Cooperative Games

Lecture 8: Simple Games

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Today

- End of the discussion about the Shapley Value
- Simple games: a class of TU games for modeling voting.
- Measuring the power of a voter: Shapley Shubik, Banzhaff and Co.

The Shapley value $Sh_i(N,v)$ of the TU game (N,v) for player i is

$$Sh_i(N,v) = \sum_{\mathcal{C} \subseteq N \setminus \{i\}} \frac{|\mathcal{C}|!(|N|-|\mathcal{C}|-1)!}{|N|!} \left(v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})\right).$$

- **Symmetry or substitution (SYM):** If $\forall (i,j) \in N$, $\forall \mathcal{C} \subseteq N \setminus \{i,j\}$, $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$ then $\phi_i(N,v) = \phi_j(N,v)$
- **Dummy (DUM):** If $\forall \mathcal{C} \subseteq N \setminus \{i\}$ $v(\mathcal{C}) = v(\mathcal{C} \cup \{i\})$, then $\phi_i(N,v) = 0$.
- **Additivity (ADD):** Let $(N, u+v)$ be a TU game defined by $\forall \mathcal{C} \subseteq N$, $(u+v)(N) = u(N) + v(N)$. $\phi(u+v) = \phi(u) + \phi(v)$.

Theorem

The Shapley value is the unique value function ϕ that satisfies (SYM), (DUM) and (ADD).

Discussion about the axioms

- SYM: it is desirable that two subsitute agents obtain the same value ✔
- DUM: it is desirable that an agent that does not bring anything in the cooperation does not get any value. ✔
- What does the addition of two games mean?
  - + if the payoff is interpreted as an expected payoff, ADD is a desirable property.
  - + for cost-sharing games, the interpretation is intuitive: the cost for a joint service is the sum of the costs of the separate services.
  - - there is no interaction between the two games.
  - - the structure of the game $(N, v+w)$ may induce a behavior that has may be unrelated to the behavior induced by either games (N,v) or (N,w).
- The axioms are independent. If one of the axiom is dropped, it is possible to find a different value function satisfying the remaining two axioms.

Let (N,v) and (N,v) be two TU games.

- **Marginal contribution:** A value function ϕ satisfies marginal contribution axiom iff for all $i \in N$,
  if for all $\mathcal{C} \subseteq N \setminus \{i\}$ $v(\mathcal{C} \cup \{i\}) - v(C) = u(\mathcal{C} \cup \{i\}) - u(C)$,
  then $\phi(u) = \phi(v)$.

The value of a player depends only on its marginal contribution.

Theorem (H.P. Young)

The Shapley value is the unique value function that satisfies symmetry and marginal contribution axioms.

We refer by $v \setminus i$ the TU game $(N \setminus \{i\}, v_{\setminus i})$ where $v_{\setminus i}$ is the restriction of v to $N \setminus \{i\}$.

- **Balanced contribution:** A value function ϕ satisfies balanced contribution iff for all $(i,j) \in N^2$
  $\phi_i(v) - \phi_i(v \setminus j) = \phi_j(v) - \phi_j(v \setminus i)$.

For any two agents, the amount that each agent would win or lose if the other "leaves the game" should be the same.

Theorem (R Myerson)

The Shapley value is the unique value function that satisfies the balanced contribution axiom.

Some properties

Theorem

For superadditive games, the Shapley value is an imputation.

Lemma

For convex game, the Shapley value is in the core.

Proofs

- Let (N, v) be a superadditive TU game.
  By superadditivity, $\forall i \in N$, $\forall \mathcal{C} \subseteq N \setminus \{i\}$
  $v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) > v(\{i\})$. Hence, for each marginal vector, an agent i gets at least $v(\{i\})$. The same is true for the Shapley value as it is the average over all marginal vectors.
- Let (N, v) be a convex game.
  We know that all marginal vectors are in the core (to show that convex games have non-empty core, we used one marginal vector and showed it was in the core).
  The core is a convex set.
  The average of a finite set of points in a convex set is also in the set.
  Finally, the Shapley value is in the core.

Summary

- **pros**
  - The Shapley value is a value function, i.e., it **always exists** and is **unique**.
  - When the valuation function is **superadditive**, the Shapley value is **individually rational**, i.e., it is an imputation.
  - When the valuation function is **convex**, the Shapley value is also group rational, hence, it is in the **core**.
  - The Shapley value is the unique value function satisfying some axioms.
- **cons**
  - The nature of the Shapley value is combinatorial.

Simple games

Simple Games

Definition (Simple games)

A game (N,v) is a **Simple game** when

the valuation function takes two values

- 1 for a winning coalitions
- 0 for the losing coalitions

v satisfies *unanimity*: $v(N) = 1$

v satisfies *monotonicity*: $S \subseteq T \Rightarrow v(S) \leqslant v(T)$

One can represent the game by stating all the wining coalitions. Thanks to monotonicity, it is sufficient to only write down the minimal winning coalitions defined as follows:

Definition (Minimal winning coalition)

Let (N,v) be a TU game. A coalition $\mathcal{C}$ is a **minimal winning coalition** iff $v(\mathcal{C}) = 1$ and $\forall i \in \mathcal{C}$, $v(\mathcal{C} \setminus \{i\}) = 0$.

Example

$N = \{1,2,3,4\}$.

We use majority voting, and in case of a tie, the decision of player 1 wins.

The set of winning coalitions is $\{\{1,2\},\{1,3\},\{1,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\}$.

The set of minimal winning coalitions is $\{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$.

Formal definition of common terms in voting

Definition (Dictator)

Let (N, v) be a simple game. A player $i \in N$ is a **dictator** iff $\{i\}$ is a winning coalition.

Note that with the requirements of simple games, it is possible to have more than one dictator!

Definition (Veto Player)

Let (N, v) be a simple game. A player $i \in N$ is a **veto** player if $N \setminus \{i\}$ is a losing coalition. Alternatively, i is a **veto** player iff for all winning coalition $\mathcal{C}$, $i \in \mathcal{C}$.

It also follows that a veto player is member of every minimal winning coalitions.

Definition (blocking coalition)

A coalition $\mathcal{C} \subseteq N$ is a **blocking coalition** iff $\mathcal{C}$ is a losing coalition and $\forall S \subseteq N \setminus \mathcal{C}$, $S \setminus \mathcal{C}$ is a losing coalition.

A class of simple games

Definition (weighted voting games)

A game $(N, w_{i\in N}, q)$ is a **weighted voting game** when v satisfies unanimity, monotonicity and the valuation function is defined as

$$v(S) = \begin{cases} 1 \text{ when } \sum_{i\in S} w_i \geqslant q \\ 0 \text{ otherwise} \end{cases}$$

Unanimity requires that $\sum_{i\in N} w_i \geqslant q$.
If we assume that $\forall i \in N \; w_i \geqslant 0$, monotonicity is guaranteed.
For the rest of the lecture, we will assume $w_i \geqslant 0$.

We will note a weighted voting game $(N, w_{i\in N}, q)$ as $[q; w_1, \ldots, w_n]$.

A weighted voting game is a **succint** representation, as we only need to define a weight for each agent and a threshold.

Example 1

Let us consider the game $[q; 4,2,1]$.

- $q = 1$: minimal winning coalitions: {1},{2},{3}
- $q = 2$: minimal winning coalitions: {1},{2}
- $q = 3$: minimal winning coalitions: {1},{2,3}
- $q = 4$: minimal winning coalition: {1}
- $q = 5$: minimal winning coalitions: {1,2},{1,3}
- $q = 6$: minimal winning coalition: {1,2}
- $q = 7$: minimal winning coalition: {1,2,3}

for $q = 4$ ("majority" weight), 1 is a dictator, 2 and 3 are dummies.

Examples

- Let us consider the game [10; 7,4,3,3,1].

  The set of minimal winning coalitions is {{1,2}{1,3}{1,4}{2,3,4}}

  Player 5, although it has some weight, is a dummy.

  Player 2 has a higher weight than player 3 and 4, but it is clear that player 2, 3 and 4 have the same influence.
- Let us consider the game [51; 49,49,2]

  The set of winning coalition is {{1,2},{1,3},{2,3}}.

  It seems that the players have symmetric roles, but it is not reflected in their weights.

Weighted voting game is a strict subclass of voting games. i.e., all voting games are **not** weighted voting games.

Example: Let $(\{1,2,3,4\},v)$ a voting game such that the set of minimal winning coalitions is $\{\{1,2\},\{3,4\}\}$. Let us assume we can represent (N,v) with a weighted voting game $[q;\, w_1,w_2,w_3,w_4]$.

$v(\{1,2\}) = 1$ then $w_1 + w_2 \geqslant q$
$v(\{3,4\}) = 1$ then $w_3 + w_4 \geqslant q$
$v(\{1,3\}) = 0$ then $w_1 + w_3 < q$
$v(\{2,4\}) = 0$ then $w_2 + w_4 < q$

But then, $w_1 + w_2 + w_3 + w_4 < 2q$ and $w_1 + w_2 + w_3 + w_4 \geqslant 2q$, which is impossible. Hence, (N,v) cannot be represented by a weighted voting game.✔

Theorem

Let (N,v) be a simple game. Then

$$Core(N,v) = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} x \text{ is an imputation} \\ x_i = 0 \text{ for each non-veto player } i \end{array} \right\}$$

Proof

- $\subseteq$ Let $x \in Core(N,v)$. By definition $x(N) = 1$. Let i be a non-veto player. $x(N \setminus \{i\}) \geqslant v(N \setminus \{i\}) = 1$. Hence $x(N \setminus \{i\}) = 1$ and $x_i = 0$.
- $\supseteq$ Let x be an imputation and $x_i = 0$ for every non-veto player i. Since $x(N) = 1$, the set V of veto players is non-empty and $x(V) = 1$.

  Let $\mathcal{C} \subseteq N$. If $\mathcal{C}$ is a winning coalition then $V \subseteq \mathcal{C}$, hence $x(\mathcal{C}) \geqslant v(\mathcal{C})$. Otherwise, $v(\mathcal{C})$ is a losing coalition (which may contain veto players), and $x(\mathcal{C}) \geqslant v(\mathcal{C})$. Hence, x is group rational.

□

Theorem

A simple game (N,v) is convex iff it is a unanimity game (N,v_V) where V is the set of veto players.

Proof

A game is convex iff $\forall S,T \subseteq N \; v(S)+v(T) \leqslant v(S\cap T)+v(S\cup T)$.

- $\Rightarrow$ Let us assume (N,v) is convex.

  If S and T are winning coalitions, $S\cup T$ is a winning coalition by monotonicity. Then, we have $2 \leqslant 1+v(S\cap T)$ and it follows that $v(S\cap T)=1$. The intersection of two winning coalitions is a winning coalition.

  Moreover, from the definition of veto players, the intersection of all winning coalitions is the set V of veto players. Hence, $v(V)=1$.

  By monotonicity, if $V\subseteq \mathcal{C}$, $v(\mathcal{C})=1$ ✔

  Otherwise, $V \nsubseteq \mathcal{C}$. Then there must be a veto player $i\notin \mathcal{C}$, and it must be the case that $v(\mathcal{C})=0$ ✔

  Hence, for all coalition $\mathcal{C}\subseteq N$, $v(\mathcal{C})=1$ iff $V\subseteq \mathcal{C}$.

□

Proof

(continuation)

$\Leftarrow$ Let (N, v_V) a unanimity game. Let us prove it is a convex game. Let $S \subseteq N$ and $T \subseteq N$, and we want to prove that $v(S) + v(T) \leqslant v(S \cup T) + v(S \cap T)$.

- case $V \subseteq S \cap T$: Then $V \subseteq S$ and $V \subseteq T$, and we have $2 \leqslant 2$ ✔
- case $V \not\subseteq S \cap T \wedge V \subseteq S \cup T$:
  - if $V \subseteq S$ then $V \not\subseteq T$ and $1 \leqslant 1$ ✔
  - if $V \subseteq T$ then $V \not\subseteq S$ and $1 \leqslant 1$ ✔
  - otherwise $V \not\subseteq S$ and $V \not\subseteq T$, and then $0 \leqslant 1$ ✔
- case $V \not\subseteq S \cup T$: then $0 \leqslant 0$ ✔

For all cases, $v(S) + v(T) \leqslant v(S \cup T) + v(S \cap T)$, hence a unanimity game is convex.

In addition, all members of V are veto players.

□

Convex simple games are the games with a single minimal winning coalition.

Shapley-Shubik power index

Definition (Pivotal or swing player)

Let (N,v) be a simple game. A agent i is **pivotal** or **a swing agent** for a coalition $\mathcal{C} \subseteq N \setminus \{i\}$ if agent i turns the coalition $\mathcal{C}$ from a losing to a winning coalition by joining $\mathcal{C}$, i.e., $v(\mathcal{C}) = 0$ and $v(\mathcal{C} \cup \{i\}) = 1$.

Given a **permutation** σ on N, there is a single pivotal agent.

The Shapley-Shubik index of an agent i is the percentage of permutation in which i is pivotal, i.e.

$$I_{SS}(N,v,i) = \sum_{\mathcal{C} \subseteq N \setminus \{i\}} \frac{|\mathcal{C}|!(|N|-|C|-1)!}{|N|!} \left(v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})\right).$$

"For each permutation, the pivotal player gets a point."

The Shapley-Shubik power index is the Shapley value.
The index corresponds to the expected marginal utility assuming all join orders to form the grand coalitions are equally likely.

Banzhaff power index

Let (N,v) be a TU game.

- We want to count the **number of coalitions** in which an agent is **a swing agent**.
- For each coalition, we determine which agent is a swing agent (more than one agent may be pivotal).
- The **raw Banzhaff index** of a player i is $\beta_i = \frac{\sum_{\mathcal{C} \subseteq N \setminus \{i\}} v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})}{2^{n-1}}$.
- For a simple game (N,v), $v(N) = 1$ and $v(\emptyset) = 0$, at least one player i has a power index $\beta_i \neq 0$. Hence, $B = \sum_{j \in N} \beta_j > 0$.
- The **normalized Banzhaff index** of player i for a simple game (N,v) is defined as $I_B(N,v,i) = \frac{\beta_i}{B}$.

The index corresponds to the expected marginal utility assuming all coalitions are equally likely.

Examples: [7; 4,3,2,1]

{1,2,3,4}
{1,2,4,3}
{1,3,2,4}
{1,3,4,2}
{1,4,2,3}
{1,4,3,2}
{2,1,3,4}
{2,1,4,3}
{2,3,1,4}
{2,3,4,1}
{2,4,1,3}
{2,4,3,1}
{3,1,2,4}
{3,1,4,2}
{3,2,1,4}
{3,2,4,1}
{3,4,1,2}
{3,4,2,1}
{4,1,2,3}
{4,1,3,2}
{4,2,1,3}
{4,2,3,1}
{4,3,1,2}
{4,3,2,1}

	1	2	3	4
Sh	$\frac{7}{12}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$

winning coalitions:
{1,2}
{1,2,3}
{1,2,4}
{1,3,4}
{1,2,3,4}

	1	2	3	4
β	$\frac{5}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$I_B(N,v,i)$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

The Shapley value and Banzhaff index may be different.

- Coleman indices: all winning coalitions are equally likely. Let $\mathcal{W}(N,v)$ be the set of all winning coalitions.
- The power of **collectivity to act**: P_{act} is the probability that a winning vote arise.

$$P_{act} = \frac{|\mathcal{W}(N,v)|}{2^n}$$

- The power **to prevent** an action: $P_{prevent}$ captures the power of i to prevent a coalition to win by withholding its vote.

$$P_{prevent} = \frac{\sum_{\mathcal{C}\subseteq N\setminus\{i\}} v(\mathcal{C}\cup\{i\}) - v(\mathcal{C})}{|\mathcal{W}(N,v)|}$$

- The power **to initiate** an action: P_{init} captures the power of i to join a losing coalition so that it becomes a winning one.

$$P_{init} = \frac{\sum_{\mathcal{C}\subseteq N\setminus\{i\}} v(\mathcal{C}\cup\{i\}) - v(\mathcal{C})}{2^n - |\mathcal{W}(N,v)|}.$$

- Maybe only minimal winning coalitions are important to measure the power of an agent (non-minimal winning coalitions may form, but only the minimal ones are important to measure power).
- Let (N, v) be a simple game, $i \in N$ be an agent. $\mathcal{M}(N, v)$ denotes the set of minimal winning coalitions, $\mathcal{M}_i(N, v)$ denotes the set of minimal winning coalitions containing i.
- The **Deegan-Packel** power index of player i is:

$$I_{DP}(N, v, i) = \frac{1}{|\mathcal{M}(N, v)|} \sum_{\mathcal{C} \in \mathcal{M}_i(N, v)} \frac{1}{|\mathcal{C}|}.$$

- The **public good index** of player i is defined as

$$I_{PG}(N, v, i) = \frac{|\mathcal{M}_i(N, v)|}{\sum_{j \in N} |\mathcal{M}_j(N, v)|}.$$

$[4; 3,2,1,1]$

$$\mathcal{W} = \left\{ \begin{array}{c} \{1,2\},\{1,3\},\{1,4\}, \\ \{1,2,3\},\{1,2,4\},\{1,3,4\}, \\ \{2,3,4\},\{1,2,3,4\}\} \end{array} \right\}$$

$$\mathcal{M} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$$

$[5; 3,2,1,1]$

$$\mathcal{W} = \left\{ \begin{array}{c} \{1,2\},\{1,2,3\},\{1,2,4\}, \\ \{1,3,4\},\{1,2,3,4\}\} \end{array} \right\}$$

$$\mathcal{M} = \{\{1,2\},\{1,3,4\}\}$$

	1	2	3	4		1	2	3	4
β	$\frac{6}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	β	$\frac{5}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
I_B	$\frac{6}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	I_B	$\frac{5}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{1}{10}$
P_{act}	$\frac{8}{16} = \frac{1}{2}$				P_{act}	$\frac{5}{16}$			
$P_{prevent}$	$\frac{6}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$P_{prevent}$	$\frac{5}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
P_{init}	$\frac{6}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	P_{init}	$\frac{5}{11}$	$\frac{3}{11}$	$\frac{1}{11}$	$\frac{1}{11}$
I_{DP}	$\frac{1}{4} \cdot \frac{3}{2}$	$\frac{1}{4} \cdot (\frac{1}{2} + \frac{1}{3})$	$\frac{1}{4} \cdot (\frac{1}{2} + \frac{1}{3})$	$\frac{1}{4} \cdot (\frac{1}{2} + \frac{1}{3})$	I_{DP}	$\frac{1}{2} \cdot (\frac{1}{2} + \frac{1}{3})$	$\frac{1}{2} \cdot \frac{1}{2}$	$\frac{1}{2} \cdot \frac{1}{3}$	$\frac{1}{2} \cdot \frac{1}{3}$
I_{PG}	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	I_{PG}	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Summary

- We introduced the simple games
- We considered few examples
- We studied some power indices

Coming next

- Representation and Complexitity issues
- Are there some succint representations for some classes of games.
- How hard is it to compute a solution concept?