Cooperative games: lecture 3

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Let N be the set of customers, and let 0 be the supplier. Let $N_* = N \cup \{0\}$. for $(i, j) \in N_*^2$, $i \neq j$, let $c_{i,j}$ be the cost of connecting i and j by the edge e_{ij}. Let (N, c) be the corresponding cost game, which is called a minimum cost spanning tree game ((MCST) game).

Theorem 1. *Every minimum cost spanning tree game has a non-empty core.*

Proof. Let us define a cost distribution x and then we will show that x is in the core.

Let $T = (N, E_N)$ a minimum cost spanning tree for the graph $\left(N_*, c_{\{ij\} \subseteq N_*^2}\right)$. Let i be a customer. Since T is a tree, there is a unique path $(0, a_1, \ldots, a_k, i)$ from 0 to i. The cost paid by agent i is defined by $x_i = c_{a_k,i}$.

This cost allocation is efficient by construction of x.

We need to show the cost allocation is group rational, i.e. for all coalition S, we have $x(S) \leq v(S)$ (it is a cost, which explains the inequality). Let $S \subset N$ and $T_S = (S \cup \{0\}, E_s)$ be a minimum cost spanning tree of the graph $\left(S \cup \{0\}, c_{\{ij\} \in S \cup \{0\}}\right)$. Let extand the tree T_S to a graph $T_S^+ = (N_*, E_N^+)$ by adding the remaining customers $N \setminus S$, and for each customer $i \in N \setminus S$, we add the edge of E_N ending in i, i.e., we add the edge (a_k, i). The graph T_S^+ has $|S| + |N \setminus S|$ edges an is connected. Hence, T_S^+ is a spanning tree. Now, we note that $c(S) + x(N \setminus S) = \sum_{e_{ij} \subseteq E_N^+} c_{ij} \geq \sum_{e_{ij} \subseteq E_N} = c(N) = x(N)$. The inequality is due to the fact that T_S^+ is a spanning tree, and T is a minimum spanning tree. It follows that $x(S) \leq v(S)$. ✔ □

A **market** is a quadruple (N, M, A, F) where

- N is a set of traders
- M is a set of m continuous good
- $A = (a_i)_{i\in N}$ is the initial endowment vector
- $F = (f_i)_{i\in N}$ is the valuation function vector
- $v(S) = \max \left\{ \sum_{i\in S} f_i(x_i) \;\middle|\; x_i \in \mathbb{R}_+^m, \sum_{i\in S} x_i = \sum_{i\in S} a_i \right\}$
- we further assume that the f_i are continuous and concave.

Theorem 2. *Every Market Game is balanced*

Proof. $f : \mathbb{R}^n \to \mathbb{R}$ is concave iff $\forall \alpha \in [0,1]$, $\forall (x,y) \in \mathbb{R}^n$, $f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha) f(y)$. It follows from this definition that for $f : \mathbb{R} \to \mathbb{R}$, $\forall x \in \mathbb{R}^n$, $\forall \lambda \in \mathbb{R}_+^n$ such that $\sum_{i=1}^n \lambda_i = 1$, we have $f(\sum_{i=1}^n \lambda_i x_i) \le \sum_{i=1}^n \lambda_i f(x_i)$.

Since the f_is are continuous, $\sum_{i\in S} f_i(x_i)$ is a continuous mapping from $\left\{(x_i)_{i\in S} \mid \forall i \in \mathbb{R}_+^k, \forall x_i \in \mathbb{R}_+^k, \sum_{i\in S} x_i = \sum_{i\in S} a_i\right\}$ to $\mathbb{R}$, hence, the maximum exists.

For a coalition $S \subseteq N$, let $x^S = \langle x_1^S, \ldots, x_n^S \rangle$ be the endowment that achieves the maximum value for the coalition S, i.e., $v(S) = \sum_{i\in S} f_i(x_i^S)$. In other words, the members of S have made some trades that have improved the value of the coalition S up to its maximal value.

Let λ be a balanced map. Let $y \in \mathbb{R}_+^n$ defined as follows: $y_i = \sum_{S\in\mathcal{C}_i} \lambda_S x_i^S$ where $\mathcal{C}_i$ is the set of coalitions that contains agent i.

First, note that y is a feasible payoff function.

$$\begin{aligned}
\sum_{i\in N} y_i &= \sum_{i\in N}\sum_{S\in\mathcal{C}_i} \lambda_S x_i^S = \sum_{S\subseteq N}\sum_{i\in S} \lambda_S x_i = \sum_{S\subseteq N} \lambda_S \sum_{i\in S} x_i^S \\
&= \sum_{S\subseteq N} \lambda_S \sum_{i\in S} a_i \text{ since } x_i^S \text{ was achieved by a sequence of trades within the members of } S \\
&= \sum_{i\in N} a_i \sum_{S\in\mathcal{C}_i} \lambda_S \\
&= \sum_{i\in N} a_i \text{ as } \lambda \text{ is balanced, (i.e., the sum of the weights over all coalitions of one agent sums up to 1)}
\end{aligned}$$

then, by definition of v, we have $v(N) \ge \sum_{i\in N} f_i(y_i)$. ✔

The f_i are concave and since $\sum_{S\in\mathcal{C}_i} \lambda_S = 1$, we have $f_i(\sum_{S\in\mathcal{C}_i} \lambda_S x_i^S) \ge \sum_{S\in\mathcal{C}_i} \lambda_S f_i(x_i^S)$. It follows:

$v(N) \ge \sum_{i\in N} f_i(y_i) \ge \sum_{i\in N} f_i(\sum_{S\in\mathcal{C}_i} \lambda_S x_i^S) \ge \sum_{i\in N}\sum_{S\in\mathcal{C}_i} \lambda_S f_i(x_i^S) \ge \sum_{S\subseteq N} \lambda_S \sum_{i\in S} f_i(x_i^S) \ge \sum_{S\subseteq N} \lambda_S v(S)$. This inequality proves that the game is balanced. ✔ □