# Weighted Voting Games

## and the tension between weight and power

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# Outline

# Introduction



## What are we going to talk about?

- Voting within EU (is it *fair*?);
- How to develop a fair game.

## Weighted Voting Games (WVG)?

A game $(N, w_{i \in N}, q)$ is a *weighted voting game* if:

- $v$ satisfies unanimity ($\sum_{i \in N} w_i \geq q$)
- $v$ satisfies monotonicity ($\forall i \in N : w_i \geq 0$)
- $v$ is defined as follows:

$$v(C) = \begin{cases} 1 & \text{if } \sum_{i \in C} w_i \geq q \\ 0 & \text{otherwise} \end{cases}$$

- representation: $[q; w_1, \ldots, w_n]$

# Power Indices

Several power indices have been proposed to represent the power of each player, among which:

- **Shapley-Shubik index**, $I_{SS}(N, v, i)$
  'For each permutation, the pivotal player gets a point.'

$$I_{SS}(N, v, i) = \sum_{C \subseteq N \setminus \{i\}} \frac{|C|!(|N| - |C| - 1)!}{|N|!}(v(C \cup \{i\}) - v(C))$$

- **(raw) Banzhaf index**, $\beta_i$
  'For each coalition in which a player is pivotal, it gets a point.'

$$\beta_i = \frac{\sum_{C \subseteq N \setminus \{i\}} \left(v(C \cup \{i\}) - v(C)\right)}{2^{n-1}}$$

# EU: History of Voting Methods

- EU consists of 25 member countries.
- In Nice (2000) a new voting scheme was developed to improve the decision process of the EU. This game is called the *Nice Rule*.
- In Brussels (2004) a new voting scheme was approved by the EU. This game is called the *European Constitution Rule*. It is also extended to 27 member countries (Romania and Bulgaria)
- Algaba et al. (2007) analyse both games and try to find a game that is fairer.

# Nice Game (1)

Representing the Nice Game:
$v_1 = [232; 29, 29, 29, 29, 27, 27, 13, 12, 12, 12, 12, 12, 10, 10, 7, 7, 7, 7, 7, 4, 4, 4, 4, 4, 3]$,
with member states sorted by decreasing population.

Additional requirements on a winning coalition:

- $v_2$: it must contain at least 13 members;
  $v_2 = [13; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1]$
- $v_3$: it must contain at least 62% of the population of the EU.
  $v_3 = [620; 182, 131, 131, 130, 126, 91, 84, 36, 24, 23, 23, 22, 22, 20, 18, 12, 12, 11, 9, 8, 5, 4, 3, 2, 1, 1]$, where $w_{3,i}$ is proportional to the population of $i$ ($\sum_{i \in N} w_{3,i} = 1000$)

# Nice Game (2)

A coalition $S$ must be winning in each of $v_1$, $v_2$ and $v_3$.
Notation: $v_1 \wedge v_2 \wedge v_3$. This is called a *weighted 3-majority game*.

$$(v_1 \wedge v_2 \wedge v_3)(S) = \begin{cases} 1 & \text{if } w_i(S) \geq q_i \text{ where } 1 \leq i \leq 3 \\ 0 & \text{otherwise} \end{cases}$$

Here $v_i = [q_i; w_{i,1}, \ldots, w_{i,n}]$ and $w_i(S) = \sum_{j \in S} w_{i,j}$.

# European Constitution Game (1)



Representing the European Constitution Game:

Similar to the Nice Game, except for:

- $v_2'$: 15 countries are needed to win: $v_2$ with $q_2 \leftarrow 15$
- $v_3'$: These 15 countries must sum up to at least 65% of the population of the EU: $v_3$ with $q_3 \leftarrow 650$.
- $bc$ : The minimum number of countries to block a coalition is 4.

# European Constitution Game (2)



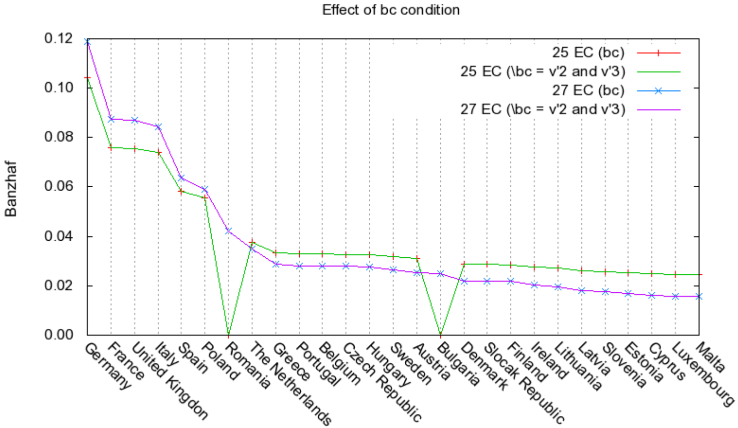
Representing the European Constitution Game:

It is a bit harder than representing the Nice game. But it is shown in the paper that if *bc* is the game [22; 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1] ($S$ winning if $|S| \geq 22$) then the game can be represented as

$$\left(v_2' \wedge v_3'\right) \vee bc$$

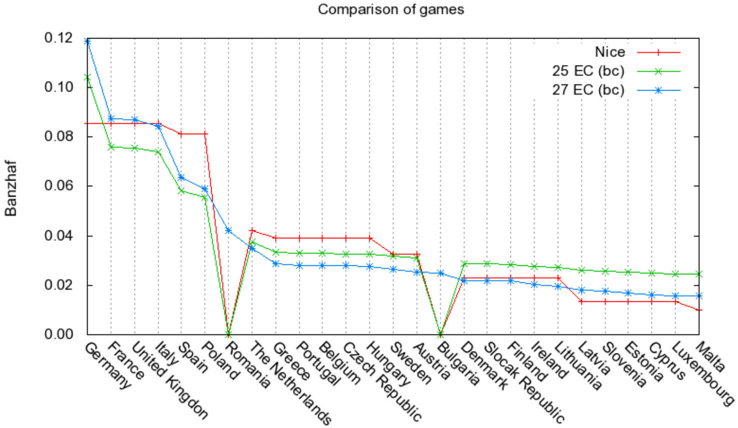
I will see if this blocking clause has an effect at all.

# Effect of Blocking Clause




Hence, the blocking game does not have any effect and just makes the total game more complicated.

# Comparison of Power in the two Games




The power is distributed differently in each game. Which one is fairest?

# What do we want? (1)

- We want each person in each member country to have equal power.
- There is a two tier system:
  - Each person votes for a representative to send to the EU meeting;
  - The representatives from each country vote.
- How to distribute the weight of each representative based on the number of people in the country he represents?

# What do we want? (2)





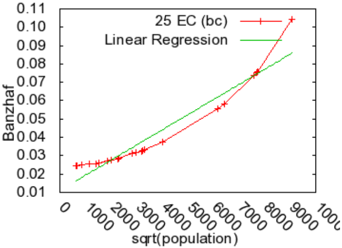
If $\beta'_x$ is the normalized Banzhaf index for a person in a country $i$ in EU with population $n_i$, and $\beta'_i$ is the normalized Banzhaf index of a representative for $i$, then Felsenthal and Machover have shown that:

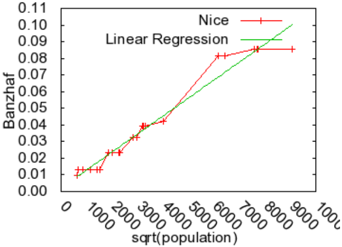
$$\beta'_x \propto \beta'_i \sqrt{\frac{2}{\pi n_i}}$$

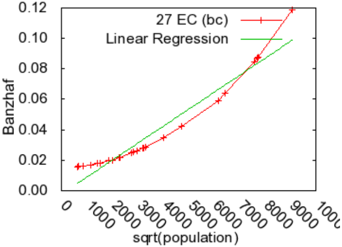
Thus the Banzhaf index of each representative $\beta'_i$ should be $\propto \sqrt{n_i}$ for each person in the EU to have equal power.
Let's see if this is the case for any of the games presented here.

# Fair games?

Fair games should have all points on the regression line since we want $\beta_i' \propto \sqrt{n_i}$. This is not the case for any of the current voting systems.



# Inverse Problem





### Inverse Problem?

Given a power index $\vec{p}$, find a WVG such that
the power of each player $i$ is as close as possible to $p_i$.

### Solution: *(de Keijzer et al. 2010)*

1. Enumerate all WVGs of $n$ players;
2. Compute for each WVG its power index;
3. Output the WVG which power index is closest to $\vec{p}$.

### Has anybody solved this before?

Not really, there are some hill climbing algorithms,
but they do not guarantee an optimal outcome.

# Terminology

Representing WVGs:
$(q : w_1, \ldots, w_n) \iff (N, W_{min}) \iff (N, L_{max})$

Representation Languages:
$\mathcal{L}_{weights}$; $\mathcal{L}_{W,min}$; $\mathcal{L}_{L,max}$

Linear Games:

- *desirability relation* $\succeq_D$, where:
  $i \succeq_D j$ iff $\forall S \subseteq N \backslash \{i, j\} : v(S \cup \{i\}) \geq v(S \cup \{j\})$
- $(N, v)$ is *linear* iff $\forall i, j \in N : i \succeq_D j$ or $j \succeq_D i$
  $\Longrightarrow$ every WVG is a linear game;
- $(q : w_1, \ldots, w_n)$ is a *canonical WVG* iff $1 \succeq_D \cdots \succeq_D n$.

# Observations

- the number of *weighted representations* for each WVG is infinite;
- there is exactly one $(N, W_{min})$-*representation* for each WVG;
- the number of WVGs of $n$ players is finite,
  because there are only finitely many sets of MWCs for $n$ players.

# Naive Algorithm





**Algorithm 1:** Solving the Inverse Problem

**Input**: target power index $\vec{p} = (p_1, \ldots, p_n)$

**Output**: $l \in \mathcal{L}_{W,min}$ such that $f(G_l)$ is as close as possible to $\vec{p}$

**begin**

  `bestgame` $\leftarrow 0$;

  `besterror` $\leftarrow \infty$;

  **forall** $l \in \mathcal{L}_{W,min}$ **do**

    Compute $f(G_l) = (f(G_l, 1), \ldots, f(G_l, n))$;

    `error` $\leftarrow \sum_{i=1}^{n}(f(G_l, i) - p_i)^2$;

    **if** `error` $<$ `besterror` **then**

      `bestgame` $\leftarrow l$;

      `besterror` $\leftarrow$ `error`;

  **return** `bestgame`;

**end**

# Running Time

- power index function $f$: $\leq$ exponential in $n$;
  (for all known power indices)
- enumeration of $\mathcal{L}_{W,min}$: doubly exponential in $n$.

Solution:
Improving the enumeration method.

# Improvements on Enumeration (1)

- focus on $\mathcal{G}_{cwvg}(n)$

Define poset $(\mathcal{G}_{cwvg}(n), \supseteq_{MWC})$ as follows:
$G_1 \supseteq_{MWC} G_2$ iff $W_{min}(G_1) \supseteq W_{min}(G_2)$,
where $G_1, G_2 \in \mathcal{G}_{cwvg}(n)$

Define rank function $\rho : \mathcal{G}_{cwvg}(n) \to \mathbb{Z}$ as follows:
$\rho(G) := |W_{min}(G)|$

Claim: $(\mathcal{G}_{cwvg}(n), \supseteq_{MWC})$ is graded under $\rho$
i.e. $\forall G_1, G_2 \in \mathcal{G}_{cwvg}(n) : \rho(G_1) = \rho(G_2) - 1$ if $G_1$ *covers* $G_2$

$G_1$ *covers* $G_2$ iff
$G_1 \subseteq_{MWC} G_2$ and there is no $G_3$ such that $G_1 \subseteq_{MWC} G_3 \subseteq_{MWC} G_2$

Proof omitted.

# Improvements on Enumeration (2)





Definition of $\mathtt{tr}(C, i)$:

Let $(N, v)$ be a canonical WVG and $C \subseteq N$. Let $p_i$ be the $i$th highest-numbered player among the players in $C$. Define $\mathtt{tr}(C, i)$ as follows:

$$\mathtt{tr}(C, i) := \begin{cases} C \backslash \{p_i, \ldots, n\} & \text{if } 0 < i \leq |C| \\ C & \text{if } i = 0 \\ \text{undefined} & \text{otherwise} \end{cases}$$

Claim:

$\forall G_1, G_2 \in \mathcal{G}_{cwvg}(n)$ such that $G_1$ covers $G_2$ in $(\mathcal{G}_{cwvg}(n), \supseteq_{MWC})$ : there is a $C \in L_{max}(G_1)$ and an $i \in \mathbb{N}$ with $0 \leq i \leq n$ such that $W_{min}(G_2) = W_{min}(G_1) \cup \mathtt{tr}(C, i)$.

Proof on the blackboard (if time allows).

# Enumeration Algorithm

**Algorithm 2:** Enumerating the class of $n$-agent canonical WVGs

```
begin
    output (1 : 0, ..., 0);
    games[0] ← {∅};
    for i = 1 to (n choose ⌊n/2⌋) do
        forall W_min ∈ games[i − 1] do
            L_max ← computeMLCs(W_min);
            forall C ∈ L_max do
                for j = 1 to n do
                    if isweighted(W_min ∪ tr(C, i)) then
                        if W_min ∪ tr(C, i) passes the
                        duplicates-check then
                            output the weighted representation of
                            the voting game with MWCs
                            W_min ∪ tr(C, i);
                            append W_min ∪ tr(C, i) to games[i];
end
```

# Observations

- duplicates-check is necessary,
  because $(\mathcal{G}_{cwvg}(n), \supseteq_{MWC})$ is not a tree;
- running time: exponential in $n$.

Solution Algorithm:

Incorporating Algorithm 2 into Algorithm 1 gives an exact anytime algorithm for solving the Inverse Problem. This solution algorithm runs in time exponential in $n$.

# Conclusion

- exact anytime algorithm for solving the Inverse Problem;
- algorithm runs in time exponential in the number of players;
- however,