Cooperative Games

Lecture 7: The Kernel (end) and The Shapley Value

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Today

- We prove one property of the kernel, and we consider an algorithm to compute an element in the kernel
- We introduce a solution concept called the Shapley value.

Theorem

Let $(N, v, \mathcal{S})$ a game with coalition structure, and let $\mathcal{I}mp \neq \emptyset$. Then we have:

- (i) $Nu(N, v, \mathcal{S}) \subseteq K(N, v, \mathcal{S})$
  Proof ✔
- (ii) $K(N, v, \mathcal{S}) \subseteq BS(N, v, \mathcal{S})$
  Proof ✘

Proof of (ii)

Let $(N,v,\mathcal{S})$ a TU game with coalition structure. Let $x \in K(N,v,\mathcal{S})$. We want to prove that $x \in BS(N,v,\mathcal{S})$. To do so, we need to show that for any objection (P,y) from any player i against any player j at x, there is a counter objection (Q,z) to (P,y). For the bargaining set, An **objection of i against** j is a pair (P,y) where

- $P \subseteq N$ is a coalition such that $i \in P$ and $j \notin P$.
- $y \in \mathbb{R}^p$ where p is the size of P
- $y(P) \leqslant v(P)$ (y is a feasible payoff for members of P)
- $\forall k \in P$, $y_k \geqslant x_k$ and $y_i > x_i$

An **counter-objection to** (P,y) is a pair (Q,z) where

- $Q \subseteq N$ is a coalition such that $j \in Q$ and $i \notin Q$.
- $z \in \mathbb{R}^q$ where q is the size of Q
- $z(Q) \leqslant v(Q)$ (z is a feasible payoff for members of Q)
- $\forall k \in Q$, $z_k \geqslant x_k$
- $\forall k \in Q \cap P$ $z_k \geqslant y_k$

Proof of (ii)

Let (P,y) be an objection of player i against player j to x. $i \in P$, $j \notin P$, $y(P) \leqslant v(P)$ and $y(P) > x(P)$.We choose $y(P) = v(P)$.

- $x_j = v(\{j\})$: Then $(\{j\}, v(\{j\}))$ is a counter objection to (P,y). ✔
- $x_j > v(\{j\})$: Since $x \in K(N,v,\mathcal{S})$ we have $s_{ji}(x) \geqslant s_{ij}(x) \geqslant v(P) - x(P) \geqslant y(P) - x(P)$ since $i \in P$, $j \notin P$.
  Let $Q \subseteq N$ such that $j \in Q$, $i \notin Q$ and $s_{ji}(x) = v(Q) - x(Q)$.
  We have $v(Q) - x(Q) \geqslant y(P) - x(P)$. Then, we have

$$
\begin{aligned}
v(Q) &\geqslant y(P) + x(Q) - x(P) \\
&\geqslant y(P \cap Q) + y(P \setminus Q) + x(Q \setminus P) - x(P \setminus Q) \\
&> y(P \cap Q) + x(Q \setminus P) \text{ since } i \in P \setminus Q,\ y(P \setminus Q) > x(P \setminus Q)
\end{aligned}
$$

  Let us define z as follows $\begin{cases} x_k \text{ if } k \in Q \setminus P \\ y_k \text{ if } k \in Q \cap P \end{cases}$

  (Q,z) is a counter-objection to (P,y). ✔

Finally $x \in BS(N,v,\mathcal{S})$.

Computing a kernel-stable payoff distribution

- There is a transfer scheme converging to an element in the kernel.
- It may require an infinite number of small steps.
- We can consider the ϵ-kernel where the inequality are defined up to an arbitrary small constant ϵ.

R. E. Stearns. **Convergent transfer schemes for n-person games**. *Transactions of the American Mathematical Society*, 1968.

Computing a kernel-stable payoff distribution

Algorithm 1: Transfer scheme converging to a ϵ-Kernel-stable payoff distribution for the CS $\mathcal{S}$

```
compute-ε-Kernel-Stable(N, v, S, ε)
repeat
    for each coalition C ∈ S do
        for each member (i,j) ∈ C, i ≠ j do        // compute the maximum surplus
            // for two members of a coalition in S
            s_ij ← max_{R⊆N|(i∈R, j∉R)} v(R) − x(R)
    δ ← max_{(i,j)∈C², C∈S} s_ij − s_ji;
    (i*, j*) ← argmax_{(i,j)∈N²} (s_ij − s_ji);
    if (x_j* − v({j}) < δ/2) then                 // payment should be individually rational
        d ← x_j* − v({j*});
    else
        d ← δ/2;
    x_i* ← x_i* + d;
    x_j* ← x_j* − d;
until δ/v(S) ⩽ ε ;
```

- The complexity for one side-payment is $O(n \cdot 2^n)$.
- Upper bound for the number of iterations for converging to an element of the ϵ-kernel: $n \cdot log_2(\frac{\delta_0}{\epsilon \cdot v(S)})$, where δ_0 is the maximum surplus difference in the initial payoff distribution.
- To derive a polynomial algorithm, the number of coalitions must be bounded. For example, only consider coalitions which size is bounded in $[K_1, K_2]$. The complexity of the truncated algorithm is $O(n^2 \cdot n_{coalitions})$ where $n_{coalitions}$ is the number of coalitions with size in$[K_1, K_2]$, which is a polynomial of order K_2.

• M. Klusch and O. Shehory. **A polynomial kernel-oriented coalition algorithm for rational information agents**. In *Proceedings of the Second International Conference on Multi-Agent Systems*, 1996.

• O. Shehory and S. Kraus. **Feasible formation of coalitions among autonomous agents in non-superadditve environments**. *Computational Intelligence*, 1999.

Summary

- We saw another way to use the excess to make objections and counter-objections.
- We defined the kernel.
- We proved that both the kernel and the bargaining set are non-empty if the set of imputations is non-empty.

pros:
- If the set of imputations is non-empty, the nucleolus, kernel, bargaining set are non-empty.
- There is an algorithm to compute a payoff in the kernel.

cons: The algorithm is not polynomial

The Shapley value

Lloyd S. Shapley. **A Value for *n*-person Games**. In *Contributions to the Theory of Games, volume II (Annals of Mathematical Studies)*, 1953.

Definition (marginal contribution)

The **marginal contribution** of agent i for a coalition $\mathcal{C} \subseteq N \setminus \{i\}$ is $mc_i(\mathcal{C}) = v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})$.

$\langle mc_1(\emptyset), mc_2(\{1\}), mc_3(\{1,2\})\rangle$ is an efficient payoff distribution for any game $(\{1,2,3\}, v)$. This payoff distribution may model a dynamic process in which 1 starts a coalition, is joined by 2, and finally 3 joins the coalition $\{1,2\}$, and where the incoming agent gets its marginal contribution.

An agent's payoff depends on which agents are already in the coalition. This payoff may not be **fair**. To increase fairness,one could take the average marginal contribution over all possible joining orders.

Let σ represent a joining order of the grand coalition N, i.e., σ is a permutation of $\langle 1, \ldots, n\rangle$.
We write $mc(\sigma) \in \mathbb{R}^n$ the payoff vector where agent i obtains $mc_i(\{\sigma(j) \mid j < i\})$. The vector mc is called a **marginal vector**.

Shapley value: version based on marginal contributions

Let (N,v) be a TU game. Let $\Pi(N)$ denote the set of all permutations of the sequence $\langle 1,\ldots,n\rangle$.

$$Sh(N,v) = \frac{\sum_{\sigma\in\Pi(N)} mc(\sigma)}{n!}$$

the Shapley value is a **fair** payoff distribution based on marginal contributions of agents averaged over joining orders of the coalition.

An example

$$N = \{1,2,3\},\ v(\{1\}) = 0,\ v(\{2\}) = 0,\ v(\{3\}) = 0,$$
$$v(\{1,2\}) = 90,\ v(\{1,3\}) = 80,\ v(\{2,3\}) = 70,$$
$$v(\{1,2,3\}) = 120.$$

	1	2	3
$1 \leftarrow 2 \leftarrow 3$	0	90	30
$1 \leftarrow 3 \leftarrow 2$	0	40	80
$2 \leftarrow 1 \leftarrow 3$	90	0	30
$2 \leftarrow 3 \leftarrow 1$	50	0	70
$3 \leftarrow 1 \leftarrow 2$	80	40	0
$3 \leftarrow 2 \leftarrow 1$	50	70	0
total	270	240	210
Shapley value	45	40	35

Let $y = \langle 50, 40, 30 \rangle$

$\mathcal{C}$	$e(\mathcal{C},x)$	$e(\mathcal{C},y)$
$\{1\}$	-45	0
$\{2\}$	-40	0
$\{3\}$	-35	0
$\{1,2\}$	5	0
$\{1,3\}$	0	0
$\{2,3\}$	-5	0
$\{1,2,3\}$	120	0

This example shows that the Shapley value may not be in the core, and may not be the nucleolus.

- There are $|\mathcal{C}|!$ permutations in which all members of $\mathcal{C}$ precede i.
- There are $|N \setminus (\mathcal{C} \cup \{i\})|!$ permutations in which the remaining members succede i, i.e. $(|N|-|\mathcal{C}|-1)!$.

The Shapley value $Sh_i(N,v)$ of the TU game (N,v) for player i can also be written

$$Sh_i(N,v) = \sum_{\mathcal{C} \subseteq N \setminus \{i\}} \frac{|\mathcal{C}|!(|N|-|\mathcal{C}|-1)!}{|N|!} \left(v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \right).$$

Using definition, the sum is over $2^{|N|-1}$ instead of $|N|!$.

Notion of value

Definition (value function)

Let $\mathcal{G}_N$ the set of all TU games (N,v). A **value function ϕ** is a function that assigns to each TU game (N,v) an efficient allocation, i.e. $\phi : \mathcal{G}_N \to \mathbb{R}^{|N|}$ such that $\phi(N,v)(N) = v(N)$.

- The Shapley value is a value function.
- None of the concepts presented thus far were a value function (the nucleolus is guaranteed to be non-empty only for games with a non-empty set of imputations).

Some interesting properties

Let (N,v) and (N,u) be TU games and ϕ be a value function.

- **Symmetry or substitution (SYM):** If $\forall (i,j) \in N$, $\forall \mathcal{C} \subseteq N \setminus \{i,j\}$, $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$ then $\phi_i(N,v) = \phi_j(N,v)$
- **Dummy (DUM):** If $\forall \mathcal{C} \subseteq N \setminus \{i\}$ $v(\mathcal{C}) = v(\mathcal{C} \cup \{i\})$, then $\phi_i(N,v) = 0$.
- **Additivity (ADD):** Let $(N, u+v)$ be a TU game defined by $\forall \mathcal{C} \subseteq N$, $(u+v)(N) = u(N) + v(N)$.
  $\phi(u+v) = \phi(u) + \phi(v)$.

Theorem

The Shapley value is the unique value function ϕ that satisfies (SYM), (DUM) and (ADD).

Unanimity game

Let N be a set of agents and $T \subseteq N \setminus \emptyset$.
The **unanimity game** (N, v_T) is defined as follows:

$$\forall \mathcal{C} \subseteq N,\ v_T(\mathcal{C}) = \begin{cases} 1, \text{ if } T \subseteq \mathcal{C}, \\ 0 \text{ otherwise.} \end{cases}$$

We note that

- if $i \in N \setminus T$, i is a null player.
- if $(i,j) \in T^2$, i and j are substitutes.

Lemma

The set $\{v_T \mid T \subseteq N \setminus \emptyset\}$ is a linear basis of $\mathcal{G}_N$.

This means that a TU game (N, v) can be represented by a unique set of values $(\alpha_T)_{T \subseteq N \setminus \emptyset}$ such that

$$\forall \mathcal{C} \subseteq N,\ v(\mathcal{C}) = \left(\sum_{T \subseteq N \setminus \emptyset} \alpha_T v_T \right)(\mathcal{C}).$$

Proof of the lemma

There are $2^n - 1$ unanimity games and the dimension of $\mathcal{G}_N$ is also $2^n - 1$.
We only need to prove that the unanimity games are linearly independent.
Towards a contradiction, let us assume that $\sum_{T \subseteq N \setminus \emptyset} \alpha_T v_T = 0$ where $(\alpha_T)_{T \subseteq N \setminus \emptyset} \neq 0_{\mathbb{R}^{2^n - 1}}$.
Let T_0 be a minimal set in $\{T \subseteq N \mid \alpha_T \neq 0\}$.
Then, $\left(\sum_{T \subseteq N \setminus \emptyset} \alpha_T v_T\right)(T_0) = \alpha_{T_0} \neq 0$, which is a contradiction.

Proof of the theorem: Uniqueness (1/2)

Let ϕ a feasible solution on $\mathcal{G}_N$ that is non-empty and satisfies the axioms SYM, DUM and ADD. Let us prove that ϕ is a value function.
Let $(Nv,) \in \mathcal{G}_N$.

- if $v = 0_{\mathcal{G}_N}$, all players are dummy. Since the solution is non-empty, $0^{\mathbb{R}^{|N|}}$ is the unique member of $\phi(N,v)$.
- otherwise, $(N,-v) \in \mathcal{G}_N$.
  Let $x \in \phi(N,v)$ and $y \in \phi(N,-v)$.
  By ADD, $x+y \in \phi(v-v)$, and then, $x = -y$ is unique.
  Moreover, $x(N) \leqslant v(N)$ as ϕ is a feasible solution.
  Also $y(N) \leqslant -v(N)$.
  Since $x = -y$, we have $v(N) \leqslant x(N) \leqslant v(N)$,
  i.e. x is efficient.

Hence, ϕ is a value function.

Proof of the theorem: Uniqueness (2/2)

Let $T \subseteq N \setminus \emptyset$ and $\alpha \in \mathbb{R}$. Let us prove that $\phi(N, \alpha \cdot v_T)$ is uniquely defined.

- Let $i \notin T$. We have trivially $T \subseteq \mathcal{C}$ iff $T \subseteq \mathcal{C} \cup \{i\}$. Then $\forall \mathcal{C} \subseteq N \setminus \{i\}$, $\alpha v_T(\mathcal{C}) = \alpha v_T(\mathcal{C} \cup \{i\})$.Hence, all agent $i \notin T$ are dummies. By DUM, $\forall i \notin T$, $\phi_i(N, \alpha \cdot v_T) = 0$.
- Let $(i,j) \in T^2$.Then for all $\mathcal{C} \subseteq N \setminus \{i,j\}$, $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$.By SYM, $\phi_i(N, \alpha \cdot v_T) = \phi_j(N, \alpha \cdot v_T)$.
- Since ϕ is a value function, it is efficient. Then, $\sum_{i \in N} \phi_i(N, \alpha \cdot v_T) = \alpha v_T(N) = \alpha$.
  Hence, $\forall i \in T$, $\phi_i(N, \alpha \cdot v_T) = \frac{\alpha}{|T|}$.

This proves that $\phi(N, \alpha \cdot v_T)$ is uniquely defined.Since any TU game (N, v) can be written as $\sum_{T \subseteq N \setminus \emptyset} \alpha_T v_T$ and because of ADD, there is a unique value function that satisfies the three axioms.

Proof of the theorem: Existence

We need to show that the Shapley value satisfies the three axioms. Let (N,v) a TU game.

- Let us assume that $\forall \mathcal{C} \subseteq N \setminus \{i,j\}$, we have $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$. Then $\forall \mathcal{C} \subseteq N \setminus \{i,j\}$, we have
  - $mc_i(\mathcal{C}) = mc_j(\mathcal{C})$
  - $v(\mathcal{C} \cup \{i,j\}) - v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{i,j\}) - v(\mathcal{C} \cup \{j\})$, hence, we have $mc_j(\mathcal{C} \cup \{j\}) = mc_i(\mathcal{C} \cup \{i\})$.

  ➡ $Sh_i(N,v) = Sh_j(N,v)$, Sh satisfies SYM.
- Let us assume there is an agent i such that for all $\mathcal{C} \subseteq N \setminus \{i\}$ we have $v(\mathcal{C}) = v(\mathcal{C} \cup \{i\})$.Then, each marginal contribution of player i is zero, and it follows that $Sh_i(N,v) = 0$. Sh satisfies DUM.
- Sh is clearly additive.

✔

Coming next

- Voting games and power indices.