Lecture 3

Characterization of games with non-empty core and games with coalition structures

3.1 Characterization of games with a non-empty core

We saw that the core may be empty, but that some classes of games have a non-empty core. The next issue is whether we can characterize the games with non-empty core. It turns out that the answer is yes, and the characterization has been found independently by Bondareva (1963) and Shapley (1967), resulting in what is now known as the Bondareva Shapley theorem. This result connects results from linear programming with the concept of the core. We will first describe the theorem and provide an intuition about the proof. Then, we will use this theorem to show that market games have a non-empty core.

3.1.1 Bondareva-Shapley theorem

Let $\mathcal{C} \subseteq N$. The *characteristic vector* $\chi_{\mathcal{C}}$ of $\mathcal{C}$ is the member of $\mathbb{R}^N$ defined by

$$\chi^i_{\mathcal{C}} = \begin{cases} 1 \text{ if } i \in \mathcal{C} \\ 0 \text{ if } i \in N \setminus \mathcal{C} \end{cases}$$

3.1.1. DEFINITION. [Map] A *map* is a function $2^N \setminus \emptyset \to \mathbb{R}_+$ that gives a positive weight to each coalition.

3.1.2. DEFINITION. [Balanced map] A function $\lambda : 2^N \setminus \emptyset \to \mathbb{R}_+$ is a *balanced map* iff $\sum_{\mathcal{C} \subseteq N} \lambda(\mathcal{C})\chi_{\mathcal{C}} = \chi_N$.

A map is balanced when the amount received over all the coalitions containing an agent i sums up to 1. We provide an example in Table 3.1 for a three-player game.

3.1.3. DEFINITION. [Balanced game] A game is *balanced* iff for each balanced map λ we have $\sum_{\mathcal{C} \subseteq N, \mathcal{C} \neq \emptyset} \lambda(\mathcal{C})v(\mathcal{C}) \leq v(N)$.

	1	2	3
$\{1,2\}$	$\frac{1}{2}$	$\frac{1}{2}$	0
$\{1,3\}$	$\frac{1}{2}$	0	$\frac{1}{2}$
$\{2,3\}$	0	$\frac{1}{2}$	$\frac{1}{2}$

$\lambda(\mathcal{C}) = \begin{cases} \frac{1}{2} \text{ if } |\mathcal{C}| = 2 \\ 0 \text{ otherwise} \end{cases}$

Each of the column sums up to 1.

$\frac{1}{2}\chi_{\{1,2\}} + \frac{1}{2}\chi_{\{1,3\}} + \frac{1}{2}\chi_{\{2,3\}} = \chi_{\{1,2,3\}}$

Table 3.1: Example of a balanced map for $n = 3$

3.1.4. THEOREM (BONDAREVA-SHAPLEY THEOREM). *A TU game has a non-empty core iff it is balanced.*

This theorem completely characterizes the set of games with a non-empty core. However, it is not always easy or feasible to check that it is a balanced game. The notion of map may not appear to be very intuitive. We will see in the next section that the concept of the map comes from linear programming.

Given a TU game (N, v), the definition of the core uses a set of linear constraints: $Core(N, v) = \{x \in \mathbb{R}^n \mid x(\mathcal{C}) \geq v(\mathcal{C}) \text{ for all } \mathcal{C} \subseteq N\}$. The idea is to use results from linear optimization to characterize the class of games with a non-empty core. We will use $\mathcal{V}(N) = \mathcal{V}$ to represent the set of all valuation functions on 2^N and $\mathcal{V}_{Core} = \{v \in \mathcal{V} \mid Core(N, v) \neq \emptyset\}$ is the set of games with non-empty core.

3.1.2 A very brief introduction to linear programming

A *linear program* has the following form: $\begin{cases} \max c^T x \\ \text{subject to} \begin{cases} Ax \leq b, \\ x \geq 0 \end{cases} \end{cases}$ where

- $x \in \mathbb{R}^n$ is a vector of n variables
- $c \in \mathbb{R}^n$ is the objective function
- A is a $m \times n$ matrix
- $b \in \mathbb{R}^n$ is a vector of size n

A and b represent the *linear constraints*. Let us look at a simple example:

maximize $8x_1 + 10x_2 + 5x_3$

$$\text{subject to} \begin{cases} 3x_1 & + & 4x_2 & + & 2x_3 & \leq & 7 & (1) \\ x_1 & + & x_2 & + & x_3 & \leq & 2 & (2) \end{cases}$$

$$A = \begin{pmatrix} 3 & 4 & 2 \\ 1 & 1 & 1 \end{pmatrix}, b = \begin{pmatrix} 7 \\ 2 \end{pmatrix}, c = \begin{pmatrix} 8 \\ 10 \\ 5 \end{pmatrix}.$$

A *feasible solution* is a solution that satisfies the constraints. For our example, we have:

- $\langle 0, 1, 1 \rangle$ is feasible, with objective function value 15.
- $\langle 1, 1, 0 \rangle$ is feasible, with objective function value 18, hence it is a better solution.

Next, we introduce the notion of the *dual* of a LP: it is another linear program which goal is to find an upper bound to the objective function of the original LP. Let us first look at our example and let us consider the following two linear transformations:

$(1) \times 1 + (2) \times 6 \Rightarrow 9x_1 + 10x_2 + 8x_3 \leq 19$

$(1) \times 2 + (2) \times 2 \Rightarrow 8x_1 + 10x_2 + 6x_3 \leq 18$

The coefficients are as large as in the obective function, hence the bound is an upper bound for the objective function. The solution cannot be better than 18, and we found one, so we have solved the problem! ✔

Primal	Dual
$\begin{cases} \max c^T x \\ \text{subject to } \begin{cases} Ax \leq b, \\ x \geq 0 \end{cases} \end{cases}$	$\begin{cases} \min y^T b \\ \text{subject to } \begin{cases} y^T A \geq c^T, \\ y \geq 0 \end{cases} \end{cases}$

3.1.5. THEOREM (DUALITY THEOREM). *When the primal and the dual are feasible, they have optimal solutions with equal value of their objective function.*

3.1.3 Linear Programming and the core

We consider the following *linear programming* problem:

$$(LP) \begin{cases} \min x(N) \\ \text{subject to } x(\mathcal{C}) \geq v(\mathcal{C}) \text{ for all } \mathcal{C} \subseteq N, S \neq \emptyset \end{cases}$$

$v \in \mathcal{V}_{core}$ iff the value of (LP) is $v(N)$.

The dual of (LP):

$$(DLP) \begin{cases} \max \sum_{\mathcal{C} \subseteq N} y_\mathcal{C} v(\mathcal{C}) \\ \text{subject to } \begin{cases} \sum_{\mathcal{C} \subseteq N} y_\mathcal{C} \chi_\mathcal{C} = \chi_N \text{ and,} \\ y_\mathcal{C} \geq 0 \text{ for all } \mathcal{C} \subseteq N, \mathcal{C} \neq \emptyset. \end{cases} \end{cases}$$

It follows from the duality theorem of linear programming:

(N, v) has a non empty core iff $v(N) \geq \sum_{\mathcal{C} \subseteq N} y_\mathcal{C} v(\mathcal{C})$ for all feasible vector $(y_\mathcal{C})_{\mathcal{C} \subseteq N}$ of (DLP). In the constraint of (DLP), we can recognize a balance map.

3.1.4 Application to market games

One example of coalitional games coming from the field of economics is a market game. This game models an environment where there is a given, fixed quantity of a set of continuous good. Initially, these goods are distributed among the players in an arbitrary way. The quantity of each good is called the endowment of the good. Each agent i has a valuation function that takes as input a vector describing its endowment for each good and that output a utility for possessing these goods (the agents do not perform any transformation, i.e., the goods are conserved as they are). To increase their utility, the agents are free to trade goods. When the agents are forming a coalition, they are trying to allocate the goods such that the social welfare of the coalition (i.e. the sum of the utility of each member of the coalition) is maximized. We now provide the formal definition.

A *market* is a quadruple (N, M, A, F) where

- N is a set of traders
- M is a set of m continuous good
- $A = (a_i)_{i \in N}$ is the initial endowment vector
- $F = (f_i)_{i \in N}$ is the valuation function vector, each f_i is continuous and concave.
- $v(S) = \max \left\{ \sum_{i \in S} f_i(x_i) \;\middle|\; x_i \in \mathbb{R}^m_+, \sum_{i \in S} x_i = \sum_{i \in S} a_i \right\}$
- we further assume that the f_i are continuous and concave.

Let us assume that the players form the grand coalition: all the players are in the market and try to maximize the sum of utility of the market. How should this utility be shared amond the players? One way to answer this question is by using an allocation that is in the core. One interesting property is that the core of such game is guaranteed to be non-empty, and one way to prove it is to use the Bondareva-Shapley theorem.

3.1.6. THEOREM. *Every Market Game is balanced*

Proof.

$f : \mathbb{R}^n \to \mathbb{R}$ is concave iff $\forall \alpha \in [0, 1]$, $\forall (x, y) \in \mathbb{R}^n$, $f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha) f(y)$. It follows from this definition that for $f : \mathbb{R} \to \mathbb{R}$, $\forall x \in \mathbb{R}^n$, $\forall \lambda \in \mathbb{R}^n_+$ such that $\sum_{i=1}^n \lambda_i = 1$, we have $f(\sum_{i=1}^n \lambda_i x_i) \leq \sum_{i=1}^n \lambda_i f(x_i)$.

Since the f_is are continuous, $\sum_{i \in S} f_i(x_i)$ is a continuous mapping from T=$\left\{ (x_i)_{i \in S} \mid \forall i \in \mathbb{R}^k_+, \forall x_i \in \mathbb{R}^k_+, \sum_{i \in S} x_i = \sum_{i \in S} a_i \right\}$ to $\mathbb{R}$. Moreover, T is compact (it is closed and bounded). Thanks to the extreme value theorem from calculus, we conclude that $\sum_{i \in S} f_i(x_i)$ attains a maximum.

For a coalition $S \subseteq N$, let $x^S = \langle x^S_1, \ldots, x^S_n \rangle$ be the endowment that achieves the maximum value for the coalition S, i.e., $v(S) = \sum_{i \in S} f_i(x^S_i)$. In other words, the members of S have made some trades that have improved the value of the coalition S up to its maximal value.

Let λ be a balanced map. Let $y \in \mathbb{R}^n_+$ defined as follows: $y_i = \sum_{S \in \mathcal{C}_i} \lambda_S x^S_i$ where $\mathcal{C}_i$ is the set of coalitions that contains agent i.

First, note that y is a feasible payoff function.

$$\begin{aligned}
\sum_{i\in N} y_i &= \sum_{i\in N}\sum_{S\in\mathcal{C}_i} \lambda_S x_i^S = \sum_{S\subseteq N}\sum_{i\in S} \lambda_S x_i^S = \sum_{S\subseteq N} \lambda_S \sum_{i\in S} x_i^S \\
&= \sum_{S\subseteq N} \lambda_S \sum_{i\in S} a_i \text{ since } x_i^S \text{ was achieved by a sequence of trades within the members of } S \\
&= \sum_{i\in N} a_i \sum_{S\in\mathcal{C}_i} \lambda_S \\
&= \sum_{i\in N} a_i \text{ as } \lambda \text{ is balanced, } \left(\begin{array}{l}\text{i.e., the sum of the weights over all coalitions} \\ \text{of one agent sums up to 1}\end{array}\right)
\end{aligned}$$

Then, by definition of v, we have $v(N) \geq \sum_{i\in N} f_i(y_i)$. ✔

The f_i are concave and since $\sum_{S\in\mathcal{C}_i} \lambda_S = 1$, we have

$$f_i(\sum_{S\in\mathcal{C}_i} \lambda_S x_i^S) \geq \sum_{S\in C_i} \lambda_S f_i(x_i^S).$$

It follows:

$$v(N) \geq \sum_{i\in N} f_i(y_i) \geq \sum_{i\in N} f_i(\sum_{S\in\mathcal{C}_i} \lambda_S x_i^S) \geq \sum_{i\in N}\sum_{S\in\mathcal{C}_i} \lambda_S f_i(x_i^S) \geq \sum_{S\subseteq N} \lambda_S \sum_{i\in S} f_i(x_i^S) \geq \sum_{S\subseteq N} \lambda_S v(S).$$

This inequality proves that the game is balanced. ✔ □

3.2 Extension of the core

There are few extensions to the concept of the Core. As discussed above, one main issue of the Core is that it can be empty. In particular, a member of a coalition may block the formation so as to gain a very small payoff. When the cost of building a coalition is considered, it can be argued that it is not worth blocking a coalition for a small utility gain. The strong and weak ϵ-Core concepts model this possibility. The constraints defining the strong (respectively the weak) ϵ-Core become $\forall T \subseteq N, x(T) \geq v(T) - \epsilon$, (respectively $\forall T \subseteq N, x(T) \geq v(T) - |T| \cdot \epsilon$). In the weak Core, the minimum amount of utility required to block a coalition is per player, whereas for the strong Core, it is a fixed amount. If one picks ϵ large enough, the strong or weak ϵ-core will exist. When decreasing the value of ϵ, there will be a threshold $\epsilon^\star$ such that for $\epsilon < \epsilon^\star$ the ϵ core ceases to be non-empty. This special ϵ-core is then called the the *least core*.

3.3 Games with Coalition Structure

Thus far, we stated that the grand coalition is formed. With this definition, checking whether the core is empty amounts to checking whether the grand coalition is stable. In many studies in economics, the superadditivity of the valuation function is not explicitly stated, but it is implicitly assumed and hence, it makes sense to consider only the grand coalition. But when the valuation function is not superadditive, agents may have an incentive to form a different partition.

We recall that a coalition structure (CS) is a partition of the grand coalitions. If $\mathcal{S}$ is a CS, then $\mathcal{S} = \{\mathcal{C}_1, \dots, \mathcal{C}_m\}$ where each $\mathcal{C}_i$ is a coalition such that $\cup_{i=1}^m \mathcal{C}_i = N$ and $i \neq j \Rightarrow \mathcal{C}_i \cap \mathcal{C}_j = \emptyset$.

Aumann and Drèze discuss why the coalition formation process may generate a CS that is not the grand coalition [1]. One reason they mention is that the valuation may not be superadditive (and they provide some discussion about why it may be the case). Another reason is that a CS may "reflect considerations that are excluded from the formal description of the game by necessity (impossibility to measure or communicate) or by choice" [1]. For example, the affinities can be based on location, or trust relations, etc.

3.3.1. DEFINITION. [Game with coalition structure] A *game with coalition structure* is a triplet (N, v, S), where (N, v) is a TU game, and S is a particular CS. In addition, transfer of utility is only permitted within (not between) the coalitions of S, i.e., $\forall \mathcal{C} \in S, x(\mathcal{C}) \leq v(\mathcal{C})$.

Another way to understand this definition is to consider that the problems of deciding which coalition forms and how to share the coalition's payoff are decoupled: the choice of the coalition is made first and results in the CS. Only the payoff distribution choice is left open. The agents are allowed to refer to the value of coalition with agents oustide of their coalition (i.e., opportunities they would get outside of their coalition) to negotiate a better payoff. Aumann and Drèze use an example of researchers in game theory that want to work in their own country, i.e., they want to belong to the coalition of game theorists of their country. They can refer to offers from foreign countries in order to negotiate their salaries. Note that the agents' goal is not to change the CS, but only to negotiate a better payoff for themselves.

First, we need to define the set of possible payoffs: the payoff distributions such that the sum of the payoff of the members of a coalition in the CS does not exceed the value of that coalition. More formally:

3.3.2. DEFINITION. [Feasible payoff] Let $(N, v, \mathcal{S})$ be a TU game with CS. The set of *feasible payoff distributions* is $X_{(N,v,\mathcal{S})} = \{x \in \mathbb{R}^n \mid \forall \mathcal{C} \in \mathcal{S} x(\mathcal{C}) \leq v(\mathcal{C})\}$.

A payoff distribution x is *efficient* with respect to a CS $\mathcal{S}$ when $\forall \mathcal{C} \in \mathcal{S}$, $\sum_{i \in \mathcal{C}} x_j = v(\mathcal{C})$. A payoff distribution is an *imputation* when it is efficient (with respect to the current CS) and individually rational (i.e., $\forall i \in N$, $x_i \geq v(\{i\})$). The set of all imputations for a CS $\mathcal{S}$ is denoted by $\mathcal{I}mp(\mathcal{S})$. We can now state the definition of the core:

3.3.3. DEFINITION. [Core] The core of a game $(N, v, \mathcal{S})$ is the set of all PCs $(\mathcal{S}, x)$ such that $x \in \mathcal{I}mp(\mathcal{S})$ and $\forall \mathcal{C} \subseteq N$, $\sum_{i \in \mathcal{C}} x_j \geq v(\mathcal{C})$, i.e.,

$$core(N, v, \mathcal{S}) = \{x \in \mathbb{R}^n \mid (\forall \mathcal{C} \in \mathcal{S}, x(\mathcal{C}) \leq v(\mathcal{C})) \wedge (\forall \mathcal{C} \subseteq N, x(\mathcal{C}) \geq v(\mathcal{C}))\}.$$

We now provide a theorem by Aumann and Drèze which shows that the core satisfies a desirable properties: if two agents can be substituted, then a core allocation must provide them identical payoffs.

3.3.4. DEFINITION. [Substitutes] Let (N, v) be a game and $(i, j) \in N^2$. Agents i and j are *substitutes* iff $\forall \mathcal{C} \subseteq N \setminus \{i, j\}$, $v(\mathcal{C} \cup \{i\}) = v(\mathcal{C} \cup \{j\})$.

Since the agents have the same impact on all coalitions that do not include them, it would be fair if they obtained the same payoff. For the core of a game in CS, this is indeed the case.

3.3.5. THEOREM. *Let $(N, v, \mathcal{S})$ be a game with coalition structure, let i and j be substitutes, and let $x \in core(N, v, \mathcal{S})$. If i and j belong to different members of $\mathcal{S}$, then $x_i = x_j$.*

Proof. Let $(i, j) \in N^2$ be substitutes, $\mathcal{C} \in \mathcal{S}$ such that $i \in \mathcal{C}$ and $j \notin \mathcal{C}$. Let $x \in Core(N, v, \mathcal{S})$. Since i and j are substitutes, we have

$$v((\mathcal{C} \setminus \{i\}) \cup \{j\}) = v((\mathcal{C} \setminus \{i\}) \cup \{i\}) = v(\mathcal{C}).$$

Since $x \in Core(N, v, \mathcal{S})$, we have $\forall \mathcal{C} \subseteq N$, $x(\mathcal{C}) \geq v(\mathcal{C})$, we apply this to the coalition $(\mathcal{C} \setminus \{i\}) \cup \{j\}$:
$0 \geq v((\mathcal{C} \setminus \{i\}) \cup \{j\}) - x((\mathcal{C} \setminus \{i\}) \cup \{j\}) = v(\mathcal{C}) - x(\mathcal{C}) + x_i - x_j$. Since $\mathcal{C} \in \mathcal{S}$ and $x \in Core(N, v, \mathcal{S})$, we have $x(\mathcal{C}) = v(\mathcal{C})$. We can then simplified the previous expression and we obtain $x_j \geq x_i$.

Since i and j play symmetric roles, we have also $x_i \geq x_j$ and finally, we obtain $x_i = x_j$. ✔ □

Aumann and Drèze made a link from a game with CS to a special superadditive game $(N, \hat{v})$ called the superadditive cover [1].

3.3.6. DEFINITION. [Superadditive cover] The *superadditive cover* of (N, v) is the game $(N, \hat{v})$ defined by

$$\begin{cases} \hat{v}(\mathcal{C}) = \max\limits_{\mathcal{P} \in \mathscr{S}_\mathcal{C}} \left\{ \sum\limits_{T \in \mathcal{P}} v(T) \right\} \forall \mathcal{C} \subseteq N \setminus \emptyset \\ \hat{v}(\emptyset) = 0 \end{cases}$$

In other words, $\hat{v}(\mathcal{C})$ is the maximal value that can be generated by any partition of $\mathcal{C}$[1]. The superadditive cover is a superadditive game. The following theorem, from [1] shows that a necessary condition for $(N, v, \mathcal{S})$ to have a non empty core is that $\mathcal{S}$ is an optimal CS.

3.3.7. THEOREM. *Let $(N, v, \mathcal{S})$ be a game with coalition structure. Then*

a) $Core(N, v, \mathcal{S}) \neq \emptyset$ *iff* $Core(N, \hat{v}) \neq \emptyset \wedge \hat{v}(N) = \sum_{\mathcal{C}\in\mathcal{S}} v(\mathcal{C})$

b) *if* $Core(N, v, \mathcal{S}) \neq \emptyset$, *then* $Core(N, v, \mathcal{S}) = Core(N, \hat{v})$

Proof. Proof of part **a)**

$\Rightarrow$ Let $x \in Core(N, v, \mathcal{S})$. We show that $x \in Core(N, \hat{v})$ as well. Let $\mathcal{C} \subseteq N \setminus \emptyset$ and $P_\mathcal{C} \in \mathscr{S}_\mathcal{C}$ be a partition of $\mathcal{C}$. By definition of the core, for every $S \subseteq N$ we have $x(S) \geq v(S)$. The payoff of coalition $\mathcal{C}$ is

$$x(\mathcal{C}) = \sum_{i\in\mathcal{C}} x_i = \sum_{S\in P_\mathcal{C}} x(S) \geq \sum_{S\in P_\mathcal{C}} v(S),$$

which is valid for all partitions of $\mathcal{C}$. Hence, $x(\mathcal{C}) \geq \max_{\mathcal{P}_\mathcal{C}\in\mathscr{S}_\mathcal{C}} \sum_{S\in\mathcal{P}_\mathcal{C}} v(S) = \hat{v}(\mathcal{C})$.

We have just proved $\forall \mathcal{C} \subseteq N \setminus \emptyset$, $x(\mathcal{C}) \geq \hat{v}(\mathcal{C})$, and so x is *group rational*. ✔

We now need to prove that $\hat{v}(N) = \sum_{\mathcal{C}\in\mathcal{S}} v(\mathcal{C})$.

$x(N) = \sum_{\mathcal{C}\in\mathcal{S}} v(\mathcal{C})$ since x is in the core of $(N, v, \mathcal{S})$ (efficient). Applying the inequality above, we have $x(N) = \sum_{\mathcal{C}\in\mathcal{S}} v(\mathcal{C}) \geq \hat{v}(N)$.
Applying the definition of the valuation function $\hat{v}$, we have $\hat{v}(N) \geq \sum_{\mathcal{C}\in\mathcal{S}} v(\mathcal{C})$.
Consequently, $\hat{v}(N) = \sum_{\mathcal{C}\in\mathcal{S}} v(\mathcal{C})$ and it follows that x is *efficient* for the game $(N, \hat{v})$✔

Hence $x \in Core(N, \hat{v})$.

$\Leftarrow$ Let's assume $x \in Core(N, \hat{v})$ and $\hat{v}(N) = \sum_{\mathcal{C}\in\mathcal{S}} v(\mathcal{C})$. We need to prove that $x \in Core(N, v, \mathcal{S})$.

[1]Note that for the grand coalition, we have $\hat{v}(N) = \max_{\mathcal{P}\in\mathscr{S}_N} \left\{\sum_{T\in\mathcal{P}} v(T)\right\}$, i.e., $\hat{v}(N)$ is the maximum value that can be produced by N. We call it the *value of the optimal coalition structure*. For some application, on issue (that will be studied later) is to find this value.

For every $\mathcal{C} \subseteq N$, $x(\mathcal{C}) \geq \hat{v}(\mathcal{C})$ since x is in the core of $Core(N, \hat{v})$. Then $x(\mathcal{C}) \geq \max\limits_{\mathcal{P}_\mathcal{C} \in \mathscr{S}_\mathcal{C}} \sum\limits_{S \in \mathcal{P}_\mathcal{C}} v(S) \geq v(\mathcal{C})$ using $\{\mathcal{C}\}$ as a partition of $\mathcal{C}$, which proves x is group rational. ✔

$x(N) = \hat{v}(N) = \sum_{\mathcal{C} \in \mathcal{S}} v(\mathcal{C})$ since x is efficient. It follows that $\forall \mathcal{C} \in \mathcal{S}$, we must have $x(\mathcal{C}) = v(\mathcal{C})$, which proves x is feasible for the CS $\mathcal{S}$, and that x is efficient.✔

Hence, $x \in Core(N, v, \mathcal{S})$. ✔

proof of part **b)**:

We have just proved that $x \in Core(N, \hat{v})$ implies that $x \in Core(N, v, \mathcal{S})$ and $x \in Core(N, v, \mathcal{S})$ implies that $x \in Core(N, \hat{v})$. This proves that if $Core(N, v, \mathcal{S}) \neq \emptyset$, $Core(N, \hat{v}) = Core(N, v, \mathcal{S})$.

□

Bibliography

[1] Robert J. Aumann and Jacques H Drèze. Cooperative games with coalition structures. *International Journal of Game Theory*, 3(4):217–237, 1974.

[2] Donald B. Gillies. *Some theorems on n-person games*. PhD thesis, Department of Mathematics, Princeton University, Princeton, N.J., 1953.

[3] James P. Kahan and Amnon Rapoport. *Theories of Coalition Formation*. Lawrence Erlbaum Associates, Publishers, 1984.

[4] Martin J. Osborne and Ariel Rubinstein. *A Course in Game Theory*. The MIT Press, 1994.

[5] Bezalel Peleg and Peter Sudhölter. *Introduction to the theory of cooperative cooperative games*. Springer, 2nd edition, 2007.

[6] Tuomas W. Sandholm, Kate S. Larson, Martin Andersson, Onn Shehory, and Fernando Tohmé. Coalition structure generation with worst case guarantees. *Artificial Intelligence*, 111(1–2):209–238, 1999.