Lecture 2

The Core

Let us assume that we have a TU game (N, v) and that we want to form the grand coalition. The Core, which was first introduced by Gillies [1], is the most attractive and natural way to define stability. A payoff distribution is in the *Core* when no group of agents has any incentive to form a different coalition. This is a strong condition for stability, so strong that some games may have an empty core. In this lecture, we will first introduce the definition of the core and consider some graphical representation for games with up to three players. Then, we will present some games that are guaranteed to have a non-empty core. Finally, we will present a theorem that characterizes games with non-empty core: the Bondareva-Shapley theorem. We will give some intuition about the proof, relying on results from linear programming, and we will use this theorem to show that market games have a non-empty core.

2.1 Definition and graphical representation for games with up to three players

We consider a TU game (N, v). We assume that all the agents cooperate by forming the *grand coalition* and that they receive a payoff distribution x. We want the grand coalition to be stable, i.e., no agent should have an incentive to leave the grand coalition. We will say that x is in the core of the game (N, v) when no group of agents has an incentive to leave the grand coalition and form a separate coalition.

2.1.1. DEFINITION. [Core] A payoff distribution $x \in \mathbb{R}^n$ is in the *Core* of a game (N, v) iff x is an imputation that is group rational, i.e., $Core(N, v) = \{x \in \mathbb{R}^n \mid \sum_{i \in N} x_i = v(N) \ \wedge \ \forall \mathcal{C} \subseteq N \ x(\mathcal{C}) \geq v(\mathcal{C})\}$

A payoff distribution is in the Core when no group of agents has any interest in rejecting it, i.e., no group of agents can gain by forming a different coalition. Note that this condition has to be true for all subsets of N (group rationality). As a special case, this ensures individual rationality. Another way to define the Core is in terms of excess:

2.1.2. DEFINITION. [Core] The Core is the set of payoff distribution $x \in \mathbb{R}^n$, such that $\forall R \subset N$, $e(R, x) \leq 0$

In other words, a PC is in the Core when there exists no coalition that has a positive excess. This definition is attractive as it shows that no coalition has any complaint: each coalition's demand can be granted.

In the first definition of the core, we see that the payoff distribution satisfies weak linear inequalities: for each coalition $\mathcal{C} \subseteq N$, we have $v(\mathcal{C}) \leq x(\mathcal{C})$. The Core is therefore closed and convex, and we can try to represent it geometrically.

Let us consider the following two-player game $(\{1, 2\}, v)$ where $v(\{1\}) = 5$, $v(\{2\}) = 5$, and $v(\{1, 2\}) = 20$. The core of the game is a segment defined as follows: $core(N, v) = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 5, x_2 \geq 5, x_1 + x_2 = 20\}$ and is represented in Figure 2.1. This example shows that, although the game is symmetric, most of the payoffs in the core are not fair. Core allocations focus on stability only and they may not be fair.

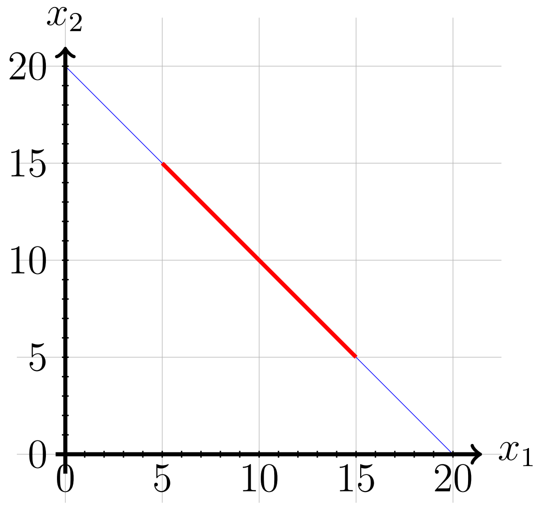
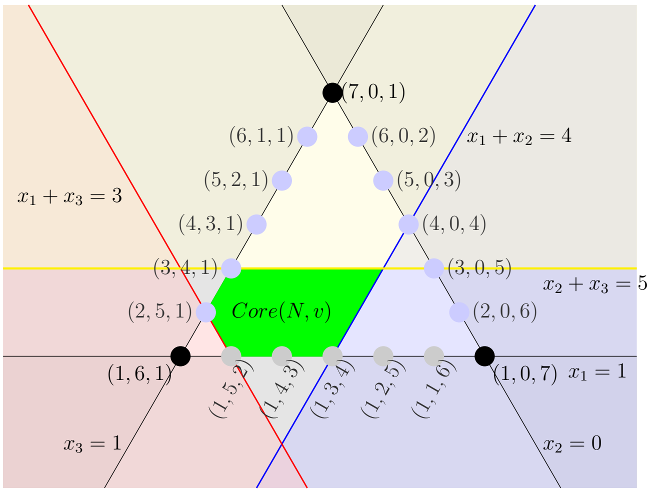


Figure 2.1: Example of a core allocation

It is possible to represent the core for game with three agents. For a game $(\{1, 2, 3\}, v)$, the efficiency condition is $v(\{1, 2, 3\}) = x_1 + x_2 + x_3$, which is a plane in a 3-dimensional space. On this plane, we can draw the conditions for individual rationality and for group rationality. Each of these conditions partitions the space into two regions separated by a line: one region is incompatible with a core allocation, the other region is not. The core is the intersection of all the compatible regions. Figure 2.2 represents the core of a three-player game.

There are, however, multiple concerns associated with using the notion of the Core. First, the Core can be empty: the conflicts captured by the characteristic function cannot satisfy all the players simultaneously. When the Core is empty, at least one player is



$$\begin{array}{ll} v(\{1\}) = 1 & v(\{1,2\}) = 4 \\ v(\{2\}) = 0 & v(\{1,3\}) = 3 \\ v(\{3\}) = 1 & v(\{2,3\}) = 5 \\ v(\emptyset) = 0 & v(\{1,2,3\}) = 8 \end{array}$$

Figure 2.2: Example of a three-player game: The core is the area in green

dissatisfied by the utility allocation and therefore blocks the coalition. Let us consider the following example from [2]: $v(\{A, B\}) = 90$, $v(\{A, C\}) = 80$, $v(\{B, C\}) = 70$, and $v(N) = 120$. In this case, the Core is the PC where the grand coalition forms and the associated payoff distribution is $(50, 40, 30)$. If $v(N)$ is increased, the size of the Core also increases. But if $v(N)$ decreases, the Core becomes empty.

Exercise: How can you modify the game in Figure 2.2 so that the core becomes empty?

2.2 Games with non-empty core

In the previous section, we saw that some games have an empty core. In this section, we provide examples of some classes of games that are guaranteed to have a non-empty core. In the following we will show that convex games and minimum cost spanning tree games have a non empty core.

We start introducing an example that models bankruptcy: individuals have claims in a resource, but the value of the resource is not sufficient to meet all of the claims (e.g., a man leaves behind an estate worth less than the value of its debts). The problem is then to share the value of the estate among all the claimants. The value of a coalition $\mathcal{C}$ is defined as the amount of the estate which is not claimed by the complement of $\mathcal{C}$, in other words $v(\mathcal{C})$ is the amount of the estate that the coalition $\mathcal{C}$ is guaranteed to obtain.

2.2.1. DEFINITION. Bankruptcy game A *Bankruptcy game* (N, E, v) where N is the set of claimants, $E \in \mathbb{R}_+$ is the estate and $c \in \mathbb{R}^n_+$ is the claim vector (i.e., c_i is the claim of the i^{th} claimant. The valuation function $v : 2^N \to \mathbb{R}$ is defined as follows. For a coalition of claimants $\mathcal{C}$, $v(\mathcal{C}) = \max \left\{0, E - \sum_{i \in N \setminus \mathcal{C}} c_i \right\}$.

First, we show that a bankruptcy game is convex.

2.2.2. THEOREM. *Every bankruptcy game is convex.*

Proof. Let (N, E, c) be a bankruptcy game. Let $S \subseteq T \subseteq N$, and $i \notin T$. We want to show that

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T),$$

or equivalently that

$$v(S \cup \{i\}) + v(T) \leq v(T \cup \{i\}) + v(S).$$

For all $C \subseteq N$, we note $c(C) = \sum_{j \in C} c_j$, then we can write:
$E - \sum_{j \in N \setminus C} c_j = E - \sum_{j \in N} c_j + \sum_{j \in C} c_i = E - c(N) + c(C).$
Let $\Delta = E - \sum_{j \in N} c_j = E - c(N)$. We have $E - \sum_{j \in N \setminus C} c_j = \Delta + c(C)$.
First, observe that $\forall (x, y) \in \mathbb{R}^2$, $\max\{0, x\} + \max\{0, y\} = \max\{0, x, y, x + y\}$.

$$\begin{aligned} v(S \cup \{i\}) + v(T) &= \max\left\{0, E - \sum_{j \in N \setminus (S \cup \{i\})} c_j\right\} + \max\left\{0, E - \sum_{j \in N \setminus T} c_j\right\} \\ &= \max\{0,\ \Delta + c(S) + c_i\} + \max\{0,\ \Delta + c(T)\} \\ &= \max\{0,\ \Delta + c(S) + c_i,\ \Delta + c(T),\ 2\Delta + c(S) + c_i + c(T)\} \end{aligned}$$

$$\begin{aligned} v(T \cup \{i\}) + v(S) &= \max\left\{0, E - \sum_{j \in N \setminus (T \cup \{i\})} c_j\right\} + \max\left\{0, E - \sum_{j \in N \setminus S} c_j\right\} \\ &= \max\{0,\ \Delta + c(T) + c_i\} + \max\{0,\ \Delta + c(S)\} \\ &= \max\{0,\ \Delta + c(T) + c_i,\ \Delta + c(S),\ 2\Delta + c(T) + c_i + c(S)\} \end{aligned}$$

Then, note that since $S \subseteq T$, $c(S) \leq c(T)$. Then
$\max\{0,\ \Delta + c(T) + c_i,\ \Delta + c(S),\ 2\Delta + c(T) + c_i + c(S)\} =$
$\max\{0,\ \Delta + c(T) + c_i,\ 2\Delta + c(T) + c_i + c(S)\}$.
We also have:
$\Delta + c(S) + c_i \leq \Delta + c(T) + c_i$.
$\Delta + c(T) \leq \Delta + c(T) + c_i$.

It follows that $\max\{0,\ \Delta + c(S) + c_i,\ \Delta + c(T),\ 2\Delta + c(S) + c_i + c(T)\}$
$\leq \max\{0,\ \Delta + c(T) + c_i,\ 2\Delta + c(T) + c_i + c(S)\}$
which proves that $v(S \cup \{i\}) + v(T) \leq v(T \cup \{i\}) + v(S)$. ✔ □

Now, we show an important property of convex games: they are guaranteed to have a non-empty core. We define a payoff distribution where each agent gets its marginal contribution, given that the agents enter the grand coalition one at a time in a given order, and we show that this payoff distribution is an imputation that is group rational.

2.2.3. THEOREM. *A convex game has a non-empty core.*

Proof. Let us assume a convex game (N, v). Let us define a payoff vector x in the following way: $x_1 = v(\{1\})$ and for all $i \in \{2, \ldots, n\}$, $x_i = v(\{1, 2, \ldots, i\}) - v(\{1, 2, \ldots, i-1\})$. In other words, the payoff of the i^{th} agent is its marginal contribution to the coalition consisting of all previous agents in the order $\{1, 2, \ldots, i-1\}$.

Let us prove that the payoff vector is *efficient* by writing up and summing the payoff of all agents:

$$\begin{array}{rcl} x_1 & = & v(\{1\}) \\ x_2 & = & v(\{1,2\} - v(\{1\}) \\ & \ldots & \\ x_i & = & v(\{1,2,\ldots,i\}) - v(\{1,2,\ldots,i-1\}) \\ & \ldots & \\ x_n & = & v(\{1,2,\ldots,n\}) - v(\{1,2,\ldots,n-1\}) \\ \hline \sum_{i\in N} x_n & = & v(\{1,2,\ldots,n\}) = v(N) \end{array}$$

By summing these n equalities, we obtain the efficiency condition:
$\sum_{i\in N} x_n = v(\{1, 2, \ldots, n\}) = v(N)$.✔

Let us prove that the payoff vector is *individually rational*. By convexity, we have $v(\{i\}) - v(\emptyset) \le v(\{1, 2, \ldots, i\}) - v(\{1, 2, \ldots, i-1\})$, hence $v(\{i\}) \le x_i$. ✔

Finally, let us prove that the payoff vector is *group rational*. Let $C \subseteq N$, $C = \{a_1, a_2, \ldots, a_k\}$ and let us consider that $a_1 < a_2 < \ldots < a_k$. It is obvious that $\{a_1, a_2, \ldots, a_k\} \subseteq \{1, 2, \ldots, a_k\}$. Using the convexity assumption, we obtain the following:

$$\begin{array}{rcl} v(\{a_1\}) - v(\emptyset) & \le & v(\{1,2,\ldots,a_1\}) - v(\{1,2,\ldots,a_1-1\}) = x_{a_1} \\ v(\{a_1,a_2\}) - v(\{a_1\}) & \le & v(\{1,2,\ldots,a_2\}) - v(\{1,2,\ldots,a_2-1\}) = x_{a_2} \\ & \ldots & \\ v(\{a_1,a_2,\ldots,a_l\}) - v(\{a_1,a_2,\ldots,a_{l-1}\}) & \le & v(\{1,2,\ldots,a_l\}) - v(\{1,2,\ldots,a_l-1\}) = x_{a_l} \\ & \ldots & \\ v(\{a_1,a_2,\ldots,a_k\}) - v(\{a_1,a_2,\ldots,a_{k-1}\}) & \le & v(\{1,2,\ldots,a_k\}) - v(\{1,2,\ldots,a_k-1\}) = x_{a_k} \\ \hline v(C) = v(\{a_1,a_2,\ldots,a_k\}) & \le & \sum_{i=1}^{k} x_{a_k} = x(C) \end{array}$$

By summing these k inequalities, we obtain:
$v(C) = v(\{a_1, a_2, \ldots, a_k\}) \le \sum_{i=1}^{k} x_{a_k} = x(C)$, which is the group rationality condition. ✔ □

Another example of games that have a non-empty core are the class of *minimum cost spanning tree game*. Let N be the set of customers, and let 0 be the supplier. Let us define $N_* = N \cup \{0\}$. For $(i,j) \in N_*^2$, $i \neq j$, let $c_{i,j}$ be the cost of connecting i and j by the edge e_{ij}. Let (N, c) be the corresponding cost game, which is called a minimum cost spanning tree game.

2.2.4. THEOREM. *Every minimum cost spanning tree game has a non-empty core.*

Proof. Let us define a cost distribution x and then we will show that x is in the core.

Let $T = (N, E_N)$ a minimum cost spanning tree for the graph $\big(N_*, c_{\{ij\}\subseteq N_*^2}\big)$. Let i be a customer. Since T is a tree, there is a unique path $(0, a_1, \ldots, a_k, i)$ from 0 to i. The cost paid by agent i is defined by $x_i = c_{a_k,i}$.

This cost allocation is efficient by construction of x.

We need to show the cost allocation is group rational, i.e., for all coalition S, we have $x(S) \leq v(S)$ (it is a cost, which explains the inequality).

Let $S \subset N$ and $T_S = (S \cup \{0\}, E_s)$ be a minimum cost spanning tree of the graph $\big(S \cup \{0\}, c_{\{ij\}\in S\cup\{0\}}\big)$. Let extand the tree T_S to a graph $T_S^+ = (N_*, E_N^+)$ by adding the remaining customers $N \setminus S$, and for each customer $i \in N \setminus S$, we add the edge of E_N ending in i, i.e., we add the edge (a_k, i). The graph T_S^+ has $|S| + |N \setminus S|$ edges an is connected. Hence, T_S^+ is a spanning tree. Now, we note that $c(S) + x(N \setminus S) = \sum_{e_{ij}\subseteq E_N^+} c_{ij} \geq \sum_{e_{ij}\subseteq E_N} = c(N) = x(N)$. The inequality is due to the fact that T_S^+ is a spanning tree, and T is a minimum spanning tree. It follows that $x(S) \leq v(S)$. ✔ □

2.3 Characterization of games with a non-empty core

We saw that the core may be empty, but that some classes of games have a non-empty core. The next issue is whether we can characterize the games with non-empty core. It turns out that the answer is yes, and the characterization has been found independently by Bondareva (1963) and Shapley (1967), resulting in what is now known as the Bondareva Shapley theorem. This result connects results from linear programming with the concept of the core. We will first describe the theorem and provide an intuition about the proof. Then, we will use this theorem to show that market games have a non-empty core.

2.3.1 Bondareva-Shapley theorem

Let $\mathcal{C} \subseteq N$. The *characteristic vector* $\chi_\mathcal{C}$ of $\mathcal{C}$ is the member of $\mathbb{R}^N$ defined by $\chi_\mathcal{C}^i = \begin{cases} 1 \text{ if } i \in \mathcal{C} \\ 0 \text{ if } i \in N \setminus \mathcal{C} \end{cases}$

2.3.1. DEFINITION. [Map] A *map* is a function $2^N \setminus \emptyset \to \mathbb{R}_+$ that gives a positive weight to each coalition.

2.3.2. DEFINITION. [Balanced map] A function $\lambda : 2^N \setminus \emptyset \to \mathbb{R}_+$ is a *balanced map* iff $\sum_{\mathcal{C}\subseteq N} \lambda(\mathcal{C})\chi_{\mathcal{C}} = \chi_N$

A map is balanced when the amount received over all the coalitions containing an agent i sums up to 1. We provide an example in Table 2.1 for a three-player game.

	1	2	3
$\{1,2\}$	$\frac{1}{2}$	$\frac{1}{2}$	0
$\{1,3\}$	$\frac{1}{2}$	0	$\frac{1}{2}$
$\{2,3\}$	0	$\frac{1}{2}$	$\frac{1}{2}$

$$\lambda(\mathcal{C}) = \begin{cases} \frac{1}{2} \text{ if } |\mathcal{C}| = 2 \\ 0 \text{ otherwise} \end{cases}$$

Each of the column sums up to 1.
$\frac{1}{2}\chi_{\{1,2\}} + \frac{1}{2}\chi_{\{1,3\}} + \frac{1}{2}\chi_{\{2,3\}} = \chi_{\{1,2,3\}}$

Table 2.1: Example of a balanced map for $n = 3$

2.3.3. DEFINITION. [Balanced game] A game is *balanced* iff for each balanced map λ we have $\sum_{\mathcal{C}\subseteq N, \mathcal{C}\neq\emptyset} \lambda(\mathcal{C})v(\mathcal{C}) \leq v(N)$.

2.3.4. THEOREM (BONDAREVA-SHAPLEY THEOREM). *A TU game has a non-empty core iff it is balanced.*

This theorem completely characterizes the set of games with a non-empty core. However, it is not always easy or feasible to check that it is a balanced game.

Given a TU game (N, v), the core is defined as follows: $Core(N, v) = \{x \in \mathbb{R}^n \mid x(\mathcal{C}) \geq v(\mathcal{C}) \text{ for all } \mathcal{C} \subseteq N\}$. Under this definition, the core is defined by a set of linear constraints. The idea is to use results from linear optimization to characterize the class of games with a non-empty core. We will use $\mathcal{V}(N) = \mathcal{V}$ to represent the set of all valuation functions on 2^N and $\mathcal{V}_{Core} = \{v \in \mathcal{V} \mid Core(N, v) \neq \emptyset\}$ is the set of games with non-empty core.

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Cooperative Games

Lecture 2: The core

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- **A problem:** Imagine you can do a project alone, or with friends. **Which** friend to choose?
  - Talented, hard working, easy to work with, etc.
  - But your friends are having the same reasoning.
- A condition for a coalition to form:

  all agents prefer to be in it.

  i.e., none of the participants wishes she were in a different coalition or by herself ⇒ **Stability**
- Stability is a necessary but not sufficient condition, (e.g., there may be multiple stable coalitions).
- The **core** is a stability concepts where no agents prefer to deviate to form a different coalition.
- For simplicity, we will only consider the problem of the stability of the grand coalition:

⇒ Is the grand coalition stable ⇔ Is the core non-empty

Today

- Definition of the core
- Some geometrical representation of the core for games with up to three agents
- Convex games and the core

Definition (valuation or characteristic function)

A *valuation function* v associates a real number $v(\mathcal{C})$ to any subset $\mathcal{C}$, i.e., $v: 2^N \to \mathbb{R}$

Definition (TU game)

A TU game is a pair (N, v) where N is a set of agents and where v is a valuation function.

Definition (Imputation)

An **imputation** is a payoff distribution x that is efficient and individually rational, i.e.:

- $\sum_{i \in N} x_i = v(N)$ (efficiency)
- for all $i \in N$, $x_i \geqslant v(\{i\})$ (individual rationality)

Definition (Group rationality)

$\forall \mathcal{C} \subseteq N$, $\sum_{i \in \mathcal{C}} x(i) \geqslant v(\mathcal{C})$

The core relates to the stability of the grand coalition:
No group of agents has any incentive to change coalition.

Definition (*core* of a game (N,v))

Let (N,v) be a TU game, and assume we form the grand coalition N.
The **core** of (N,v) is the set:

$$Core(N,v) = \{x \in \mathbb{R}^n \mid x \text{ is a group rational imputation}\}$$

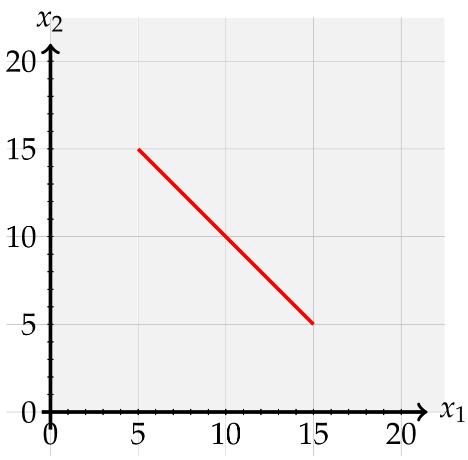
Equivalently,

$$Core(N,v) = \{x \in \mathbb{R}^n \mid x(N) \leqslant v(N) \wedge x(\mathcal{C}) \geqslant v(\mathcal{C})\ \forall \mathcal{C} \subseteq N\}$$

Weighted graph games

$$N = \{1,2\}$$
$$v(\{1\}) = 5, v(\{2\}) = 5$$
$$v(\{1,2\}) = 20$$

$$core(N,v) = \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \geqslant 5, x_2 \geqslant 5, x_1 + x_2 = 20\}$$



The core may not be fair: the core only considers stability.

three-player majority game

$$N = \{1,2,3\}$$
$$v(\{i\}) = 0$$
$$v(\{\mathcal{C}\}) = \alpha \text{ for } |\mathcal{C}| = 2$$
$$v(N) = 1$$

$$(x_1, x_2, x_3) \in Core(N,v) \Leftrightarrow \begin{cases} \forall i \in N,\ x_i \geqslant 0 \\ \forall (i,j) \in N^2\ i \neq j,\ x_i + x_j \geqslant \alpha \\ \sum_{i \in N} x_i = 1 \end{cases}$$
$$\Leftrightarrow \begin{cases} \forall i \in N\ 0 \leqslant x_i \leqslant 1 - \alpha & (1) \\ \sum_{i \in N} x_i = 1 & (2) \end{cases}$$

$Core(N,v)$ is nonempty iff $\alpha \leqslant \frac{2}{3}$
(by summing (1) for all $i \in N$ and using (2))

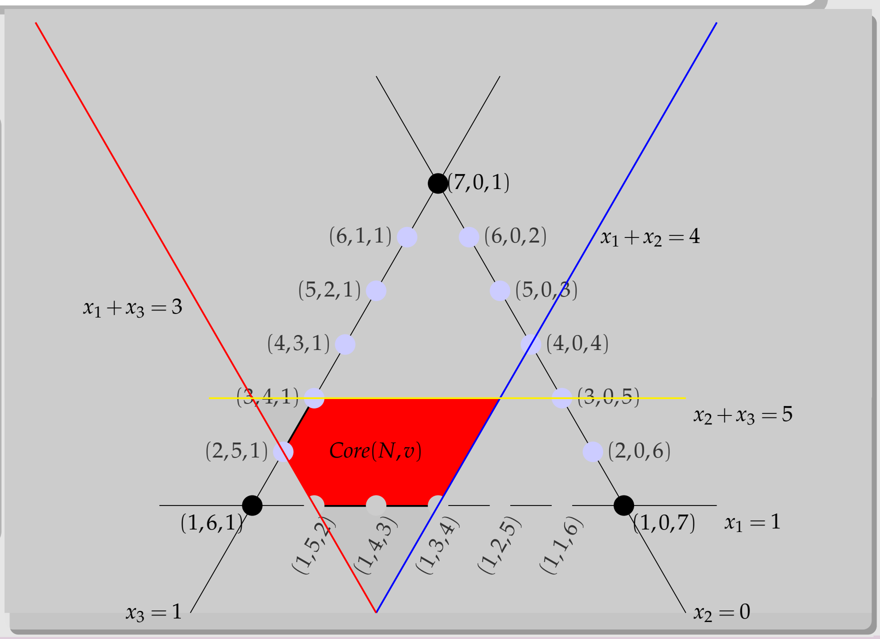
what happens when $\alpha > \frac{2}{3}$ and the core is empty?

Example with barycentric coordinate

$$v(\emptyset) = 0 \quad \begin{matrix} v(\{1\}) = 1 \\ v(\{2\}) = 0 \\ v(\{3\}) = 1 \end{matrix} \quad \begin{matrix} v(\{1,2\}) = 4 \\ v(\{1,3\}) = 3 \\ v(\{2,3\}) = 5 \end{matrix} \quad v(\{1,2,3\}) = 8$$

set of imputations $\mathcal{I} = \left\{ \sum_{i=1}^{3} x_i = 8, x_1 \geqslant 1, x_2 \geqslant 0, x_3 \geqslant 1 \right\}$

$\mathcal{I}$ is a triangle with vertices: $(7,0,1)$, $(1,6,1)$, $(1,0,7)$.
On the plane: $x_1 + x_2 + x_3 = 8$



Issues with the core

- The core may not always be non-empty
- When the core is not empty, it may not be 'fair'
- It may not be easy to compute
- ➩ Are there classes of games that have a non-empty core?
- ➩ Is it possible to characterize the games with non-empty core?

Definition (Convex games)

A game (N,v) is **convex** iff
$\forall \mathcal{C} \subseteq \mathcal{T}$ and $i \notin \mathcal{T}$, $v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \leqslant v(\mathcal{T} \cup \{i\}) - v(\mathcal{T})$.

TU-game is convex if the marginal contribution of each player increases with the size of the coalition he joins.

Bankruptcy game (E,c) $E \geqslant 0$ is the estate, there are n claimants and $c \in \mathbb{R}^n_+$ is the claim vector (c_i is the claim of the i^{th} claimant). $v(\mathcal{C}) = \max\{0, E - \sum_{i \in N \setminus \mathcal{C}} c_i\}$

Theorem

Each bankruptcy game is convex

Theorem

A convex game has a non-empty core

Proof for convexity of a bankruptcy market

Proof for characterization of a convex game

Proof for non-emptyness of the core for convex games

Minimum cost spanning tree games

- N be the set of customers
- 0 be the supplier
- $N_* = N \cup \{0\}$
- $c_{i,j}$ is the cost of connecting i and j by the edge e_{ij} for $(i,j) \in N_*^2$, $i \neq j$
- for a coalition $\mathcal{C}$, $T_\mathcal{C} = (\mathcal{C}, E_\mathcal{C})$ is the minimum cost spanning tree spanning over the set of edges $\mathcal{C} \cup \{0\}$.
- the cost function is $c(S) = \sum_{(i,j) \in E_\mathcal{C}} c_{ij}$
- A **minimum cost spanning tree game** is the associated cost game

Theorem

Every minimum cost spanning tree game has a non-empty core.

The Bondareva Shapley theorem:
a characterization of games with non-empty core.

The theorem was proven independently by
O. Bondareva (1963) and L. Shapley (1967).

Let $\mathcal{C} \subseteq N$. The **characteristic vector** $\chi_\mathcal{C}$ of $\mathcal{C}$ is the member of $\mathbb{R}^N$ defined by $\chi^i_\mathcal{C} = \begin{cases} 1 \text{ if } i \in \mathcal{C} \\ 0 \text{ if } i \in N \setminus \mathcal{C} \end{cases}$

A **map** is a function $2^N \setminus \emptyset \to \mathbb{R}_+$ that gives a positive weight to each coalition.

Definition (Balanced map)

A function $\lambda : 2^N \setminus \emptyset \to \mathbb{R}_+$ is a **balanced map** iff $\sum_{\mathcal{C} \subseteq N} \lambda(\mathcal{C}) \chi_\mathcal{C} = \chi_N$

A map is balanced when the amount received over all the coalitions containing an agent i sums up to 1.

Example: $n = 3$, $\lambda(\mathcal{C}) = \begin{cases} \frac{1}{2} \text{ if } |\mathcal{C}| = 2 \\ 0 \text{ otherwise} \end{cases}$

	1	2	3
{1,2}	$\frac{1}{2}$	$\frac{1}{2}$	0
{1,3}	$\frac{1}{2}$	0	$\frac{1}{2}$
{2,3}	0	$\frac{1}{2}$	$\frac{1}{2}$

Each of the column sums up to 1.
$\frac{1}{2}\chi_{\{1,2\}} + \frac{1}{2}\chi_{\{1,3\}} + \frac{1}{2}\chi_{\{2,3\}} = \chi_{\{1,2,3\}}$

Characterization of games with non-empty core

Definition (Balanced game)

A game is **balanced** iff for each balanced map λ we have $\sum_{\mathcal{C} \subseteq N, \mathcal{C} \neq \emptyset} \lambda(\mathcal{C}) v(\mathcal{C}) \leqslant v(N)$.

Theorem (Bondareva Shapley)

A TU game has a non-empty core iff it is balanced.

Summary

- We introduced the core: a stability solution concept.
- We looked at some examples and geometrical representation
- We saw that the core can be empty.
- We proved that convex games have a non-empty core.
- We proved that Minimum Cost Spanning Tree game have a non-empty core
- We started to look at a characterization of the Bondareva-Shapley theorem

Coming next

- Characterization of games with non-empty core (Bondareva Shapley theorem), informal introduction to linear programming.
- Application of Bondareva-Shapley to market games.
- Other games with non-empty core.
- Computational complexity of the core.