

Lecture 4

The bargaining set.

The notion of core stability is maybe the most natural way to describe stability. However, some games have an empty core. If agents adopts the notion of the core, they will be unable to reach an agreement about the payoff distribution. If they still want to benefit from the cooperation with other agents, they need to relax the stability requirements. In this lecture, we will see that the notion of bargaining set is one way to reach an argument and to maintain some notion of stability (though of course, it is a weaker version of stability).

The definition of the bargaining set is due to Davis and Maschler [2]. This notion is about the stability of a given coalition structure (CS). The agents do not try to change the nature of the CS, but simply to find a way to distribute the value of the different coalitions between the members of each coalition. Let us assume that a payoff distribution is proposed. Some agents may form an objection against this payoff distribution by showing a problem of that distribution and offering a different payoff distribution that eliminates this issue (or improve the situation). If all other agents agree with this objection, the payoff distribution should change. However, some other agents may form a counter-objection showing some shortcomings of the objection. The idea of stability in this context is to ensure that, for each possible objection, there exists a counter-objection. When this is the case, there is no ground for changing the payoff distribution, which provides some stability. In the following, we will describe the precise notion of objections and counter-objections.

4.1 Objections, counter-objections and the prebargaining set

Let (N, v, \mathcal{S}) be a game with coalition structure and x an imputation. For the bargaining set, an *objection* from an agent i against a payoff distribution x is targeting a particular agent j , in the hope of obtaining a payment from j . The goal of agent i is to show that agent j gets too much payoff as there are some ways in which some agents

but j can benefit. The objection can take the following (informal) form:

I get too little in the imputation x , and agent j gets too much! I can form a coalition that excludes j in which some members benefit and all members are at least as well off as in x .

We recall that in a game with coalition structure (N, v, \mathcal{S}) , the agents do not try to change the CS, but only obtain a better payoff. The set $X_{(N, v, \mathcal{S})}$ of *feasible* payoff vectors for (N, v, \mathcal{S}) is defined as $X_{(N, v, \mathcal{S})} = \{x \in \mathbb{R}^n \mid \forall \mathcal{C} \in \mathcal{S}, \sum_{i \in \mathcal{C}} x_i \leq v(\mathcal{C})\}$. We are now ready to formally define an objection.

4.1.1. DEFINITION. [Objection] Let (N, v, \mathcal{S}) be a game with coalition structure, $x \in X_{(N, v, \mathcal{S})}$, $\mathcal{C} \in \mathcal{S}$ be a coalition, and i and j two distinct members of \mathcal{C} ($(i, j) \in \mathcal{C}^2$, $i \neq j$). An *objection of i against j* is a pair (P, y) where

- $P \subseteq N$ is a coalition such that $i \in P$ and $j \notin P$.
- $y \in \mathbb{R}^p$ where p is the size of P
- $y(P) \leq v(P)$ (y is a feasible payoff distribution for the agents in P)
- $\forall k \in P$, $y_k \geq x_k$ and $y_i > x_i$ (agent i strictly benefits from y , and the other members of P do not do worse in y than in x .)

An objection is a pair (P, y) that is announced by an agent i against a particular agent j and a payoff distribution x . It can be understood as a *potential threat* to form coalition P , which contains i but not j . If the agents in P really deviate, agent i will benefit (strictly), and the other agents in P are guaranteed not to be worse off (and may even benefit from the deviation). The goal is not to change the CS, but simply to update the payoff distribution. In this case, agent i is calling for a transfer of utility from agent j to agent i .

The agent that is targeted by the threat may try to show that it deserves the payoff x_j . To do so, its goal is to show that, if the threat was implemented, there is another deviation that would ensure that j can still obtain x_j and that no agent (except maybe agent i) would be worse off. In that case, we say that the objection is *ineffective*. Agent j can summarize its argument by saying:

I can form a coalition that excludes agent i in which all agents are at least as well off as in x , and as well off as in the payoff proposed by i for those who were offered to join i in the argument.

The formal definition of a counter-objection is the following.

4.1.2. DEFINITION. [Counter-objection] A *counter-objection* of agent j to the objection (P, y) of agent i is a pair (Q, z) where

- $Q \subseteq N$ is a coalition such that $j \in Q$ and $i \notin Q$.

- $z \in \mathbb{R}^q$ where q is the size of Q
- $z(Q) \leq v(Q)$ (z is a feasible payoff distribution for the agents in Q)
- $\forall k \in Q, z_k \geq x_k$ (the members of Q get at least the value in x)
- $\forall k \in Q \cap P, z_k \geq y_k$ (the members of Q which are also members of P get at least the value promised in the objection)

In a counter-objection, agent j must show that it can protect its payoff x_j in spite of the existing objection of i . Agents in the deviating coalition Q should improve their payoff compared to x . For those who were also members of the deviating coalition P with agent i , they should make sure that they obtain a better payoff than in y . In this way, all agents in P and Q benefit. Note that agent i is in P and not in Q , and consequently, i may be worse off in this counter-objection.

When an objection has a counter-objection, no agent will be willing to follow agent i and implement the threat. Hence, the agents do not have any incentives to change the payoff distribution and the payoff is stable.

4.1.3. DEFINITION. [Stability] Let (N, v, \mathcal{S}) a game with coalition structure. A vector $x \in X_{(N, v, \mathcal{S})}$ is *stable* iff for each objection at x there is a counter-objection.

The definition of the pre-bargaining set is then simply the set of payoff distributions that are stable. Note that they are stable for the specific definition of objections and counter-objections we presented (we can, and we will, think about other ways to define objections and counter-objections).

4.1.4. DEFINITION. Pre-bargaining set The *pre-bargaining set* (*preBS*) is the set of all stable members of $X_{(N, v, \mathcal{S})}$.

We will explain later the presence of the prefix pre in the definition. For now, we need to wonder about the relationship with the core. The idea was to relax the stability requirements of the core. The following lemma states that indeed, we have only relaxed them:

4.1.5. LEMMA. Let (N, v, \mathcal{S}) a game with coalition structure, we have

$$\text{Core}(N, v, \mathcal{S}) \subseteq \text{preBS}(N, v, \mathcal{S}).$$

Proof. Let us assume that the core of (N, v, \mathcal{S}) is non-empty and that $x \in \text{Core}(N, v, \mathcal{S})$. Given the payoff x , no agent i has any objection against any other agent j . Hence, there are no objections, and the payoff is stable according to the pre-bargaining set. \square

4.2 An example

Let us now consider an example using a 7-player simple majority game, i.e., we consider the game $(\{1, 2, 3, 4, 5, 6, 7\}, v)$ defined as follows:

$$v(\mathcal{C}) = \begin{cases} 1 & \text{if } |\mathcal{C}| \geq 4 \\ 0 & \text{otherwise} \end{cases}$$

Let us consider $x = \langle -\frac{1}{5}, \frac{1}{5}, \dots, \frac{1}{5} \rangle$. It is clear that $x(N) = 1$. Let us now prove that x is in the pre-bargaining set of the game $(N, v, \{N\})$.

First note that objections within members of $\{2, 3, 4, 5, 6, 7\}$ will have a counter-objection by symmetry (i.e., for $i, j \in \{2, 3, 4, 5, 6, 7\}$, if i has an objection (P, y) against j , j can use the counter-objection (Q, z) with $Q = P \setminus \{i\} \cup \{j\}$ and $z_k = y_k$ for $k \in P \setminus \{i\}$ and $z_j = y_i$).

Hence, we only have to consider two type of objections (P, y) : the ones of 1 against a member of $\{2, 3, 4, 5, 6, 7\}$, and the ones from a members of $\{2, 3, 4, 5, 6, 7\}$ against i . We are going to treat only the first case, we leave the second as an exercise.

Let us consider an objection (P, y) of agent i against a member of $\{2, 3, 4, 5, 6, 7\}$. Since the members $\{2, \dots, 7\}$ play symmetric roles, we consider an objection of 1 against 7 using successively $P = \{1, 2, 3, 4, 5, 6\}$, $P = \{1, 2, 3, 4, 5\}$, $P = \{1, 2, 3, 4\}$, $P = \{1, 2, 3\}$, $P = \{1, 2\}$ and $P = \{1\}$. For each case, we will look for a counter-objection (Q, z) of player 7.

- We consider that $P = \{1, 2, 3, 4, 5, 6\}$. We need to find the payoff vector $y \in \mathbb{R}^6$ so that (P, y) is an objection.

$$y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \dots, \frac{1}{5} + \alpha_6 \rangle,$$

The conditions for (P, y) to be an objection are the following:

- each agent is as well off as in x : $\alpha > -\frac{1}{5}, \alpha_i \geq 0$
- y is feasible for coalition P : $\sum_{i=2}^6 (\alpha_i + \frac{1}{5}) + \alpha \leq 1$.

w.l.o.g $0 \leq \alpha_2 \leq \alpha_3 \leq \alpha_4 \leq \alpha_5 \leq \alpha_6$.

$$\text{Then } \sum_{i=2}^6 \left(\frac{1}{5} + \alpha_i \right) + \alpha = \frac{5}{5} + \sum_{i=2}^6 \alpha_i + \alpha = 1 + \sum_{i=2}^6 \alpha_i + \alpha \leq 1.$$

$$\text{Then } \sum_{i=2}^6 \alpha_i \leq -\alpha < \frac{1}{5}.$$

- ⇒ We need to find a counter-objection for (P, y) .

claim: we can choose $Q = \{2, 3, 4, 7\}$ and $z = \langle \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5} + \alpha_4, \frac{1}{5} + \alpha_5 \rangle$

$$z(Q) = \frac{1}{5} + \alpha_2 + \frac{1}{5} + \alpha_3 + \frac{1}{5} + \alpha_4 + \frac{1}{5} + \alpha_5 = \frac{4}{5} + \sum_{i=2}^5 \alpha_i \leq 1 \text{ since } \sum_{i=2}^5 \alpha_i \leq \sum_{i=2}^6 \alpha_i < \frac{1}{5} \text{ so } z \text{ is feasible.}$$

It is clear that $\forall i \in Q, z_i \geq x_i$ ✓ and that $\forall i \in Q \cap P, z_i \geq y_i$ ✓

Hence, (Q, z) is a counter-objection. ✓

- Now, let us consider that $P = \{1, 2, 3, 4, 5\}$. The vector $y = \langle \alpha, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5} + \alpha_4, \frac{1}{5} + \alpha_5 \rangle$ is an objection when

$$\alpha > -\frac{1}{5}, \alpha_i \geq 0, \sum_{i=2}^5 (\frac{1}{5} + \alpha_i) + \alpha \leq 1$$

$$\text{This time, we have } \sum_{i=2}^5 (\frac{1}{5} + \alpha_i) + \alpha = \frac{4}{5} + \sum_{i=2}^5 \alpha_i + \alpha \leq 1$$

$$\text{then } \sum_{i=2}^5 \alpha_i \leq 1 - \frac{4}{5} - \alpha = \frac{1}{5} - \alpha \text{ and finally } \sum_{i=2}^5 \alpha_i \leq \frac{1}{5} - \alpha < \frac{2}{5}.$$

⇒ We need to find a counter-objection to (P, y)

claim: we can choose $Q = \{2, 3, 6, 7\}$, $z = \langle \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3, \frac{1}{5}, \frac{1}{5} \rangle$

It is clear that $\forall i \in Q, z_i \geq x_i$ ✓ and $\forall i \in P \cap Q, z_i \geq y_i$ (for agent 2 and 3).

$z(Q) = \frac{1}{5} + \alpha_2 + \frac{1}{5} + \alpha_3 + \frac{1}{5} + \frac{1}{5} = \frac{4}{5} + \alpha_2 + \alpha_3$. We have $\alpha_2 + \alpha_3 < \frac{1}{5}$, otherwise, we would have $\alpha_2 + \alpha_3 \geq \frac{1}{5}$ and since the α_i are ordered, we would then have

$$\sum_{i=2}^5 \alpha_i \geq \frac{2}{5}, \text{ which is not possible. Hence } z(Q) \leq 1 \text{ which proves } z \text{ is feasible}$$

✓

Using similar arguments, we find a counter-objection for each other objections (you might want to fill in the details at home).

- $P = \{1, 2, 3, 4\}$, $y = \langle \alpha, \frac{1}{5} + \alpha_1, \frac{1}{5} + \alpha_2, \frac{1}{5} + \alpha_3 \rangle$, $\alpha > -\frac{1}{5}, \alpha_i \geq 0, \sum_{i=2}^4 \alpha_i + \alpha \leq \frac{2}{5} \Rightarrow \sum_{i=2}^4 \alpha_i \leq \frac{2}{5} - \alpha < \frac{3}{5}$.

⇒ $Q = \{2, 5, 6, 7\}$, $z = \langle \frac{1}{5} + \alpha_2, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \rangle$ since $\alpha_2 \leq \frac{1}{5}$

- $|P| \leq 3$ $P = \{1, 2, 3\}$, $v(P) = 0$, $y = \langle \alpha, \alpha_1, \alpha_2 \rangle$, $\alpha > -\frac{1}{5}, \alpha_i \geq 0, \alpha_1 + \alpha_2 \leq -\alpha < \frac{1}{5}$

⇒ $Q = \{4, 5, 6, 7\}$, $z = \langle \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5} \rangle$ will be a counter-objection (1 cannot provide more than $\frac{1}{5}$ to any other agent).

- For each possible objection of 1, we found a counter-objection. Using similar arguments, we can find a counter-objection to any objection of player 7 against player 1.

⇒ $x \in \text{preBS}(N, v, \mathcal{S})$. ✓

4.3 The bargaining set

In the example, agent 1 gets $-\frac{1}{5}$ when $v(\mathcal{C}) \geq 0$ for all coalition $\mathcal{C} \subseteq N$! This shows that the pre-bargaining set may *not* be individually rational.

Let $I(N, v, \mathcal{S}) = \{x \in X_{(N, v, \mathcal{S})} \mid x_i \geq v(\{i\}) \forall i \in N\}$ be the *set of individually rational payoff vector* in $X_{(N, v, \mathcal{S})}$. Given most of the games, the set of individually rational payoffs is non-empty. For some classes of games, we can show it is the case, as shown by the following lemma.

4.3.1. LEMMA. *If a game is weakly superadditive, $I(N, v, \mathcal{S}) \neq \emptyset$.*

Proof. A game (N, v) is weakly superadditive when $\forall \mathcal{C} \subseteq N$ and $i \notin \mathcal{C}$, we have that $v(\mathcal{C}) + v(\{i\}) \leq v(\mathcal{C} \cup \{i\})$. Let us consider that, for $\mathcal{C} \in \mathcal{S}$, each agent in \mathcal{C} gets its marginal contribution given an ordering of agents in \mathcal{C} . Since for all $\mathcal{C} \subset N$, $i \notin \mathcal{C}$ $v(\mathcal{C} \cup \{i\}) - v(\mathcal{C}) \geq v(\{i\})$, we know that $x_i \geq v(\{i\})$. Hence, there exists a payoff distribution in $I(N, v, \mathcal{S})$. \square

Since we want a solution concept to be at least an imputation (i.e., efficient and individually rational), we define the bargaining set to be the set of payoff distributions that are individually rational and in the pre-bargaining set.

4.3.2. DEFINITION. Bargaining set Let (N, v, \mathcal{S}) a game in coalition structure. The *bargaining set* (BS) is defined by

$$BS(N, v, \mathcal{S}) = I(N, v, \mathcal{S}) \cap preBS(N, v, \mathcal{S}).$$

Of course, this restriction does not have any negative impact on the relationship between the core and the bargaining set.

4.3.3. LEMMA. *We have $Core(N, v, \mathcal{S}) \subseteq BS(N, v, \mathcal{S})$.*

This lemma shows that we have relaxed the requirements of core stability. One question is whether we have relaxed them enough to guarantee that this new stability concept is guaranteed to be non-empty. The answer to this question is yes!

4.3.4. THEOREM. *Let (N, v, \mathcal{S}) a game with coalition structure. Assume that $I(N, v, \mathcal{S}) \neq \emptyset$. Then the bargaining set $BS(N, v, \mathcal{S}) \neq \emptyset$.*

It is possible to give a direct proof of this theorem (for example, see the proof in Section 4.2 in *Introduction to the Theory of Cooperative Games* [6]). We will not present this proof now, but we will prove this theorem differently in a coming lecture.

4.4 One issue with the bargaining set

We relaxed the requirements of core stability to come up with a solution concept that is guaranteed to be non-empty. In doing so, the stable payoff distributions may have some issues, and we are going to consider one using the example of a weighted voting game. We first recall the definition.

4.4.1. DEFINITION. weighted voting games A game $(N, w_{i \in N}, q, v)$ is a *weighted voting game* when v satisfies unanimity, monotonicity and the valuation function is defined as

$$v(S) = \begin{cases} 1 & \text{when } \sum_{i \in S} w_i \geq q \\ 0 & \text{otherwise} \end{cases}$$

We note such a game by $(q : w_1, \dots, w_n)$

We consider the 6-player weighted majority game $(3:1,1,1,1,1,0)$. Agent 6 is a null player since its weight is 0, in other words, its presence does not affect by any means the decision taken by the other agents. Nevertheless the following payoff distribution is in the bargaining set $x = \langle \frac{1}{7}, \dots, \frac{1}{7}, \frac{2}{7} \rangle \in BS(N, v)$! This may be quite surprising as the null player receives the most payoff, but none of the other agents are able to provide an objection that is not countered!

One of the desirable properties of a payoff was to be *reasonable from above*. We recall that x is *reasonable from above* if $\forall i \in N \ x^i \leq mc_i^{max}$ where $mc_i^{max} = \max_{C \subseteq N \setminus \{i\}} v(C \cup \{i\}) - v(C)$. mc_i^{max} is the strongest *threat* that an agent can use against a coalition. It is desirable that no agent gets more than mc_i^{max} , as it never contributes more than mc_i^{max} in any coalition i can join. The previous example shows that the bargaining set is not *reasonable from above*: the dummy agent gets more than $\max_{C \subseteq N \setminus \{6\}} (v(C \cup \{6\}) - v(C)) = 0$.✘

Proof.

This proof will be part of homework 2. □

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