Cooperative Games

Lecture 8: Simple Games

Stéphane Airiau

ILLC - University of Amsterdam

Today

- Simple games: a class of TU games for modeling voting.
- Measuring the power of a voter: Shapley Shubik, Banzhaff and Co.

Simple games

Simple Games

Definition (Simple games)

A game (N,v) is a **Simple game** when

the valuation function takes two values

- 1 for a winning coalitions
- 0 for the losing coalitions

v satisfies *unanimity*: $v(N) = 1$

v satisfies *monotonicity*: $S \subseteq T \Rightarrow v(S) \leqslant v(T)$

One can represent the game by stating all the wining coalitions. Thanks to monotonicity, it is sufficient to only write down the minimal winning coalitions defined as follows:

Definition (Minimal winning coalition)

Let (N,v) be a TU game. A coalition $\mathcal{C}$ is a **minimal winning coalition** iff $v(\mathcal{C}) = 1$ and $\forall i \in \mathcal{C}$, $v(\mathcal{C} \setminus \{i\}) = 0$.

Example

$N = \{1,2,3,4\}$.

We use majority voting, and in case of a tie, the decision of player 1 wins.

The set of winning coalitions is $\{\{1,2\},\{1,3\},\{1,4\},\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\},\{1,2,3,4\}\}$.

The set of minimal winning coalitions is $\{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$.

Formal definition of common terms in voting

Definition (Dictator)

Let (N, v) be a simple game. A player $i \in N$ is a **dictator** iff $\{i\}$ is a winning coalition.

Note that with the requirements of simple games, it is possible to have more than one dictator!

Definition (Veto Player)

Let (N, v) be a simple game. A player $i \in N$ is a **veto** player if $N \setminus \{i\}$ is a losing coalition. Alternatively, i is a **veto** player iff for all winning coalition $\mathcal{C}$, $i \in \mathcal{C}$.

It also follows that a veto player is member of every minimal winning coalitions.

Definition (blocking coalition)

A coalition $\mathcal{C} \subseteq N$ is a **blocking coalition** iff $\mathcal{C}$ is a losing coalition and $\forall S \subseteq N \setminus \mathcal{C}$, $S \setminus \mathcal{C}$ is a losing coalition.

Definition (weighted voting games)

A game $(N, w_{i \in N}, q)$ is a **weighted voting game** when v satisfies unanimity, monotonicity and the valuation function is defined as

$$v(S) = \begin{cases} 1 \text{ when } \sum_{i \in S} w_i \geqslant q \\ 0 \text{ otherwise} \end{cases}$$

Unanimity requires that $\sum_{i \in N} w_i \geqslant q$.
If we assume that $\forall i \in N \; w_i \geqslant 0$, monotonicity is guaranteed.
For the rest of the lecture, we will assume $w_i \geqslant 0$.

We will note a weighted voting game $(N, w_{i \in N}, q)$ as $[q; w_1, \ldots, w_n]$.

A weighted voting game is a **succinct** representation, as we only need to define a weight for each agent and a threshold.

Example 1

Let us consider the game $[q; 4, 2, 1]$.

- $q = 1$: minimal winning coalitions: {1},{2},{3}
- $q = 2$: minimal winning coalitions: {1},{2}
- $q = 3$: minimal winning coalitions: {1},{2,3}
- $q = 4$: minimal winning coalition: {1}
- $q = 5$: minimal winning coalitions: {1,2},{1,3}
- $q = 6$: minimal winning coalition: {1,2}
- $q = 7$: minimal winning coalition: {1,2,3}

for $q = 4$ ("majority" weight), 1 is a dictator, 2 and 3 are dummies.

Examples

- Let us consider the game [10; 7,4,3,3,1].

  The set of minimal winning coalitions is {{1,2}{1,3}{1,4}{2,3,4}}

  Player 5, although it has some weight, is a dummy.

  Player 2 has a higher weight than player 3 and 4, but it is clear that player 2, 3 and 4 have the same influence.
- Let us consider the game [51; 49,49,2]

  The set of winning coalition is {{1,2},{1,3},{2,3}}.

  It seems that the players have symmetric roles, but it is not reflected in their weights.

Weighted voting game is a strict subclass of voting games. i.e., all voting games are **not** weighted voting games.

Example: Let $(\{1,2,3,4\},v)$ a voting game such that the set of minimal winning coalitions is $\{\{1,2\},\{3,4\}\}$. Let us assume we can represent (N,v) with a weighted voting game $[q;\, w_1,w_2,w_3,w_4]$.

$v(\{1,2\}) = 1$ then $w_1 + w_2 \geqslant q$
$v(\{3,4\}) = 1$ then $w_3 + w_4 \geqslant q$
$v(\{1,3\}) = 0$ then $w_1 + w_3 < q$
$v(\{2,4\}) = 0$ then $w_2 + w_4 < q$

But then, $w_1 + w_2 + w_3 + w_4 < 2q$ and $w_1 + w_2 + w_3 + w_4 \geqslant 2q$, which is impossible. Hence, (N,v) cannot be represented by a weighted voting game.✔

Theorem

Let (N,v) be a simple game. Then

$$Core(N,v) = \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{l} x \text{ is an imputation} \\ x_i = 0 \text{ for each non-veto player } i \end{array} \right\}$$

Proof

- $\subseteq$ Let $x \in Core(N,v)$. By definition $x(N) = 1$. Let i be a non-veto player. $x(N \setminus \{i\}) \geqslant v(N \setminus \{i\}) = 1$. Hence $x(N \setminus \{i\}) = 1$ and $x_i = 0$.
- $\supseteq$ Let x be an imputation and $x_i = 0$ for every non-veto player i. Since $x(N) = 1$, the set V of veto players is non-empty and $x(V) = 1$.

  Let $\mathcal{C} \subseteq N$. If $\mathcal{C}$ is a winning coalition then $V \subseteq \mathcal{C}$, hence $x(\mathcal{C}) \geqslant v(\mathcal{C})$. Otherwise, $v(\mathcal{C})$ is a losing coalition (which may contain veto players), and $x(\mathcal{C}) \geqslant v(\mathcal{C})$. Hence, x is group rational.

□

Theorem

A simple game (N,v) is convex iff it is a unanimity game (N,v_V) where V is the set of veto players.

Proof

A game is convex iff $\forall S,T \subseteq N \; v(S)+v(T) \leqslant v(S\cap T)+v(S\cup T)$.

- $\Rightarrow$ Let us assume (N,v) is convex.

  If S and T are winning coalitions, $S\cup T$ is a winning coalition by monotonicity. Then, we have $2 \leqslant 1+v(S\cap T)$ and it follows that $v(S\cap T)=1$. The intersection of two winning coalitions is a winning coalition.

  Moreover, from the definition of veto players, the intersection of all winning coalitions is the set V of veto players. Hence, $v(V)=1$.

  By monotonicity, if $V\subseteq \mathcal{C}$, $v(\mathcal{C})=1$ ✔

  Otherwise, $V \nsubseteq \mathcal{C}$. Then there must be a veto player $i \notin \mathcal{C}$, and it must be the case that $v(\mathcal{C})=0$ ✔

  Hence, for all coalition $\mathcal{C}\subseteq N$, $v(\mathcal{C})=1$ iff $V\subseteq \mathcal{C}$.

□

Proof

(continuation)

$\Leftarrow$ Let (N, v_V) a unanimity game. Let us prove it is a convex game. Let $S \subseteq N$ and $T \subseteq N$, and we want to prove that $v(S)+v(T) \leqslant v(S\cup T)+v(S\cap T)$.

- case $V \subseteq S\cap T$: Then $V \subseteq S$ and $V \subseteq T$, and we have $2 \leqslant 2$ ✔
- case $V \not\subseteq S\cap T \wedge V \subseteq S\cup T$:
  - if $V \subseteq S$ then $V \not\subseteq T$ and $1 \leqslant 1$ ✔
  - if $V \subseteq T$ then $V \not\subseteq S$ and $1 \leqslant 1$ ✔
  - otherwise $V \not\subseteq S$ and $V \not\subseteq T$, and then $0 \leqslant 1$ ✔
- case $V \not\subseteq S\cup T$: then $0 \leqslant 0$ ✔

For all cases, $v(S)+v(T) \leqslant v(S\cup T)+v(S\cap T)$, hence a unanimity game is convex.

In addition, all members of V are veto players.

□

Convex simple games are the games with a single minimal winning coalition.

Shapley-Shubik power index

Definition (Pivotal or swing player)

Let (N,v) be a simple game. A agent i is **pivotal** or **a swing agent** for a coalition $\mathcal{C} \subseteq N \setminus \{i\}$ if agent i turns the coalition $\mathcal{C}$ from a losing to a winning coalition by joining $\mathcal{C}$, i.e., $v(\mathcal{C}) = 0$ and $v(\mathcal{C} \cup \{i\}) = 1$.

Given a **permutation** σ on N, there is a single pivotal agent.

The Shapley-Shubik index of an agent i is the percentage of permutation in which i is pivotal, i.e.

$$I_{SS}(N,v,i) = \sum_{\mathcal{C} \subseteq N \setminus \{i\}} \frac{|\mathcal{C}|!(|N|-|C|-1)!}{|N|!} \left(v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})\right).$$

"For each permutation, the pivotal player gets a point."

The Shapley-Shubik power index is the Shapley value.
The index corresponds to the expected marginal utility assuming all join orders to form the grand coalitions are equally likely.

Banzhaff power index

Let (N,v) be a TU game.

- We want to count the **number of coalitions** in which an agent is **a swing agent**.
- For each coalition, we determine which agent is a swing agent (more than one agent may be pivotal).
- The **raw Banzhaff index** of a player i is $\beta_i = \frac{\sum_{\mathcal{C} \subseteq N \setminus \{i\}} v(\mathcal{C} \cup \{i\}) - v(\mathcal{C})}{2^{n-1}}$.
- For a simple game (N,v), $v(N) = 1$ and $v(\emptyset) = 0$, at least one player i has a power index $\beta_i \neq 0$. Hence, $B = \sum_{j \in N} \beta_j > 0$.
- The **normalized Banzhaff index** of player i for a simple game (N,v) is defined as $I_B(N,v,i) = \frac{\beta_i}{B}$.

The index corresponds to the expected marginal utility assuming all coalitions are equally likely.

Examples: [7; 4,3,2,1]

{1,2,3,4}
{1,2,4,3}
{1,3,2,4}
{1,3,4,2}
{1,4,2,3}
{1,4,3,2}
{2,1,3,4}
{2,1,4,3}
{2,3,1,4}
{2,3,4,1}
{2,4,1,3}
{2,4,3,1}
{3,1,2,4}
{3,1,4,2}
{3,2,1,4}
{3,2,4,1}
{3,4,1,2}
{3,4,2,1}
{4,1,2,3}
{4,1,3,2}
{4,2,1,3}
{4,2,3,1}
{4,3,1,2}
{4,3,2,1}

	1	2	3	4
Sh	$\frac{7}{12}$	$\frac{1}{4}$	$\frac{1}{12}$	$\frac{1}{12}$

winning coalitions:
{1,2}
{1,2,3}
{1,2,4}
{1,3,4}
{1,2,3,4}

	1	2	3	4
β	$\frac{5}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
$I_B(N,v,i)$	$\frac{1}{2}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

The Shapley value and Banzhaff index may be different.

- Coleman indices: all winning coalitions are equally likely. Let $\mathcal{W}(N,v)$ be the set of all winning coalitions.
- The power of **collectivity to act**: P_{act} is the probability that a winning vote arise.

$$P_{act} = \frac{|\mathcal{W}(N,v)|}{2^n}$$

- The power **to prevent** an action: $P_{prevent}$ captures the power of i to prevent a coalition to win by withholding its vote.

$$P_{prevent} = \frac{\sum_{\mathcal{C}\subseteq N\setminus\{i\}} v(\mathcal{C}\cup\{i\}) - v(\mathcal{C})}{|\mathcal{W}(N,v)|}$$

- The power **to initiate** an action: P_{init} captures the power of i to join a losing coalition so that it becomes a winning one.

$$P_{init} = \frac{\sum_{\mathcal{C}\subseteq N\setminus\{i\}} v(\mathcal{C}\cup\{i\}) - v(\mathcal{C})}{2^n - |\mathcal{W}(N,v)|}.$$

- Maybe only minimal winning coalitions are important to measure the power of an agent (non-minimal winning coalitions may form, but only the minimal ones are important to measure power).
- Let (N,v) be a simple game, $i \in N$ be an agent. $\mathcal{M}(N,v)$ denotes the set of minimal winning coalitions, $\mathcal{M}_i(N,v)$ denotes the set of minimal winning coalitions containing i.
- The **Deegan-Packel** power index of player i is:

$$I_{DP}(N,v,i) = \frac{1}{|\mathcal{M}(N,v)|} \sum_{\mathcal{C} \in \mathcal{M}_i(N,v)} \frac{1}{|\mathcal{C}|}.$$

- The **public good index** of player i is defined as

$$I_{PG}(N,v,i) = \frac{|\mathcal{M}_i(N,v)|}{\sum_{j \in N} |\mathcal{M}_j(N,v)|}.$$

$[4; 3, 2, 1, 1]$

$$\mathcal{W} = \left\{ \begin{array}{c} \{1,2\},\{1,3\},\{1,4\}, \\ \{1,2,3\},\{1,2,4\},\{1,3,4\}, \\ \{2,3,4\},\{1,2,3,4\}\} \end{array} \right\}$$

$$\mathcal{M} = \{\{1,2\},\{1,3\},\{1,4\},\{2,3,4\}\}$$

$[5; 3, 2, 1, 1]$

$$\mathcal{W} = \left\{ \begin{array}{c} \{1,2\},\{1,2,3\},\{1,2,4\}, \\ \{1,3,4\},\{1,2,3,4\}\} \end{array} \right\}$$

$$\mathcal{M} = \{\{1,2\},\{1,3,4\}\}$$

	1	2	3	4		1	2	3	4
β	$\frac{6}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	β	$\frac{5}{8}$	$\frac{3}{8}$	$\frac{1}{8}$	$\frac{1}{8}$
I_B	$\frac{6}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	$\frac{2}{12}$	I_B	$\frac{5}{10}$	$\frac{3}{10}$	$\frac{1}{10}$	$\frac{1}{10}$
P_{act}	$\frac{8}{16} = \frac{1}{2}$				P_{act}	$\frac{5}{16}$			
$P_{prevent}$	$\frac{6}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$P_{prevent}$	$\frac{5}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	$\frac{1}{5}$
P_{init}	$\frac{6}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	$\frac{2}{8}$	P_{init}	$\frac{5}{11}$	$\frac{3}{11}$	$\frac{1}{11}$	$\frac{1}{11}$
I_{DP}	$\frac{1}{4} \cdot \frac{3}{2}$	$\frac{1}{4} \cdot \left(\frac{1}{2} + \frac{1}{3}\right)$	$\frac{1}{4} \cdot \left(\frac{1}{2} + \frac{1}{3}\right)$	$\frac{1}{4} \cdot \left(\frac{1}{2} + \frac{1}{3}\right)$	I_{DP}	$\frac{1}{2} \cdot \left(\frac{1}{2} + \frac{1}{3}\right)$	$\frac{1}{2} \cdot \frac{1}{2}$	$\frac{1}{2} \cdot \frac{1}{3}$	$\frac{1}{2} \cdot \frac{1}{3}$
I_{PG}	$\frac{3}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	$\frac{2}{9}$	I_{PG}	$\frac{2}{5}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{5}$

Summary

- We introduced the simple games
- We considered few examples
- We studied some power indices

Coming next

- Representation and Complexitity issues
- Are there some succinct representations for some classes of games.
- How hard is it to compute a solution concept?