Cooperative Games

Lecture 6: The nucleolus and the Kernel

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Today

- We start by proving that the nucleolus has at most an element.
- We introduce the kernel, another stability concept from the bargaining set family, where the excess plays a key role.
- We consider some properties of the kernel, and we present an algorithm to compute a kernel-stable payoff distribution.

For a TU game (N,v), the $Nu(N,v) \neq \emptyset$ when $\mathcal{I}mp \neq \emptyset$, which is a great property as agents will always find an agreement.

Theorem

The nucleolus has **at most one** element

In other words, there is **one** agreement which is stable according to the nucleolus.

To prove this, we need theorems 3 and 4.

Theorem (3)

Let A be a non-empty convex subset of $\mathbb{R}^m$
Then the set $\{x \in A \mid \forall y \in A \;\; x^{\blacktriangleright} \leqslant_{lex} y^{\blacktriangleright}\}$ has at most one element.

Theorem (4)

Let (N,v) be a TU game such that $\mathcal{I}mp \neq \emptyset$.
(i) $\mathcal{I}mp$ is a non-empty and convex subset of $\mathbb{R}^{|N|}$
(ii) $\{e(x) \mid x \in \mathcal{I}mp\}$ is a non-empty convex subset of $\mathbb{R}^{2^{|N|}}$

Proof of Theorem 3

Let A be a non-empty convex subset of $\mathbb{R}^m$, and
$M^{in} = \{x \in A \mid \forall y \in A \;\; x^{\blacktriangleright} \leqslant_{lex} y^{\blacktriangleright}\}$. We now prove that $|M^{in}| \leqslant 1$.

Towards a contradiction, let us assume M^{in} has at least two elements x and y, $x \neq y$. By definition of M^{in}, we must have $x^{\blacktriangleright} = y^{\blacktriangleright}$.

Let $\alpha \in (0,1)$ and σ be a permutation of $\{1, \ldots, m\}$ such that
$(\alpha x + (1-\alpha)y)^{\blacktriangleright} = \sigma(\alpha x + (1-\alpha)y) = \alpha\sigma(x) + (1-\alpha)\sigma(y)$.
Let us show by contradiction that $\sigma(x) = x^{\blacktriangleright}$ and $\sigma(y) = y^{\blacktriangleright}$.

Let us assume that either $\sigma(x) <_{lex} x^{\blacktriangleright}$ or $\sigma(y) <_{lex} y^{\blacktriangleright}$, it follows that $\alpha\sigma(x) + (1-\alpha)\sigma(y) <_{lex} \alpha x^{\blacktriangleright} + (1-\alpha)y^{\blacktriangleright} = x^{\blacktriangleright}$.
Since A is convex, $\alpha x + (1-\alpha)y \in A$. But this is a contradiction because by definition of M^{in}, $\alpha x + (1-\alpha)y \in A$ cannot be strictly smaller than $x^{\blacktriangleright}$, $y^{\blacktriangleright}$ in A. This proves $\sigma(x) = x^{\blacktriangleright}$ and $\sigma(y) = y^{\blacktriangleright}$.

Since $x^{\blacktriangleright} = y^{\blacktriangleright}$, we have $\sigma(x) = \sigma(y)$, hence $x = y$. This contradicts the fact that $x \neq y$. Hence, M^{in} cannot have at least two elements, and $|M^{in}| \leqslant 1$.

Proof Theorem 4 (i)

Let (N,v) be a TU game s.t. $\mathcal{I}mp \neq \emptyset$ (in case $\mathcal{I}mp = \emptyset$, $\mathcal{I}mp$ is trivially convex). Let $(x,y) \in \mathcal{I}mp^2$, $\alpha \in [0,1]$. Let us prove $\mathcal{I}mp$ is convex by showing that $u = \alpha x + (1-\alpha)y \in \mathcal{I}mp$, i.e., individually rational and efficient.

Individual rationality: Since x and y are individually rational, for all agents i,
$u_i = \alpha x_i + (1-\alpha)y_i \geqslant \alpha v(\{i\}) + (1-\alpha)v(\{i\}) = v(\{i\})$. Hence u is individually rational.

Efficiency: Since x and y are efficient, we have
$$\sum_{i\in N} u_i = \sum_{i\in N} \alpha x_i + (1-\alpha)y_i \geqslant \alpha \sum_{i\in N} x_i + (1-\alpha)\sum_{i\in N} y_i$$
$\sum_{i\in N} u_i \geqslant \alpha v(N) + (1-\alpha)v(N) = v(N)$, hence u is efficient.

Thus, $u \in \mathcal{I}mp$.

Proof Theorem 4 (ii)

Let (N,v) be a TU game and $\mathcal{I}mp$ its set of imputations. We need to show $\{e(z) \mid z \in \mathcal{I}mp\}$ is a non-empty convex subset of $\mathbb{R}^m$.
Let $(x,y) \in \mathcal{I}mp^2$, $\alpha \in [0,1]$, and $\mathcal{C} \subseteq N$ and we consider the sequence $\alpha e(x) + (1-\alpha)e(y)$, and we look at the entry corresponding to coalition $\mathcal{C}$.

$$\begin{aligned}
(\alpha e(x) + (1-\alpha)e(y))_{\mathcal{C}} &= \alpha e(\mathcal{C},x) + (1-\alpha)e(\mathcal{C},y) \\
&= \alpha(v(\mathcal{C}) - x(\mathcal{C})) + (1-\alpha)(v(\mathcal{C}) - y(\mathcal{C})) \\
&= v(\mathcal{C}) - (\alpha x(\mathcal{C}) + (1-\alpha)y(\mathcal{C})) \\
&= v(\mathcal{C}) - ([\alpha x + (1-\alpha)y](\mathcal{C})) \\
&= e(\alpha x + (1-\alpha)y, \mathcal{C})
\end{aligned}$$

Since the previous equality is valid for all $\mathcal{C} \subseteq N$, both sequences are equal: $\alpha e(x) + (1-\alpha)e(y) = e(\alpha x + (1-\alpha)y)$.

Since $\mathcal{I}mp$ is convex, $\alpha x + (1-\alpha)y \in \mathcal{I}mp$, it follows that $e(\alpha x + (1-\alpha)y) \in \{e(z) \mid z \in \mathcal{I}mp\}$. Hence, $\{e(z) \mid z \in \mathcal{I}mp\}$ is convex.

Proof that the nucleolus has at most one element

Let (N,v) be a TU game, and $\mathcal{I}mp$ its set of imputations.
Theorem 4(ii): $\{e(x) \mid x \in \mathcal{I}mp\}$ is a non-empty convex subset of $\mathbb{R}^{2^{|N|}}$.
Theorem 3: If A is a non-empty convex subset of $\mathbb{R}^m$, then the set $\{x \in A \mid \forall y \in A \;\; x^{\blacktriangleright} \leqslant_{lex} y^{\blacktriangleright}\}$ has at most one element.

Applying theorem 3 with $A = \{e(x) \mid x \in \mathcal{I}mp\}$ we obtain $B = \{e(x) \mid x \in \mathcal{I}mp \wedge \forall y \in \mathcal{I}mp \;\; e(x)^{\blacktriangleright} \leqslant_{lex} e(y)^{\blacktriangleright}\}$ has at most one element.
B is the image of the nucleolus under the function e. We need to make sure that an $e(x)$ corresponds to at most one element in $\mathcal{I}mp$. This is true since for $(x,y) \in \mathcal{I}mp^2$, we have $x \neq y \Rightarrow e(x) \neq e(y)$.

Hence $Nu(N,v) = \{x \mid x \in \mathcal{I}mp \wedge \forall y \in \mathcal{I}mp \;\; e(x)^{\blacktriangleright} \leqslant_{lex} e(y)^{\blacktriangleright}\}$ has at most one element!

One last stability concept from the bargaining set family:

The kernel.

M. Davis. and M. Maschler, **The kernel of a cooperative game**. *Naval Research Logistics Quarterly*, 1965.

Excess

Definition (Excess)

For a TU game (N,v), the excess of coalition $\mathcal{C}$ for a payoff distribution x is defined as $e(\mathcal{C},x) = v(\mathcal{C}) - x(\mathcal{C})$.

We saw that a positive excess can be interpreted as an amount of complaint for a coalition.
We can also interpret the excess as a potential to generate more utility.

Let (N, v) be a TU game, $\mathcal{S} \in \mathscr{S}_N$ a coalition structure and x a payoff distribution. Objections and counter-objections are exchanged between **members of the same coalition** in $\mathcal{S}$. Objections and counter-objections take the form of **coalitions**, i.e., they do not propose another payoff distribution.

Let $\mathcal{C} \in \mathcal{S}$, $k \in \mathcal{C}$, $l \in \mathcal{C}$.

Objection: A coalition $P \subseteq N$ is an objection of k against l to x iff $k \in P$, $l \notin P$ and $x_l > v(\{l\})$.

> *"P is a coalition that contains k, excludes l and which sacrifices too much (or gains too little)."*

Counter-objection: A coalition $Q \subseteq N$ is a counter-objection to the objection P of k against l at x iff $l \in Q$, $k \notin Q$ and $e(Q, x) \geqslant e(P, x)$.

> *"k's demand is not justified: Q is a coalition that contains l and excludes k and that sacrifices even more (or gains even less)."*

A first definition

Remember that the set of feasible payoff vectors for $(N, v, \mathcal{S})$ is $X_{(N,v,\mathcal{S})} = \{x \in \mathbb{R}^n \mid \text{for every } \mathcal{C} \in \mathcal{S} : x(\mathcal{C}) \leqslant v(\mathcal{C})\}$.

Definition (Kernel)

Let $(N, v, \mathcal{S})$ be a TU game in coalition structure. The **kernel** is the set of imputations $x \in X_{(N,v,\mathcal{S})}$ s.t. for any coalition $\mathcal{C} \in \mathcal{S}$, for each objection P of an agent $k \in \mathcal{C}$ over any other member $l \in \mathcal{C}$ to x, there is a counter-objection of l to P.

Another definition

Definition (Maximum surplus)

For a TU game (N,v), the **maximum surplus** $s_{k,l}(x)$ of **agent k over agent l** with respect to a payoff distribution x is the **maximum excess** from a coalition that **includes k** but does **exclude** l, i.e.,

$s_{k,l}(x) = \max_{\mathcal{C} \subseteq N \mid k \in \mathcal{C},\, l \notin \mathcal{C}} e(\mathcal{C}, x)$.

Definition (Kernel)

Let $(N,v,\mathcal{S})$ be a TU game with coalition structure. The **kernel** is the set of imputations $x \in X_{(N,v,\mathcal{S})}$ such that for every coalition $\mathcal{C} \in CS$, if $(k,l) \in \mathcal{C}^2$, $k \neq l$, then we have either $s_{kl}(x) \geqslant s_{lk}(x)$ or $x_k = v(\{k\})$.

$s_{kl}(x) < s_{lk}(x)$ calls for a transfer of utility from k to l unless it is prevented by individual rationality, i.e., by the fact that $x_k = v(\{k\})$.

Theorem

Let $(N, v, \mathcal{S})$ a game with coalition structure, and let $\mathcal{I}mp \neq \emptyset$. Then we have:

- (i) $Nu(N, v, \mathcal{S}) \subseteq K(N, v, \mathcal{S})$
- (ii) $K(N, v, \mathcal{S}) \subseteq BS(N, v, \mathcal{S})$

Theorem

Let $(N, v, \mathcal{S})$ a game with coalition structure, and let $\mathcal{I}mp \neq \emptyset$. The kernel $K(N, v, \mathcal{S})$ and the bargaining set $BS(N, v, \mathcal{S})$ of the game are non-empty.

Proof

Since the Nucleolus is non-empty when $\mathcal{I}mp \neq \emptyset$, the proof is immediate using the theorem above. □

Proof of (i)

Let $x \notin K(N,v,\mathcal{S})$, we want to show that $x \notin Nu(N,v,\mathcal{S})$.

$x \notin K(N,v,\mathcal{S})$, hence, there exists $\mathcal{C} \in CS$ and $(k,l) \in \mathcal{C}^2$ such that $s_{lk}(x) > s_{kl}(x)$ and $x_k > v(\{k\})$.
Let y be a payoff distribution corresponding to a transfer of utility $\epsilon > 0$ from k to l: $y_i = \begin{cases} x_i \text{ if } i \neq k \text{ and } i \neq l \\ x_k - \epsilon \text{ if } i = k \\ x_l + \epsilon \text{ if } i = l \end{cases}$
Since $x_k > v(\{k\})$ and $s_{lk}(x) > s_{kl}(x)$, we can choose $\epsilon > 0$ small enough s.t.

- $x_k - \epsilon > v(\{k\})$
- $s_{lk}(y) > s_{kl}(y)$

We need to show that $e(y)^{\blacktriangleright} \leqslant_{lex} e(x)^{\blacktriangleright}$.

Note that for any coalition $S \subseteq N$ s.t. $e(S,x) \neq e(S,y)$ we have either

- $k \in S$ and $l \notin S$ ($e(S,x) > e(S,y)$ since $e(S,y) = e(S,x) + \epsilon > e(S,x)$)
- $k \notin S$ and $l \in S$ ($e(S,x) < e(S,y)$ since $e(S,y) = e(S,x) - \epsilon < e(S,x)$)

Proof of (i)

Let $\{B_1(x), \dots, B_M(x)\}$ a partition of the set of all coalitions s.t.

- $(S,T) \in B_i(x)$ iff $e(S,x) = e(T,x)$. We denote by $e_i(x)$ the common value of the excess in $B_i(x)$, i.e. $e_i(x) = e(S,x)$ for all $S \in B_i(x)$.
- $e_1(x) > e_2(x) > \cdots > e_M(x)$

In other words, $e(x)^{\blacktriangleright} = \langle \underbrace{e_1(x), \dots, e_1(x)}_{|B_1(x)| times}, \dots, \underbrace{e_M(x), \dots, e_M(x)}_{|B_M(x)| times} \rangle$.

Let i^* be the minimal value of $i \in \{1, \dots, M\}$ such that there is $\mathcal{C} \in B_{i^*}(x)$ with $e(\mathcal{C},x) \neq e(\mathcal{C},y)$.

For all $i < i^*$, we have $B_i(x) = B_i(y)$ and $e_i(x) = e_i(y)$.

Proof of (i)

Since $s_{lk}(x) > s_{kl}(x)$ B_{i^*} contains

- at least one coalition S that contains l but not k, for such coalition, we must have $e(S,x) > e(S,y)$
- no coalition that contains k but not l.

If B_{i^*} contains either

- coalitions that contain both k and l
- or coalitions that do not contain both k and l

Then, for any such coalitions S, we have $e(S,x) = e(S,y)$, and it follows that $B_{i^*}(y) \subset B_{i^*}(x)$.

Otherwise, we have $e_{i^*}(y) < e_{i^*}(x)$.

In both cases, we have $e(y)$ is lexicographically less than $e(x)$, and hence y is not in the nucleolus of the game $(N, v, \mathcal{S})$.

Proof of (ii)

Let $(N, v, \mathcal{S})$ a TU game with coalition structure. Let $x \in K(N, v, \mathcal{S})$. We want to prove that $x \in BS(N, v, \mathcal{S})$. To do so, we need to show that for any objection (P, y) from any player i against any player j at x, there is a counter objection (Q, z) to (P, y).For the bargaining set, An **objection of i against j** is a pair (P, y) where

- $P \subseteq N$ is a coalition such that $i \in P$ and $j \notin P$.
- $y \in \mathbb{R}^p$ where p is the size of P
- $y(P) \leqslant v(P)$ (y is a feasible payoff for members of P)
- $\forall k \in P$, $y_k \geqslant x_k$ and $y_i > x_i$

An **counter-objection to** (P, y) is a pair (Q, z) where

- $Q \subseteq N$ is a coalition such that $j \in Q$ and $i \notin Q$.
- $z \in \mathbb{R}^q$ where q is the size of Q
- $z(Q) \leqslant v(Q)$ (z is a feasible payoff for members of Q)
- $\forall k \in Q$, $z_k \geqslant x_k$
- $\forall k \in Q \cap P$ $z_k \geqslant y_k$

Proof of (ii)

Let (P, y) be an objection of player i against player j to x. $i \in P$, $j \notin P$, $y(P) \leqslant v(P)$ and $y(P) > x(P)$.We choose $y(P) = v(P)$.

- $x_j = v(\{j\})$: Then $(\{j\}, v(\{j\}))$ is a counter objection to (P, y). ✔
- $x_j > v(\{j\})$: Since $x \in K(N, v, \mathcal{S})$ we have $s_{ji}(x) \geqslant s_{ij}(x) \geqslant v(P) - x(P) \geqslant y(P) - x(P)$ since $i \in P$, $j \notin P$.
  Let $Q \subseteq N$ such that $j \in Q$, $i \notin Q$ and $s_{ji}(x) = v(Q) - x(Q)$.
  We have $v(Q) - x(Q) \geqslant y(P) - x(P)$. Then, we have

$$\begin{aligned} v(Q) &\geqslant y(P) + x(Q) - x(P) \\ &\geqslant y(P \cap Q) + y(P \setminus Q) + x(Q \setminus P) - x(P \setminus Q) \\ &> y(P \cap Q) + x(Q \setminus P) \text{ since } i \in P \setminus Q,\ y(P \setminus Q) > x(P \setminus Q) \end{aligned}$$

  Let us define z as follows $\begin{cases} x_k \text{ if } k \in Q \setminus P \\ y_k \text{ if } k \in Q \cap P \end{cases}$

  (Q, z) is a counter-objection to (P, y). ✔

Finally $x \in BS(N, v, \mathcal{S})$.

Computing a kernel-stable payoff distribution

- There is a transfer scheme converging to an element in the kernel.
- It may require an infinite number of small steps.
- We can consider the ϵ-kernel where the inequality are defined up to an arbitrary small constant ϵ.

R. E. Stearns. **Convergent transfer schemes for n-person games**. *Transactions of the American Mathematical Society*, 1968.

Computing a kernel-stable payoff distribution

Algorithm 1: Transfer scheme converging to a ϵ-Kernel-stable payoff distribution for the CS $\mathcal{S}$

```
compute-ϵ-Kernel-Stable(N, v, S, ϵ)
repeat
    for each coalition C ∈ S do
        for each member (i,j) ∈ C, i ≠ j do        // compute the maximum surplus
            // for two members of a coalition in S
            s_ij ← max_{R⊆N|(i∈R, j∉R)} v(R) − x(R)
    δ ← max_{(i,j)∈C², C∈S} s_ij − s_ji;
    (i*, j*) ← argmax_{(i,j)∈N²} (s_ij − s_ji);
    if (x_{j*} − v({j}) < δ/2) then                // payment should be individually rational
        d ← x_{j*} − v({j*});
    else
        d ← δ/2;
    x_{i*} ← x_{i*} + d;
    x_{j*} ← x_{j*} − d;
until δ/v(S) ⩽ ϵ ;
```

- The complexity for one side-payment is $O(n \cdot 2^n)$.
- Upper bound for the number of iterations for converging to an element of the ϵ-kernel: $n \cdot log_2(\frac{\delta_0}{\epsilon \cdot v(S)})$, where δ_0 is the maximum surplus difference in the initial payoff distribution.
- To derive a polynomial algorithm, the number of coalitions must be bounded. For example, only consider coalitions which size is bounded in $[K_1, K_2]$. The complexity of the truncated algorithm is $O(n^2 \cdot n_{coalitions})$ where $n_{coalitions}$ is the number of coalitions with size in$[K_1, K_2]$, which is a polynomial of order K_2.

• M. Klusch and O. Shehory. **A polynomial kernel-oriented coalition algorithm for rational information agents**. In *Proceedings of the Second International Conference on Multi-Agent Systems*, 1996.

• O. Shehory and S. Kraus. **Feasible formation of coalitions among autonomous agents in non-superadditve environments**. *Computational Intelligence*, 1999.

Summary

- We saw another way to use the excess to make objections and counter-objections.
- We defined the kernel.
- We proved that both the kernel and the bargaining set are non-empty if the set of imputations is non-empty.

pros:
- If the set of imputations is non-empty, the nucleolus, kernel, bargaining set are non-empty.
- There is an algorithm to compute a payoff in the kernel.

cons: The algorithm is not polynomial

Coming next

- The **Shapley value**.
  It is not a stability concept, but it tries to guarantee fairness. We will see it can be defined axiomatically or using the concept of marginal contributions.