

# Cooperative Games

## Lecture 4: The Bargaining Set

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## Today

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- If agents desire the kind of stability offered by the core, they will be unable to reach an agreement.
- ⇒ they have no choice but to **relax** their stability requirements.
- Need a solution that allows agents to reach an agreement, but maintain some stability.
- Then, we will consider the bargaining set, which relaxes the requirements of the core.

A second solution concept:

## The bargaining set.

R.J. Aumann and M. Maschler. **The bargaining set for cooperative games**, in *Advances in game theory (Annals of mathematics study)*, 1964.

M. Davis and M. Maschler. **Existence of stable payoff configurations for cooperative games**, *Bulletin of the American mathematical society*, 1963.

Let  $(N, v, \mathcal{S})$  be a game with coalition structure and  $x$  an imputation.

The bargaining set models stability in the following sense:

Any **argument** from an agent  $i$  against a payoff distribution  $x$  is of the following form:

*I get too little in the imputation  $x$ , and agent  $j$  gets too much! I can form a coalition that excludes  $j$  in which some members benefit and all members are at least as well off as in  $x$ .*

The argument is **ineffective** for the bargaining set if agent  $j$  can answer the following:

*I can form a coalition that excludes agent  $i$  in which all agents are at least as well off as in  $x$ , and as well off as in the payoff proposed by  $i$  for those who were offered to join  $i$  in the argument.*

## Definition (Objection)

Let  $(N, v, \mathcal{S})$  be a game with coalition structure,  $x \in X_{(N, v, \mathcal{S})}$  (the set of all feasible payoff vectors for  $(N, v, \mathcal{S})$ ),  $\mathcal{C} \in \mathcal{S}$  be a coalition, and  $i$  and  $j$  two distinct members of  $\mathcal{C}$  ( $(i, j) \in \mathcal{C}^2, i \neq j$ ).

An **objection of  $i$  against  $j$**  is a pair  $(P, y)$  where

- $P \subseteq N$  is a coalition such that  $i \in P$  and  $j \notin P$ .
- $y \in \mathbb{R}^p$  where  $p$  is the size of  $P$
- $y(P) \leq v(P)$  ( $y$  is a feasible payoff distribution for the agents in  $P$ )
- $\forall k \in P, y_k \geq x_k$  and  $y_i > x_i$  (agent  $i$  strictly benefits from  $y$ , and the other members of  $P$  do not do worse in  $y$  than in  $x$ .)

An objection  $(P, y)$  of  $i$  against  $j$  is a **potential threat** by coalition  $P$ , which contains  $i$  but not  $j$ , to deviate from  $x$ . The goal is not to change  $\mathcal{S}$ , but to obtain a side payment from  $j$  to  $i$ , i.e., to modify  $x$  within  $X_{(N, v, \mathcal{S})}$ .

### Definition (Counter-objection)

An **counter-objection to**  $(P, y)$  is a pair  $(Q, z)$  where

- $Q \subseteq N$  is a coalition such that  $j \in Q$  and  $i \notin Q$ .
- $z \in \mathbb{R}^q$  where  $q$  is the size of  $Q$
- $z(Q) \leq v(Q)$  ( $z$  is a feasible payoff distribution for the agents in  $Q$ )
- $\forall k \in Q, z_k \geq x_k$  (the members of  $Q$  get at least the value in  $x$ )
- $\forall k \in Q \cap P, z_k \geq y_k$  (the members of  $Q$  which are also members of  $P$  get at least the value promised in the objection)

In a counter-objection, agent  $j$  must show that it can protect its payoff  $x_j$  in spite of the existing objection of  $i$ .

### Definition (Stability)

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Let  $(N, v, \mathcal{S})$  a game with coalition structure. A vector  $x \in X_{(N, v, \mathcal{S})}$  is **stable** iff for each objection at  $x$  there is a counter-objection.

### Definition (Pre-bargaining set)

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The **pre-bargaining set** (*preBS*) is the set of all stable members of  $X_{(N, v, \mathcal{S})}$ .

### Lemma

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Let  $(N, v, \mathcal{S})$  a game with coalition structure, we have  
$$\text{Core}(N, v, \mathcal{S}) \subseteq \text{preBS}(N, v, \mathcal{S}).$$

This is true since, if  $x \in \text{Core}(N, v, \mathcal{S})$ , no agent  $i$  has any objection against any other agent  $j$ .

## Example

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Let  $(N, v)$  be a 7-player simple majority game, i.e.

$$v(C) = \begin{cases} 1 & \text{if } |C| \geq 4 \\ 0 & \text{otherwise} \end{cases} .$$

Let us consider  $x = \langle -\frac{1}{5}, \frac{1}{5}, \dots, \frac{1}{5} \rangle$ . It is clear that  $x(N) = 1$ .

Let us prove that  **$x$  is in the pre-bargaining set** of the game  $(N, v, \{N\})$ .

Objections within members of  $\{2, 3, 4, 5, 6, 7\}$  will have a counter-objection by symmetry. ✓

Let us consider the objections  $(P, y)$  of 1 against another member of  $\{2, 3, 4, 5, 6, 7\}$ . Since the players  $\{2, \dots, 7\}$  play symmetric roles, we consider an objection  $(P, y)$  of 1 against 7 using successively as  $P$   $\{1, 2, 3, 4, 5, 6\}$ ,  $\{1, 2, 3, 4, 5\}$ ,  $\{1, 2, 3, 4\}$ ,  $\{1, 2, 3\}$ ,  $\{1, 2\}$  and  $\{1\}$ . We will look for a counter-objection of player 7 using  $(Q, z)$ .



- $P = \{1, 2, 3, 4, 5, 6\}$ . We need to find the payoff vector  $y \in \mathbb{R}^6$  so that  $(P, y)$  is an objection.  $y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \dots, \frac{1}{5} + \alpha_6 \rangle$ .

The conditions for  $(P, y)$  to be an objection are the following:

- each agent is as well off as in  $x$ :  $\alpha > -\frac{1}{5}$ ,  $\alpha_i \geq 0$
- $y$  is feasible for coalition  $P$ :  $\sum_{i=2}^6 \left( \alpha_i + \frac{1}{5} \right) + \alpha \leq 1$ .

w.l.o.g  $0 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \alpha_5 \leq \alpha_6$ .

$$\text{Then } \sum_{i=2}^6 \left( \frac{1}{5} + \alpha_i \right) + \alpha = \frac{5}{5} + \sum_{i=2}^6 \alpha_i + \alpha = 1 + \sum_{i=2}^6 \alpha_i + \alpha \leq 1.$$

$$\text{Then } \sum_{i=2}^6 \alpha_i \leq -\alpha < \frac{1}{5}.$$

⇒ We need to find a counter-objection for  $(P, y)$ .

**claim:** we can choose  $Q = \{2, 3, 4, 5\}$  and

$$z = \langle \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5} + \alpha_4, \frac{1}{5} + \alpha_5 \rangle$$

$$z(Q) = \frac{1}{5} + \alpha_2 + \frac{1}{5} + \alpha_3 + \frac{1}{5} + \alpha_4 + \frac{1}{5} + \alpha_5 = \frac{4}{5} + \sum_{i=2}^5 \alpha_i \leq 1 \text{ since } \sum_{i=2}^5 \alpha_i \leq \sum_{i=2}^6 \alpha_i < \frac{1}{5} \text{ so } z \text{ is feasible.}$$

It is clear that  $\forall i \in Q$ ,  $z_i \geq x_i$  ✓ and that  $\forall i \in Q \cap P$ ,  $z_i \geq y_i$  ✓

Hence,  $(Q, z)$  is a counter-objection. ✓

- $P = \{1, 2, 3, 4, 5\}$ . The vector  $y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5} + \alpha_4, \frac{1}{5} + \alpha_5 \rangle$  is an objection when

$$\alpha > -\frac{1}{5}, \alpha_i \geq 0, \sum_{i=2}^5 (\frac{1}{5} + \alpha_i) + \alpha \leq 1$$

This time, we have 
$$\sum_{i=2}^5 (\frac{1}{5} + \alpha_i) + \alpha = \frac{4}{5} + \sum_{i=2}^5 \alpha_i + \alpha \leq 1$$

then 
$$\sum_{i=2}^5 \alpha_i \leq 1 - \frac{4}{5} - \alpha = \frac{1}{5} - \alpha$$
 and finally 
$$\sum_{i=2}^5 \alpha_i \leq \frac{1}{5} - \alpha < \frac{2}{5}.$$

⇒ We need to find a counter-objection to  $(P, y)$

**claim:** we can choose  $Q = \{2, 3, 6, 7\}$ ,  $z = \langle \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5}, \frac{1}{5} \rangle$

It is clear that  $\forall i \in Q, z_i \geq x_i$  ✓ and  $\forall i \in P \cap Q, z_i \geq y_i$  (for agent 2 and 3).

$z(Q) = \frac{1}{5} + \alpha_2 + \frac{1}{5} + \alpha_3 + \frac{1}{5} + \frac{1}{5} = \frac{4}{5} + \alpha_2 + \alpha_3$ . We have

$\alpha_2 + \alpha_3 < \frac{1}{5}$ , otherwise, we would have  $\alpha_2 + \alpha_3 \geq \frac{1}{5}$  and since

the  $\alpha_i$  are ordered, we would then have 
$$\sum_{i=2}^5 \alpha_i \geq \frac{2}{5},$$
 which is

not possible. Hence  $z(Q) \leq 1$  which proves  $z$  is feasible ✓

Using similar arguments, we find a counter-objection for each other objections (you might want to fill in the details at home).

- $P = \{1, 2, 3, 4\}$ ,  $y = \langle \alpha, \frac{1}{5} + \alpha_1, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3 \rangle$ ,  $\alpha > -\frac{1}{5}$ ,  $\alpha_i \geq 0$ ,  
 $\sum_{i=2}^4 \alpha_i + \alpha \leq \frac{2}{5} \Rightarrow \sum_{i=2}^4 \alpha_i \leq \frac{2}{5} - \alpha < \frac{3}{5}$ .
- ⇒  $Q = \{2, 5, 6, 7\}$ ,  $z = \langle \frac{1}{5} + \alpha_2, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \rangle$  since  $\alpha_2 \leq \frac{1}{5}$
- $|P| \leq 3$   $P = \{1, 2, 3\}$ ,  $v(P) = 0$ ,  $y = \langle \alpha, \alpha_1, \alpha_2 \rangle$ ,  $\alpha > -\frac{1}{5}$ ,  
 $\alpha_i \geq 0$ ,  $\alpha_1 + \alpha_2 \leq -\alpha < \frac{1}{5}$
- ⇒  $Q = \{4, 5, 6, 7\}$ ,  $z = \langle \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \rangle$  will be a counter argument  
 (1 cannot provide more than  $\frac{1}{5}$  to any other agent).
- For each possible objection of 1, we found a counter-objection. Using similar arguments, we can find a counter-objection to any objection of player 7 against player 1.
- ⇒  $x \in \text{preBS}(N, v, \mathcal{S})$ . ✓

## Bargaining set

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In the example, agent 1 gets  $-\frac{1}{5}$  when  $v(\mathcal{C}) \geq 0$  for all coalition  $\mathcal{C} \subseteq N$ ! This shows that the pre-bargaining set may **not** be individually rational.

Let  $I(N, v, \mathcal{S}) = \{x \in X_{(N, v, \mathcal{S})} \mid x_i \geq v(\{i\}) \forall i \in N\}$  be the **set of individually rational payoff vector** in  $X_{(N, v, \mathcal{S})}$ .

### Lemma

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If a game is weakly superadditive,  $I(N, v, \mathcal{S}) \neq \emptyset$ .

### Definition (Bargaining set)

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Let  $(N, v, \mathcal{S})$  a game in coalition structure.

The **bargaining set (BS)** is defined by

$$BS(N, v, \mathcal{S}) = I(N, v, \mathcal{S}) \cap preBS(N, v, \mathcal{S}).$$

### Lemma

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We have  $Core(N, v, \mathcal{S}) \subseteq BS(N, v, \mathcal{S})$ .

## Theorem

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Let  $(N, v, \mathcal{S})$  a game with coalition structure. Assume that  $I(N, v, \mathcal{S}) \neq \emptyset$ . Then the bargaining set  $BS(N, v, \mathcal{S}) \neq \emptyset$ .

## Proof

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It is possible to give a direct proof of this theorem (a bit long, (Section 4.2 in **Introduction to the Theory of Cooperative Games**)).

We will show this result in a different way in the lecture about the nucleolus next week. □

B. Peleg and P. Sudhölter **Introduction to the Theory of Cooperative Games**, Springer, 2007.

## Definition (weighted voting games)

A game  $(N, w_{i \in N}, q, v)$  is a **weighted voting game** when  $v$  satisfies unanimity, monotonicity and the valuation function is defined as

$$v(S) = \begin{cases} 1 & \text{when } \sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise} \end{cases}$$

We note such a game by  $(q: w_1, \dots, w_n)$

Let  $(N, v)$  be the game associated with the 6-player weighted majority game  $(3:1,1,1,1,1,0)$ .

Agent 6 is a null/dummy player since its weight is 0.

Nevertheless  $\langle \frac{1}{7}, \dots, \frac{1}{7}, \frac{2}{7} \rangle \in BS(N, v)$ .

## Proof

This will be part of homework 2

□

Agent 6 is a dummy, however, it receives a payoff of  $\frac{2}{7}$ , which is larger than agents who are not dummy!

Remember:  $mc_i^{max} = \max_{\mathcal{C} \subseteq N \setminus \{i\}} v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})$

$x$  is **reasonable from above** if  $\forall i \in N \ x^i < mc_i^{max}$

$\Leftrightarrow mc_i^{max}$  is the strongest **threat** that an agent can use against a coalition.

The bargaining set is not **Reasonable from above**: the dummy agent gets more than  $\max_{\mathcal{C} \subseteq N \setminus \{6\}} (v(\mathcal{C} \cup \{6\}) - v(\mathcal{C})) = 0$ . **X**

### Lemma

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The core is reasonable for above and from below

### Proof

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Since the core satisfies IR, it must be reasonable from below. Let  $(N, v)$  be a game,  $x \in C(N, v)$  and  $i \in N$ . Then  $x(N) = v(N)$  and  $x(N \setminus \{i\}) \geq v(N \setminus \{i\})$ . Then

$$x_i = v(N) - x(N \setminus \{i\}) \leq v(N) - v(N \setminus \{i\}) \leq mc_i^{max}.$$

□

## Summary

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- We introduced the bargaining set, and looked at some examples.
  - pros:** it is guaranteed to be non-empty, when the core is non-empty, it is contained in the bargaining set.
  - cons:** it may not be reasonable from above.



## Coming next

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- We will consider the Nucleolus. It can also be defined in terms of objections and counter objections, but the nature of the objection is different from the bargaining set.