \subseteq BS(N,v,\mathcal{S})$.

Theorem

Let $(N, v, \mathcal{S})$ a game with coalition structure. Assume that $I(N, v, \mathcal{S}) \neq \emptyset$. Then the bargaining set $BS(N, v, \mathcal{S}) \neq \emptyset$.

Proof

It is possible to give a direct proof of this theorem (a bit long, (Section 4.2 in **Introduction to the Theory of Cooperative Games**)).
We will show this result in a different way in the lecture about the nucleolus next week. □

B. Peleg and P. Sudhölter **Introduction to the Theory of Cooperative Games**, Springer, 2007.

Definition (weighted voting games)

A game $(N, w_{i\in N}, q, v)$ is a **weighted voting game** when v satisfies unanimity, monotonicity and the valuation function is defined as

$$v(S) = \begin{cases} 1 \text{ when } \sum_{i\in S} w_i \geqslant q \\ 0 \text{ otherwise} \end{cases}$$

We note such a game by $(q : w_1, \ldots, w_n)$

Let (N, v) be the game associated with the 6-player weighted majority game (3:1,1,1,1,1,0). Agent 6 is a null player since its weight is 0. Nevertheless $x = \langle \frac{1}{7}, \ldots, \frac{1}{7}, \frac{2}{7} \rangle \in BS(N, v)$.

Show it at home (a solution will be posted online). We need to consider all objections (P, y) from the an agent against the null agent, and find a counter objection (Q, z).

Agent 6 is a dummy, however, it receives a payoff of $\frac{2}{7}$, which is larger than agents who are not dummy!

Remember: $mc_i^{max} = \max_{\mathcal{C} \subseteq N \setminus \{i\}} v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})$

x is **reasonable from above** if $\forall i \in N \; x^i < mc_i^{max}$

- ➪ mc_i^{max} is the strongest **threat** that an agent can use against a coalition.

The bargaining set is not **Reasonable from above**: the dummy agent gets more than $\max_{\mathcal{C} \subseteq N \setminus \{6\}} (v(\mathcal{C} \cup \{6\}) - v(\mathcal{C})) = 0$.✘

Summary

- We introduced the notion of games with coalition structures.
- We looked at the definition of the core, i.e., stability of the coalition structure. Games with coalition structure may have an empty core
  (e.g., $(N,v,\{N\})$ and v in exercise 4 of homework 1).
- We introduced the bargaining set, and looked at some examples.

pros: it is guaranteed to be non-empty,
when the core is non-empty, it is contained in the bargaining set.

cons: it may not be reasonable from above.

Coming next

- We will consider the Nucleolus. It can also be defined in terms of objections and counter objections, but the nature of the objection is different from the bargaining set.