Two Interesting NTU-games

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April 21, 2010

Example 1 [Stable matching]

Suppose that $N = M \dot{\cup} W$, where $|M| = |W|$. We refer to the members of M as men and to the members of W as women. A ***matching*** is a function $f : N \rightarrow N$ that pairs men and women into couples. So for all $i \in N$ we have $f(f(i)) = i$ (i is matched with $f(i)$ and $f(i)$ is matched with i) and both $f(M) = W$ and $f(W) = M$ (each i is of the opposite sex than $f(i)$).

Suppose that each member i of N has a strict preference relation $\succ_i$ over the members of the opposite sex. We call a matching f ***unstable*** if for some $m \in M$ and $w \in W$

$$w \succ_m f(m) \text{ and } m \succ_w f(w).$$

In words, a matching is unstable if a man and a woman exist who both prefer each other than the assigned partner. A matching is ***stable*** if it is not unstable.

We now interpret the above setting as an NTU game. We take as the set of outcomes X the set of all matchings. Next, we extend each strict preference relation $\succ_i$ to a non-strict preference over the set of matchings X, by putting for $f, g \in X$

$$f \succeq_i g \text{ iff } f(i) \succ_i g(i) \text{ or } f(i) = g(i).$$

That is, player i prefers the matching f over the matching g if he/she strictly prefers his/her partner in f over his/her partner in g or both f and g assign the same partner to i.

Given a coalition $S \subseteq N$ we say that it is ***closed*** under a matching f if $f(S) = S$. Finally, for each coalition $S \subseteq N$ we define $V(S)$ as the set of matchings under which S is closed. That is,

$$V(S) := X \cap \{f : N \rightarrow N \mid f(S) = S\}.$$

We call the above NTU game the ***matching game***. □

We have the following characterization of stable matchings in game-theoretic terms.

Theorem 1 *The core of the matching game consists of the set of all stable matchings.*

Proof. Suppose that a matching f is unstable. So for some $m \in M$ and $w \in W$

$$w \succ_m f(m) \text{ and } m \succ_w f(w).$$

Then the matching f is blocked by the coalition $\{m, w\}$. Indeed, let g be an arbitrary matching such that $g(m) = w$ (and hence $g(w) = m$). By definition $g \in V(\{m, w\})$ and both $g \succ_m f$ and $g \succ_w f$. So f does not belong to the core.

Suppose now that a matching f does not belong to the core. Then it is blocked by some coalition S. So a matching g exists such that S is closed under g and for all $i \in S$ we have $g(i) \succ_i f(i)$.

S is non-empty and closed under g, so $S \cap M$ is non-empty. Take some $m \in S \cap M$. Let $w := g(m)$ (and hence $g(w) = m$). We have $g(m) \succ_m f(m)$ and, since $w \in S$, $g(w) \succ_w f(w)$, i.e., $w \succ_m f(m)$ and $m \succ_w f(w)$. So f is unstable. □

The core of the matching game is non-empty. An element of the core is constructed by means of so-called ***deferred acceptance procedure***.

Example 2 [House exchange]

Suppose that each player i owns a different house and that each player i has a strict preference relation $\succ_i$ over the set H of the houses owned. We allow that players exchange houses aiming at situations in which the resulting outcome is satisfactory to all players.

To formulate this problem as an NTU-game we take as the set of outcomes the set of possible house reallocations, i.e.,

$$X := \{f : N \to H \mid f \text{ is 1-1}\}$$

and assume that $h \in X$ is the *initial* house allocation.

Then we extend each strict preference relation $\succ_i$ to a non-strict preference over the set of reallocations X, by putting for $f, g \in X$

$$f \succeq_i g \text{ iff } f(i) \succ_i g(i) \text{ or } f(i) = g(i).$$

Finally, given a coalition S we define

$$V(S) := X \cap \{f : N \to H \mid f(S) = h(S) \text{ and } f(i) = h(i) \text{ for } i \notin S\}.$$

So $V(S)$ consists of the house reallocations that exchange among the members of S their initially owned houses and leave unchanged the houses allocated to the other players.

We call the resulting NTU game the ***house exchange game***. □

The core of the house exchange game is non-empty. An element of the core is constructed by means of so-called ***top cycle trading procedure***.