# Marginal Contribution Nets for Games with Externalities

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- The value of a coalition $C$ relative to a coalition structure $\pi$ is denote by $w(C, \pi)$. The requirement here is that $C$ must be a member of the coalition structure.
- Each such pair $(C, \pi)$ is a called an embedded coalition.

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- Each pattern is of the form $\mathcal{P}_0|\mathcal{P}_1, \ldots, \mathcal{P}_k$, where each $\mathcal{P}_i$ is a boolean expression over $N$.
- There is no obligation to specify any of the elements of the rule except for "|" and "*Value*".

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- The value $w(C, \pi)$ of an embedded coalition with respect to an embedded MC-net is given by

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- $P_i$ and $P_i'$ denote the set of positive and negative literals in $\mathcal{P}_i$, respectively.

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- $$a_3 \longrightarrow 1$$
$$a_3 | a_1 \wedge a_2 \longrightarrow 1$$
$$a_1 \wedge a_2 \longrightarrow 1$$

# Expressiveness

- Let $(C_0, \pi)$ be an embedded coalition with $\pi = \{C_0, C_1, \ldots, C_k\}$.
  $p_0 \wedge \neg \bigwedge_{i \neq 0} p_i | p_1 \wedge \neg \bigwedge_{i \neq 1} p_i, p_2 \wedge \neg \bigwedge_{i \neq 2} p_i, \ldots, p_m \wedge \neg \bigwedge_{i \neq m} p_i$
  Each $p_i$ is the conjuction of agents in coalition $C_i$.

# Extended generalized Shapley value

- Given $w$ and an embedded coalition $(C, \pi)$, the *EGSV* of $w$ is $\chi_w(C, \pi) = \sum_{C \in T \subseteq \pi} \frac{(|T|-1)!(|\pi|-|T|)!}{|\pi|!}(v(\lfloor T \rfloor) - v(\lfloor T \setminus \{C\} \rfloor))$ where $\lfloor T \rfloor = \cup_{A \in T} A$ and $v(S) = w(S, \{S, N \setminus S\})$.
- This is in fact the standard Shapley value of a game $w_\pi$ where $\pi = \{C_0, C_1, \ldots, C_m\}$ and $N_\pi = \{a_{C_0}, a_{C_1}, \ldots, a_{C_m}\}$ is the set of players and the value function $w_\pi$ is defined as $w_\pi(T) = w(\lfloor T \rfloor, \{\lfloor T \rfloor, \lfloor \pi \setminus T \rfloor\}$.

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- The Shapley value of $w_\pi$ is the EGSV of the original game. We already know how to compute the Shapley value from MC-nets.

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  2. Otherwise the rule would be $\emptyset \longrightarrow 0$.

# Example

$$\pi = \{\{a_1 a_2\}, \{a_3\}\}$$

$$N_\pi = \{a_{C_1}, a_{C_2}\}$$

$$a_3 \rightarrow 1 \quad \Rightarrow \quad a_{C_2} \rightarrow 1$$

$$a_3 | a_1 \wedge a_2 \rightarrow \quad \Rightarrow \quad a_{C_2} \wedge \neg a_{C_1} \rightarrow 1$$

$$a_1 \wedge a_2 \rightarrow 1 \quad \Rightarrow \quad a_{C_1} \rightarrow 1$$

# Conclusion

- A compact and simple representation method for games with externalities was demonstrated.
- It was shown that this method expresses all such games and it is exponentially more efficient.
- A method to compute the extended generalized Shapley value using embedded MC-nets was discussed which reveals the computational power of this representation method.