Cooperative Games

Lecture 3: The core

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Today

- Characterize the set of games with non-empty core (Bondareva Shapley theorem), and we will informally introduce linear programming.
- Application of the Bondareva Shapley theorem to market games.

The Bondareva Shapley theorem:
a characterization of games with non-empty core.

The theorem was proven independently by
O. Bondareva (1963) and L. Shapley (1967).

Let $\mathcal{C} \subseteq N$. The **characteristic vector** $\chi_{\mathcal{C}}$ of $\mathcal{C}$ is the member of $\mathbb{R}^N$ defined by $\chi^i_{\mathcal{C}} = \begin{cases} 1 \text{ if } i \in \mathcal{C} \\ 0 \text{ if } i \in N \setminus \mathcal{C} \end{cases}$

A **map** is a function $2^N \setminus \emptyset \to \mathbb{R}_+$ that gives a positive weight to each coalition.

Definition (Balanced map)

A function $\lambda : 2^N \setminus \emptyset \to \mathbb{R}_+$ is a **balanced map** iff $\sum_{\mathcal{C} \subseteq N} \lambda(\mathcal{C}) \chi_{\mathcal{C}} = \chi_N$

A map is balanced when the amount received over all the coalitions containing an agent i sums up to 1.

Example: $n = 3$, $\lambda(\mathcal{C}) = \begin{cases} \frac{1}{2} \text{ if } |\mathcal{C}| = 2 \\ 0 \text{ otherwise} \end{cases}$

	1	2	3
{1,2}	$\frac{1}{2}$	$\frac{1}{2}$	0
{1,3}	$\frac{1}{2}$	0	$\frac{1}{2}$
{2,3}	0	$\frac{1}{2}$	$\frac{1}{2}$

Each of the column sums up to 1.
$\frac{1}{2}\chi_{\{1,2\}} + \frac{1}{2}\chi_{\{1,3\}} + \frac{1}{2}\chi_{\{2,3\}} = \chi_{\{1,2,3\}}$

Characterization of games with non-empty core

Definition (Balanced game)

A game is **balanced** iff for each balanced map λ we have $\sum_{\mathcal{C} \subseteq N, \mathcal{C} \neq \emptyset} \lambda(\mathcal{C}) v(\mathcal{C}) \leqslant v(N)$.

Theorem (Bondareva Shapley)

A TU game has a non-empty core iff it is balanced.

Some idea of the proof

Notations:

- Let $\mathcal{V}(N) = \mathcal{V}$ the set of all coalition functions on 2^N.
- Let $\mathcal{V}_{Core} = \{v \in \mathcal{V} \mid Core(N,v) \neq \emptyset\}$.

Can we characterize $\mathcal{V}_{Core}$?

$Core(N,v) = \{x \in \mathbb{R}^n \mid x(\mathcal{C}) \geqslant v(\mathcal{C}) \text{ for all } \mathcal{C} \subseteq N\}$

The core is defined by a set of linear constraints.

➯ The idea is to use results from linear optimization.

Linear programming

A linear program has the following form:

$$\begin{cases} \max c^T x \\ \text{subject to} \begin{cases} Ax \leqslant b, \\ x \geqslant 0 \end{cases} \end{cases}$$

- x is a vector of n variables
- c is the objective function
- A is a $m \times n$ matrix
- b is a vector of size n
- A and b represent the linear constraints

example: maximize $8x_1 + 10x_2 + 5x_3$

$$\text{subject to} \begin{cases} 3x_1 + 4x_2 + 2x_3 \leqslant 7 & (1) \\ x_1 + x_2 + x_3 \leqslant 2 & (2) \end{cases}$$

$$A = \begin{pmatrix} 3 & 4 & 2 \\ 1 & 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}, c = \begin{pmatrix} 8 \\ 10 \\ 5 \end{pmatrix}.$$

A **feasible solution** is a solution that satisfies the constraints.

Example: maximize $8x_1 + 10x_2 + 5x_3$

subject to $\begin{cases} 3x_1 + 4x_2 + 2x_3 \leqslant 7 & (1) \\ x_1 + x_2 + x_3 \leqslant 2 & (2) \end{cases}$

- $\langle 0,1,1 \rangle$ is feasible, with objective function value 15.
- $\langle 1,1,0 \rangle$ is feasible, with objective function value 18.

The **dual** of a LP: finding an upper bound to the objective function of the LP.

$(1) \times 1 + (2) \times 6$ ➯ $9x_1 + 10x_2 + 8x_3 \leqslant 19$

$(1) \times 2 + (2) \times 2$ ➯ $8x_1 + 10x_2 + 6x_3 \leqslant 18$

The coefficients are as large as in the obective function,

➯ the bound is an upper bound for the objective function.

Hence, the solution cannot be better than 18, and we found one, Problem solved! ✔

Primal	Dual
$\begin{cases} \max c^T x \\ \text{subject to} \begin{cases} Ax \leqslant b, \\ x \geqslant 0 \end{cases} \end{cases}$	$\begin{cases} \min y^T b \\ \text{subject to} \begin{cases} y^T A \geqslant c^T, \\ y \geqslant 0 \end{cases} \end{cases}$

Theorem (Duality theorem)

When the primal and the dual are feasible, they have optimal solutions with equal value of their objective function.

We consider the following **linear programming** problem:

$$(LP)\begin{cases}\min x(N) \\ \text{subject to } x(\mathcal{C}) \geqslant v(\mathcal{C}) \text{ for all } \mathcal{C} \subseteq N,\ S \neq \emptyset\end{cases}$$

$v \in \mathcal{V}_{core}$ iff the value of (LP) is $v(N)$.

The dual of (LP):

$$(DLP)\begin{cases}\max \sum_{\mathcal{C}\subseteq N} y_{\mathcal{C}} v(\mathcal{C}) \\ \text{subject to } \begin{cases}\sum_{\mathcal{C}\subseteq N} y_{\mathcal{C}} \chi_{\mathcal{C}} = \chi_N \text{ and,} \\ y_{\mathcal{C}} \geqslant 0 \text{ for all } \mathcal{C} \subseteq N,\ \mathcal{C} \neq \emptyset.\end{cases}\end{cases}$$

It follows from the duality theorem of linear programming: (N, v) has a non empty core iff $v(N) \geqslant \sum_{\mathcal{C}\subseteq N} y_{\mathcal{C}} v(\mathcal{C})$ for all feasible vector $(y_{\mathcal{C}})_{\mathcal{C}\subseteq N}$ of (DLP).

Recognize the balance map in the constraint of (DLP)

A **market** is a quadruple (N, M, A, F) where

- N is a set of traders
- M is a set of m continuous good
- $A = (a_i)_{i \in N}$ is the initial endowment vector
- $F = (f_i)_{i \in N}$ is the valuation function vector
- $v(S) = \max \left\{ \sum_{i \in S} f_i(x_i) \;\middle|\; x_i \in \mathbb{R}^m_+, \sum_{i \in S} x_i = \sum_{i \in S} a_i \right\}$
- we further assume that the f_i are continuous and concave.

Theorem

Every Market Game is balanced

Coalition Structure

Definition (Coalition Structure)

A **coalition structure (CS)** is a partition of the grand coalition into coalitions.
$\mathcal{S} = \{\mathcal{C}_1, \ldots, \mathcal{C}_k\}$ where $\cup_{i \in \{1..k\}} \mathcal{C}_i = N$ and $i \neq j \Rightarrow \mathcal{C}_i \cap \mathcal{C}_j = \emptyset$.
We note $\mathscr{S}_N$ the set of all coalition structures over the set N.

ex: $\{\{1,3,4\}\{2,7\}\{5\}\{6,8\}\}$ is a coalition structure for $n = 8$ agents.

We will study three solution concepts: the **bargaining set**, the **nucleolus** and the **kernel**. They form the **"bargaining set family"** and we will see later why. In addition, the definition of each of these solution concepts uses a CS.

We start by defining a game with coalition structure, and see how we can define the core of such a game. Then, we'll start studying the bargaining set family.

Game with Coalition Structure

Definition (TU game)

A TU game is a pair (N,v) where N is a set of agents and where v is a valuation function.

Definition (Game with Coalition Structures)

A **TU-game with coalition structure** $(N,v,\mathcal{S})$ consists of a TU game (N,v) and a CS $\mathcal{S} \in \mathscr{S}_N$.

- We assume that the players agreed upon the formation of $\mathcal{S}$ and only the payoff distribution choice is left open.
- The CS may model affinities among agents, may be due to external causes (e.g. affinities based on locations).
- The agents may refer to the value of coalitions with agents outside their coalition (i.e. opportunities they would have outside of their coalition).
- (N,v) and $(N,v,\{N\})$ represent the same game.

Definition (*core* of a game (N,v))

The core of a TU game (N,v) is defined as
$Core(N,v) = \{x \in \mathbb{R}^n \mid x(N) \leqslant v(N) \wedge x(\mathcal{C}) \geqslant v(\mathcal{C})\ \forall \mathcal{C} \subseteq N\}$

The set of **feasible** payoff vectors for $(N,v,\mathcal{S})$ is
$X_{(N,v,\mathcal{S})} = \{x \in \mathbb{R}^n \mid$ for every $\mathcal{C} \in \mathcal{S}$ $x(\mathcal{C}) \leqslant v(\mathcal{C})\}$.

Definition (Core of a game with CS)

The **core** $Core(N,v,\mathcal{S})$ of $(N,v,\mathcal{S})$ is defined by
$\{x \in \mathbb{R}^n \mid (\forall \mathcal{C} \in \mathcal{S}, x(\mathcal{C}) \leqslant v(\mathcal{C}))$ and $(\forall \mathcal{C} \subseteq N, x(\mathcal{C}) \geqslant v(\mathcal{C}))\}$

We have $Core(N,v,\{N\}) = Core(N,v)$.

The next theorems are due to Aumann and Drèze.

R.J. Aumann and J.H. Drèze. **Cooperative games with coalition structures**, *International Journal of Game Theory*, 1974

Definition (Superadditive cover)

The **superadditive cover** of (N,v) is the game $(N,\hat{v})$ defined by

$$\begin{cases} \hat{v}(\mathcal{C}) = \max_{\mathcal{P}\in\mathscr{S}_{\mathcal{C}}} \left\{ \sum_{T\in\mathcal{P}} v(T) \right\} \forall \mathcal{C} \subseteq N \setminus \emptyset \\ \hat{v}(\emptyset) = 0 \end{cases}$$

- We have $\hat{v}(N) = \max_{\mathcal{P}\in\mathscr{S}_N} \left\{ \sum_{T\in\mathcal{P}} v(T) \right\}$, i.e., $\hat{v}(N)$ is the maximum value that can be produced by N. We call it the **value of the optimal coalition structure**.
- The superadditive cover is a superadditive game (**why?**).

Theorem

Let $(N,v,\mathcal{S})$ be a game with coalition structure. Then

a) $Core(N,v,\mathcal{S}) \neq \emptyset$ iff $Core(N,\hat{v}) \neq \emptyset \wedge \hat{v}(N) = \sum_{\mathcal{C}\in\mathcal{S}} v(\mathcal{C})$

b) if $Core(N,v,\mathcal{S}) \neq \emptyset$, then $Core(N,v,\mathcal{S}) = Core(N,\hat{v})$

Definition (Substitutes)

Let (N, v) be a game and $(i, j) \in N^2$. Agents i and j are **substitutes** iff $\forall \mathcal{C} \subseteq N \setminus \{i, j\}$, $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$.

A nice property of the core related to fairness:

Theorem

Let $(N, v, \mathcal{S})$ be a game with coalition structure,
let i and j be substitutes, and let $x \in Core(N, v, \mathcal{S})$.
If i and j belong to different members of $\mathcal{S}$, then $x_i = x_j$.

Summary

- We introduced a stability solution concept: the core.
- we looked at examples:
  - individual games: some games have an empty core.
  - classes of games have a non-empty core: e.g. convex games and minimum cost spanning tree games.
- We look at a characterization of games with non-empty core: the Shapley Bondareva theorem, which relies on a result from linear programming.
- We Apply the Bondareva-Shapley to market games.
- We considered the core of games with coalition structures.

Coming next

- Bargaining sets.