Cooperative Games

Lecture 5: The nucleolus

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Today

- We consider one way to compare two imputations.
- We define the Nucleolus and look at some properties.
- We prove important properties of the nucleolus, which requires some elements of analysis.

Excess of a coalition

Definition (Excess of a coalition)

Let (N, v) be a TU game, $\mathcal{C} \subseteq N$ be a coalition, and x be a payoff distribution over N. The **excess** $e(\mathcal{C}, x)$ of coalition $\mathcal{C}$ at x is the quantity $e(\mathcal{C}, x) = v(\mathcal{C}) - x(\mathcal{C})$.

An example: let $N = \{1,2,3\}$, $\mathcal{C} = \{1,2\}$, $v(\{1,2\}) = 8$, $x = \langle 3,2,5 \rangle$,
$e(\mathcal{C}, x) = v(\{1,2\}) - (x_1 + x_2) = 8 - (3+2) = 3$.

We can interpret a positive excess ($e(\mathcal{C}, x) \geqslant 0$) as the amount of **dissatisfaction** or **complaint** of the members of $\mathcal{C}$ from the allocation x.

We can use the excess to define the core:
$Core(N, v) = \{x \in \mathbb{R}^n \mid x \text{ is an imputation and } \forall \mathcal{C} \subseteq N, e(\mathcal{C}, x) \leqslant 0\}$

This definition shows that no coalition has any complaint: each coalition's demand can be granted.

$$N = \{1,2,3\},\ v(\{i\}) = 0 \text{ for } i \in \{1,2,3\}$$
$$v(\{1,2\}) = 5,\ v(\{1,3\}) = 6,\ v(\{2,3\}) = 6$$
$$v(N) = 8$$

Let us consider two payoff vectors $x = \langle 3,3,2 \rangle$ and $y = \langle 2,3,3 \rangle$.
Let $e(x)$ denote the sequence of **excesses** of all coalitions at x.

$x = \langle 3,3,2 \rangle$

coalition $\mathcal{C}$	$e(\mathcal{C},x)$
{1}	-3
{2}	-3
{3}	-2
{1,2}	-1
{1,3}	1
{2,3}	1
{1,2,3}	0

$y = \langle 2,3,3 \rangle$

coalition $\mathcal{C}$	$e(\mathcal{C},y)$
{1}	-2
{2}	-3
{3}	-3
{1,2}	0
{1,3}	1
{2,3}	0
{1,2,3}	0

Which payoff should we prefer? x or y? Let us write the excess in the decreasing order (from the greatest excess to the smallest)

$\langle 1,1,0,-1,-2,-3,-3 \rangle$ $\qquad$ $\langle 1,0,0,0,-2,-3,-3 \rangle$

Definition (lexicographic order of $\mathbb{R}^m$ $\geqslant_{lex}$)

Let $\geqslant_{lex}$ denote the **lexicographical** ordering of $\mathbb{R}^m$,
i.e., $\forall (x,y) \in \mathbb{R}^m$, $x \geqslant_{lex} y$ iff

$$\begin{cases} \text{x=y or} \\ \exists t \text{ s. t. } 1 \leqslant t \leqslant m \text{ s. t. } \forall i \text{ s. t. } 1 \leqslant i \leqslant t\ x_i = y_i \text{ and } x_t > y_t \end{cases}$$

example: $\langle 1,1,0,-1,-2,-3,-3\rangle \geqslant_{lex} \langle 1,0,0,0,-2,-3,-3\rangle$
Let l be a sequence of m reals. We denote by $l^{\blacktriangleright}$ the **reordering** of l in **decreasing** order.

In the example, $e(x) = \langle -3,-3,-2,-1,1,1,0\rangle$ and then $e(x)^{\blacktriangleright} = \langle 1,1,0,-1,-2,-3,-3\rangle$.

Hence, we can say that y is better than x by writing

$$e(x)^{\blacktriangleright} \geqslant_{lex} e(y)^{\blacktriangleright}.$$

Some properties of $\leqslant_{lex}$ and its strict version

- $\forall x \in \mathbb{R}^m \; x \leqslant_{lex} x$ ▶
- $\forall x \in \mathbb{R}^m$ and any permutation σ of $\{1,\dots,m\}$, $\sigma(x) \leqslant_{lex} x$ ▶
- $\forall x,y,u,v \in \mathbb{R}^m$ and $\alpha > 0$
  - $x \leqslant_{lex} y \Rightarrow \alpha x \leqslant_{lex} \alpha y$
  - $x <_{lex} y \Rightarrow \alpha x <_{lex} \alpha y$
  - $(x \leqslant_{lex} y \wedge u \leqslant_{lex} v) \Rightarrow x+u \leqslant_{lex} y+v$
  - $(x <_{lex} y \wedge u \leqslant_{lex} v) \Rightarrow x+u <_{lex} y+v$
  - $x \leqslant_{lex} y$ we **cannot** conclude anything for the comparison between $-\alpha x$ and $-\alpha y$.

Definition (Nucleolus)

Let (N, v) be a TU game.
Let $\mathcal{I}mp$ be the set of all imputations.
The **nucleolus** $Nu(N, v)$ is the set
$Nu(N, v) = \left\{ x \in \mathcal{I}mp \mid \forall y \in \mathcal{I}mp \;\; e(y)^{\blacktriangleright} \geqslant_{lex} e(x)^{\blacktriangleright} \right\}$

An alternative definition in terms of objections and counter-objections

Let (N,v) be a TU game. **Objections** are made by **coalitions** instead of individual agents. Let $P \subseteq N$ be a coalition that expresses an objection.

A pair (P,y), in which $P \subseteq N$ and y is an imputation, is an **objection** to x iff $e(P,x) > e(P,y)$.

> *Our excess for coalition P is too large at x, payoff y reduces it.*

A coalition (Q,y) is a **counter-objection** to the objection (P,y) when $e(Q,y) > e(Q,x)$ and $e(Q,y) \geqslant e(P,x)$.

> *Our excess under y is larger than it was under x for coalition Q! Furthermore, our excess at y is larger than what your excess was at x!*

An imputation fails to be stable if the excess of some coalition P can be reduced without increasing the excess of some other coalition to a level at least as large as that of the original excess of P.

Definition (Nucleolus)

Let (N, v) be a TU game. The **nucleolus** is the set of imputations x such that for every objection (P, y), there exists a counter-objection (Q, y).

M.J. Osborne and A. Rubinstein. **A course in game theory**, *MIT Press*, 1994, Section 14.3.3.

Theorem

Let (N,v) be a TU game with a non-empty core. Then $Nu(N,v) \subseteq Core(N,v)$

Proof

This will be part of homework 2 □

Theorem

Let (N, v) be a superadditive game and $\mathcal{I}mp$ be its set of imputations. Then, $\mathcal{I}mp \neq \emptyset$.

Proof

Let (N, v) be a superadditive game.
Let x be a payoff distribution defined as follows:
$x_i = v(\{i\}) + \frac{1}{|N|}\left(v(N) - \sum_{j\in N} v(\{j\})\right)$.

- $v(N) - \sum_{j\in N} v(\{j\}) > 0$ since (N, v) is superadditive.
- It is clear x is individually rational ✔
- It is clear x is efficient ✔

Hence, $x \in \mathcal{I}mp$. □

Theorem (Non-emptyness of the nucleolus)

Let (N, v) be a TU game, if $\mathcal{I}mp \neq \emptyset$,
then the nucleolus $Nu(N, v)$ is **non-empty**.

Element of Analysis

Let $E = \mathbb{R}^m$ and $X \subseteq E$. $||.||$ denote a distance in E, e.g., the euclidean distance.
We consider functions of the form $u : \mathbb{N} \rightarrow \mathbb{R}^m$. Another viewpoint on u is an infinite **sequence** of elements indexed by natural numbers $(u_0, u_1, \dots, u_k, \dots)$ where $u_i \in X$.

- **convergent sequence:** A sequence (u_t) converges to $l \in \mathbb{R}^m$ iff for all $\epsilon > 0$, $\exists T \in \mathbb{N}$ s.t. $\forall t \geqslant T$, $||u_t - l|| \leqslant \epsilon$.
- **extracted sequence:** Let (u_t) be an infinite sequence and $f : \mathbb{N} \rightarrow \mathbb{N}$ be a monotonically increasing function. The sequence v is extracted from u iff $v = u \circ f$, i.e., $v_t = u_{f(t)}$.
- **closed set:** a set X is closed if and only if it contains all of its limit points.
  i.e. for all converging sequences $(x_0, x_1 \dots)$ of elements in X, the limit of the sequence has to be in X as well.
  An example: if $X = (0,1]$, $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots, \frac{1}{n}, \dots)$ is a converging sequence. However, 0 is not in X, and hence, X is not closed.
  "A closed set contains its borders".

Element of Analysis

- **bounded set:** A subset $X \subseteq \mathbb{R}^m$ is **bounded** if it is contained in a ball of finite radius, i.e. $\exists c \in \mathbb{R}^m$ and $\exists r \in \mathbb{R}^+$ s.t. $\forall x \in X \; ||x - c|| \leqslant r$.
- **compact set:** A subset $X \subseteq \mathbb{R}^m$ is a **compact** set iff from all sequences in X, we can extract a convergent sequence in X.

⇨ A set is **compact** set of $\mathbb{R}^m$ iff it is **closed** and **bounded**.

- **convex set:** A set X is convex iff $\forall (x,y) \in X^2$, $\forall \alpha \in [0,1]$, $\alpha x + (1-\alpha)y \in X$ (i.e. all points in a line from x to y is contained in X).
- **continuous function:** Let $X \subseteq \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^m$.
  f is **continuous at $x_0 \in X$** iff $\forall \epsilon \in \mathbb{R}$, $\epsilon > 0$, $\exists \delta \in \mathbb{R}$, $\delta > 0$ s.t. $\forall x \in X$ s.t. $||x - x_0|| < \delta$, we have $||f(x) - f(x_o)|| < \epsilon$, i.e.
  $\forall \epsilon > 0 \;\; \exists \delta > 0 \;\; \forall x \in X \quad ||x - x_0|| < \delta \Rightarrow ||f(x) - f(x_0)|| < \epsilon$.

Let $X \subseteq \mathbb{R}^n$.

Thm A_1 If $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous and $X \subseteq E$ is a non-empty compact subset of $\mathbb{R}^n$,
then $f(X)$ is a non-empty compact subset of $\mathbb{R}^m$.

Thm A_2 Extreme value theorem: Let X be a non-empty compact subset of $\mathbb{R}^n$, $f : X \to \mathbb{R}$ a **continuous** function.
Then f is bounded and it reaches its supremum.

Thm A_3 Let X be a non-empty compact subset of $\mathbb{R}^n$. $f : X \to \mathbb{R}$ is continuous iff for every closed subset $B \subseteq \mathbb{R}$, the set $f^{-1}(B)$ is compact.

Proof of non-emptyness of the nucleolus

Assume we have the following theorems 1 and 2 (we will prove them in the next slide).

Theorem (1)

Let A be a non-empty compact subset of $\mathbb{R}^m$.
$\{x \in A \mid \forall y \in A \; x \leqslant_{lex} y\}$ is non-empty.

Theorem (2)

Assume we have a TU game (N,v), and consider its set $\mathcal{I}mp$.
If $\mathcal{I}mp \neq \emptyset$, then set $B = \{e(x)^{\blacktriangleright} \mid x \in \mathcal{I}mp\}$ is a non-empty compact subset of $\mathbb{R}^{2^{|N|}}$

Let us take a TU game (N,v) and let us assume $\mathcal{I}mp \neq \emptyset$. Then B in theorem 2 is a non-empty compact subset of $\mathbb{R}^{2^{|N|}}$. Now let A in theorem 1 be B in theorem 2. So
$\{e(x)^{\blacktriangleright} \mid (x \in \mathcal{I}mp) \wedge (\forall y \in \mathcal{I}mp \; e(x)^{\blacktriangleright} \leqslant_{lex} e(y)^{\blacktriangleright})\}$ is non-empty. From this, it follows that:
$Nu(N,v) = \left\{x \in \mathcal{I}mp \mid \forall y \in \mathcal{I}mp \; e(y)^{\blacktriangleright} \geqslant_{lex} e(x)^{\blacktriangleright}\right\} \neq \emptyset$. ✔

Proof of theorem 2

Let (N,v) be a TU game and consider its set $\mathcal{I}mp$. Let us assume that $\mathcal{I}mp \neq \emptyset$ to prove that $B = \{e(x)^{\blacktriangleright} \mid x \in \mathcal{I}mp\}$ is a non-empty compact subset of $\mathbb{R}^{2^{|N|}}$.

First, let us prove that $\mathcal{I}mp$ is a non-empty compact subset of $\mathbb{R}^{|N|}$.

- $\mathcal{I}mp$ non-empty by assumption.
- To see that $\mathcal{I}mp$ is bounded, we need to show that for all i, x_i is bounded by some constant (independent of x). We have $v(\{i\}) \leqslant x_i$ (ind. rational) and $x(N) = v(N)$ (efficient). Then $x_i + \sum_{j=1, j\neq i}^{n} v(\{j\}) \leqslant v(N)$, hence $x_i \leqslant v(N) - \sum_{j=1, j\neq i}^{n} v(\{j\})$.
- $\mathcal{I}mp$ is closed (the boundaries of $\mathcal{I}mp$ are members of $\mathcal{I}mp$).

This proves that $\mathcal{I}mp$ is a non-empty compact subset of $\mathbb{R}^{|N|}$.

Thm A$_1$ If $f : E \longrightarrow \mathbb{R}^m$ is continuous, $X \subseteq E$ is a non-empty compact subset of $\mathbb{R}^n$, then $f(X)$ is a non-empty compact subset of $\mathbb{R}^m$.

$e()^{\blacktriangleright}$ is a continuous function and $\mathcal{I}mp$ is a non-empty and compact subset of $\mathbb{R}^{2^{|N|}}$. Using thm A$_1$, $e(\mathcal{I}mp)^{\blacktriangleright} = \{e(x)^{\blacktriangleright} | x \in \mathcal{I}mp\}$ is a non-empty compact subset of $\mathbb{R}^{2^{|N|}}$.

Proof of theorem 1

For a non-empty compact subset A of $\mathbb{R}^m$, we need to prove that the set $\{x \in A \mid \forall y \in A \;\; x \leqslant_{lex} y\}$ is non-empty.

First, let $\pi_i : \mathbb{R}^m \to \mathbb{R}$ the projection function s.t. $\pi_i(x_1, \ldots, x_m) = x_i$.

Then, let us define the following sets:

$$\begin{cases} A_0 = A \\ A_{i+1} = \underset{x \in A_i}{\operatorname{argmin}} \, \pi_{i+1}(x) \\ i \in \{0, 1, \ldots, m-1\} \end{cases}$$

- $A_0 = A$
- $A_1 = \operatorname{argmin}_{x \in A} \pi_1(x)$ is the set of elements in A with the smallest first entry in the sequence.
- $A_2 = \operatorname{argmin}_{x \in A_1} \pi_2(x)$ composed of the elements that have the smallest second entry among the elements with the smallest first entry
- …
- $A_m = \{x \in A \mid \forall y \in A \;\; x \leqslant_{lex} y\}$

We want to prove by induction that each A_i is non-empty compact subset of $\mathbb{R}^m$ for $i \in \{1, \ldots, m\}$ to prove that A_m is non-empty.

Proof of theorem 1

- $A_0 = A$ is non-empty compact of $\mathbb{R}^m$ by hypothesis ✔.
- Let us assume that A_i is a non-empty compact subset of $\mathbb{R}^m$ and let us prove that A_{i+1} is a non-empty compact subset of $\mathbb{R}^m$. π_{i+1} is a continuous function and A_i is a non-empty compact subset of $\mathbb{R}^m$.

Thm A$_2$: Extreme value theorem: Let X be a non-empty compact subset of $\mathbb{R}^m$, $f : X \to \mathbb{R}$ a **continuous** function.

Using the extreme value theorem, $\min_{x \in A_i} \pi_{i+1}(x)$ exists and it is reached in A_i, hence $\operatorname{argmin}_{x \in A_i} \pi_{i+1}(x)$ is non-empty.

Now, we need to show it is compact.

We note by $\pi_i^{-1} : \mathbb{R} \to \mathbb{R}^m$ the inverse of π_i. Let $\alpha \in \mathbb{R}$, $\pi_i^{-1}(\alpha)$ is the set of all vectors $\langle x_1, \ldots, x_{i-1}, \alpha, x_{i+1}, \ldots, x_m \rangle$ s.t. $x_j \in \mathbb{R}$, $j \in \{1, \ldots, m\}$, $j \neq i$. We can rewrite A_{i+1} as:

$$A_{i+1} = \pi_{i+1}^{-1}\left(\min_{x \in A_i} \pi_{i+1}(x)\right) \bigcap A_i$$

Proof of theorem 1

Thm A$_3$: Let X be a non-empty compact subset of $\mathbb{R}^n$.
$f : X \longrightarrow \mathbb{R}$ is continuous iff for every closed subset $B \subseteq \mathbb{R}$, the set $f^{-1}(B)$ is compact.

$$A_{i+1} = \underbrace{\underbrace{\pi_{i+1}^{-1}\left(\left\{\underbrace{\min_{x \in A_i} \pi_{i+1}(x)}_{\text{closed}}\right\}\right)}_{\text{According to Thm A}_3\text{, it is a compact subset of } \mathbb{R}^m} \bigcap A_i}_{\substack{\text{is a compact subset of } \mathbb{R}^m \text{ since} \\ \text{the intersection of two closed sets is closed and in } \mathbb{R}^m, \\ \text{and a closed subset of a compact subset of } \mathbb{R}^m \\ \text{is a compact subset of } \mathbb{R}^m \ ✔}}$$

Hence A_{i+1} is a non-empty compact subset of $\mathbb{R}^m$ and the proof is complete. □

For a TU game (N, v) the nucleolus $Nu(N, v)$ is non-empty when $\mathcal{I}mp \neq \emptyset$, which is a great property as agents will always find an agreement. But there is more!

Theorem

The nucleolus has **at most one** element

In other words, there is **one** agreement which is stable according to the nucleolus.

proof in the next lecture

Summary

- We defined the excess of a coalition at a payoff distribution, which can model the complaints of the members in a coalition.
- We used the ordered sequence of excesses over all coalitions and the lexicographic ordering to compare any two imputations.
- We defined the nucleolus for a TU game.

pros:
- If the set of imputations is non-empty, the nucleolus is non-empty.
- The nucleolus contains at most one element.
- When the core is non-empty, the nucleolus is contained in the core.

cons: Difficult to compute.

Coming next

- The **kernel**, also a member of the bargaining set family, also based on the excess.